

SHARP LEHMER MEAN BOUNDS FOR NEUMAN MEANS WITH APPLICATIONS

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Abstract. In the article, we present the best possible parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\beta_1, \beta_2, \beta_3, \beta_4$ such that the double inequalities

$$L_{\alpha_1}(a, b) < N_{AG}(a, b) < L_{\beta_1}(a, b), \quad L_{\alpha_2}(a, b) < N_{GA}(a, b) < L_{\beta_2}(a, b),$$

$$L_{\alpha_3}(a, b) < N_{QA}(a, b) < L_{\beta_3}(a, b), \quad L_{\alpha_4}(a, b) < N_{AQ}(a, b) < L_{\beta_4}(a, b)$$

hold for all $a, b > 0$ with $a \neq b$, where $L_p(a, b) = (a^{p+1} + b^{p+1}) / (a^p + b^p)$ is the p th Lehmer mean, and $N_{AG}(a, b)$, $N_{GA}(a, b)$, $N_{QA}(a, b)$ and $N_{AQ}(a, b)$ are the Neuman means. As applications, we find several sharp inequalities involving the hyperbolic, trigonometric and inverse trigonometric functions.

1. Introduction

For $p \in \mathbb{R}$ and $a, b > 0$ with $a \neq b$, the Schwab-Borchardt mean $SB(a, b)$ [1, 2] and p th Lehmer mean $L_p(a, b)$ [3] of a and b are respectively defined by

$$SB(a, b) = \begin{cases} \frac{\sqrt{b^2 - a^2}}{\arccos(a/b)}, & a < b, \\ \frac{\sqrt{a^2 - b^2}}{\cosh^{-1}(a/b)}, & a > b \end{cases}$$

and

$$L_p(a, b) = \frac{a^{p+1} + b^{p+1}}{a^p + b^p}, \quad (1.1)$$

where $\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})$ is the inverse inverse hyperbolic cosine function.

It is well known that the Lehmer mean $L_p(a, b)$ is strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, the Schwab-Borchardt mean $SB(a, b)$ is strictly increasing in both a and b , nonsymmetric and homogeneous of degree 1 with respect to a and b . Many symmetric bivariate means are special cases of the Lehmer mean or Schwab-Borchardt mean. For example, $L_{-1}(a, b) = 2ab / (a + b) = H(a, b)$ is

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the harmonic mean, $L_{-1/2}(a, b) = \sqrt{ab} = G(a, b)$ is the geometric mean, $L_0(a, b) = (a + b)/2 = A(a, b)$ is the arithmetic mean, $L_1(a, b) = (a^2 + b^2)/(a + b) = C(a, b)$ is the contraharmonic mean, $SB[G(a, b), A(a, b)] = (a - b)/[2 \arcsin((a - b)/(a + b))] = P(a, b)$ is the first Seiffert mean, $SB[A(a, b), Q(a, b)] = (a - b)/[2 \arctan((a - b)/(a + b))] = T(a, b)$ is the second Seiffert mean, $SB[Q(a, b), A(a, b)] = (a - b)/[2 \sinh^{-1}((a - b)/(a + b))] = M(a, b)$ is the Neuman-Sándor mean, $SB[A(a, b), G(a, b)] = (a - b)/[2 \tanh^{-1}((a - b)/(a + b))] = L(a, b)$ is the logarithmic mean, where $Q(a, b) = \sqrt{(a^2 + b^2)}/2$ is quadratic mean, $\sinh^{-1}(x) = \log(x + \sqrt{x^2 + 1})$ is the inverse hyperbolic sine function and $\tanh^{-1}(x) = \log[(1 + x)/(1 - x)]/2$ is the inverse hyperbolic tangent function.

Let $X(a, b)$ and $Y(a, b)$ be the symmetric bivariate means of a and b . Then the Neuman mean $N_{XY}(a, b)$ [4] is given by

$$N_{XY}(a, b) = \frac{1}{2} \left(X(a, b) + \frac{Y^2(a, b)}{SB(X(a, b), Y(a, b))} \right). \tag{1.2}$$

Let $a > b > 0$, $v = (a - b)/(a + b) \in (0, 1)$. Then the following explicit formulas and inequalities can be found in the literature [4].

$$N_{AG}(a, b) = \frac{A(a, b)}{2} \left[1 + (1 - v^2) \frac{\tanh^{-1}(v)}{v} \right], \tag{1.3}$$

$$N_{GA}(a, b) = \frac{A(a, b)}{2} \left[\sqrt{1 - v^2} + \frac{\arcsin(v)}{v} \right], \tag{1.4}$$

$$N_{AQ}(a, b) = \frac{A(a, b)}{2} \left[1 + (1 + v^2) \frac{\arctan(v)}{v} \right], \tag{1.5}$$

$$N_{QA}(a, b) = \frac{A(a, b)}{2} \left[\sqrt{1 + v^2} + \frac{\sinh^{-1}(v)}{v} \right], \tag{1.6}$$

$$L(a, b) < N_{AG}(a, b) < P(a, b) < N_{GA}(a, b) < A(a, b) \\ < M(a, b) < N_{QA}(a, b) < T(a, b) < N_{AQ}(a, b) < Q(a, b).$$

Recently, the Neuman means $N_{AG}(a, b)$, $N_{GA}(a, b)$, $N_{AQ}(a, b)$, $N_{QA}(a, b)$ and the Lehmer mean $L_p(a, b)$ have attracted the attention many researchers.

Neuman [4] proved that the double inequalities

$$\alpha_1 A(a, b) + (1 - \alpha_1) G(a, b) < N_{GA}(a, b) < \beta_1 A(a, b) + (1 - \beta_1) G(a, b), \\ \alpha_2 Q(a, b) + (1 - \alpha_2) A(a, b) < N_{AQ}(a, b) < \beta_2 Q(a, b) + (1 - \beta_2) A(a, b), \\ \alpha_3 A(a, b) + (1 - \alpha_3) G(a, b) < N_{AG}(a, b) < \beta_3 A(a, b) + (1 - \beta_3) G(a, b), \\ \alpha_4 Q(a, b) + (1 - \alpha_4) A(a, b) < N_{QA}(a, b) < \beta_4 Q(a, b) + (1 - \beta_4) A(a, b)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 2/3$, $\beta_1 \geq \pi/4$, $\alpha_2 \leq 2/3$, $\beta_2 \geq (\pi - 2)/[4(\sqrt{2} - 1)] = 0.689\dots$, $\alpha_3 \leq 1/3$, $\beta_3 \geq 1/2$, $\alpha_4 \leq 1/3$ and $\beta_4 \geq [\log(1 + \sqrt{2}) + \sqrt{2} - 2]/[2(\sqrt{2} - 1)] = 0.356\dots$

In [5], Zhang et. al. presented the best possible parameters $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1/2]$ and $\alpha_3, \alpha_4, \beta_3, \beta_4 \in [1/2, 1]$ such that the double inequalities

$$G(\alpha_1 a + (1 - \alpha_1)b, \alpha_1 b + (1 - \alpha_1)a) < N_{AG}(a, b) < G(\beta_1 a + (1 - \beta_1)b, \beta_1 b + (1 - \beta_1)a),$$

$$G(\alpha_2 a + (1 - \alpha_2)b, \alpha_2 b + (1 - \alpha_2)a) < N_{GA}(a, b) < G(\beta_2 a + (1 - \beta_2)b, \beta_2 b + (1 - \beta_2)a),$$

$$Q(\alpha_3 a + (1 - \alpha_3)b, \alpha_3 b + (1 - \alpha_3)a) < N_{QA}(a, b) < Q(\beta_3 a + (1 - \beta_3)b, \beta_3 b + (1 - \beta_3)a),$$

$$Q(\alpha_4 a + (1 - \alpha_4)b, \alpha_4 b + (1 - \alpha_4)a) < N_{AQ}(a, b) < Q(\beta_4 a + (1 - \beta_4)b, \beta_4 b + (1 - \beta_4)a)$$

hold for all $a, b > 0$ with $a \neq b$.

Qian et. al. [6] proved that the double inequalities

$$\alpha_1 A(a, b) + (1 - \alpha_1)L(a, b) < N_{AG}(a, b) < \beta_1 A(a, b) + (1 - \beta_1)L(a, b),$$

$$\alpha_2 A(a, b) + (1 - \alpha_2)P(a, b) < N_{GA}(a, b) < \beta_2 A(a, b) + (1 - \beta_2)P(a, b),$$

$$\alpha_3 Q(a, b) + (1 - \alpha_3)M(a, b) < N_{QA}(a, b) < \beta_3 Q(a, b) + (1 - \beta_3)M(a, b),$$

$$\alpha_4 Q(a, b) + (1 - \alpha_4)T(a, b) < N_{AQ}(a, b) < \beta_4 Q(a, b) + (1 - \beta_4)T(a, b)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 0$, $\beta_1 \geq 1/2$, $\alpha_2 \leq 0$, $\beta_2 \geq (\pi^2 - 8)/(4\pi - 8)$, $\alpha_3 \leq 0$, $\beta_3 \geq [\sqrt{2} \log^2(1 + \sqrt{2}) + 2 \log(1 + \sqrt{2}) - 2\sqrt{2}]/[4 \log(1 + \sqrt{2}) - 2\sqrt{2}]$, $\alpha_4 \leq 0$, $\beta_4 \geq (\pi^2 + 2\pi - 16)/(4\sqrt{2}\pi - 16)$.

In [7–11], the authors proved that $\lambda_1 = -1/6$, $\mu_1 = 0$, $\lambda_2 = -1/3$, $\mu_2 = 0$, $\lambda_3 = 0$, $\mu_3 = 1/4$, $\lambda_4 = -1/4$, $\mu_4 = 0$, $\lambda_5 = -1/6$, $\mu_5 = 0$, $\lambda_6 = 0$, $\mu_6 = 1/3$, $\lambda_7 = 0$ and $\mu_7 = 1/6$ are the best possible parameters such that the double inequalities

$$L_{\lambda_1}(a, b) < I(a, b) < L_{\mu_1}(a, b), \quad L_{\lambda_2}(a, b) < L(a, b) < L_{\mu_2}(a, b),$$

$$L_{\lambda_3}(a, b) < T^*(a, b) < L_{\mu_3}(a, b), \quad L_{\lambda_4}(a, b) < AG(a, b) < L_{\mu_4}(a, b),$$

$$L_{\lambda_5}(a, b) < P(a, b) < L_{\mu_5}(a, b), \quad L_{\lambda_6}(a, b) < T(a, b) < L_{\mu_6}(a, b),$$

$$L_{\lambda_7}(a, b) < M(a, b) < L_{\mu_7}(a, b)$$

hold for all $a, b > 0$ with $a \neq b$, where $I(a, b) = (b^b/a^a)^{1/(b-a)}/e$ is the identric mean, $T^*(a, b) = 2 \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta / \pi$ is the Toader mean [12], and $AG(a, b)$ is the classical arithmetic-geometric mean.

Let $a, b > 0$, $p \in [0, 1]$ and \mathcal{N} be a symmetric bivariate mean, then the one-parameter mean $\mathcal{N}_p(a, b)$ was defined by Neuman [13] as follows

$$\mathcal{N}_p(a, b) = \mathcal{N} \left[\frac{1+p}{2}a + \frac{1-p}{2}b, \frac{1-p}{2}a + \frac{1+p}{2}b \right].$$

Neuman [13] proved that the double inequalities

$$H_{p_1(a,b)} < N_{AG}(a, b) < H_{q_1(a,b)}, \quad H_{p_2(a,b)} < N_{GA}(a, b) < H_{q_2(a,b)},$$

$$C_{p_3(a,b)} < N_{QA}(a, b) < C_{q_3(a,b)}, \quad C_{p_4(a,b)} < N_{AQ}(a, b) < C_{q_4(a,b)},$$

for all $a, b > 0$ with $a \neq b$ if $p_1 \geq \sqrt{2}/2$, $q_1 \leq \sqrt{3}/3$, $p_2 \geq \sqrt{1-\pi}/4$, $q_2 \leq \sqrt{6}/6$, $p_3 = 0$, $q_3 \geq \sqrt{6}/6$, $p_4 \leq \sqrt{\pi-2}/2$ and $q_4 \geq \sqrt{3}/3$.

The main purpose of this paper is to present the best possible parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\beta_1, \beta_2, \beta_3, \beta_4$ such that the double inequalities

$$L_{\alpha_1}(a, b) < N_{AG}(a, b) < L_{\beta_1}(a, b), \quad L_{\alpha_2}(a, b) < N_{GA}(a, b) < L_{\beta_2}(a, b),$$

$$L_{\alpha_3}(a, b) < N_{QA}(a, b) < L_{\beta_3}(a, b), \quad L_{\alpha_4}(a, b) < N_{AQ}(a, b) < L_{\beta_4}(a, b)$$

hold for all $a, b > 0$ with $a \neq b$, and establish several sharp inequalities involving the hyperbolic, trigonometric and inverse trigonometric functions. Some complicated computations are carried out using Mathematica computer algebra system.

2. Main results

THEOREM 2.1. *The double inequality*

$$L_{\alpha_1}(a, b) < N_{AG}(a, b) < L_{\beta_1}(a, b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq -1/3$ and $\beta_1 \geq 0$.

Proof. Since $L_p(a, b)$ and $N_{AG}(a, b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a = x > 1$ and $b = 1$. Let $p \in \mathbb{R}$, then (1.1) and (1.3) lead to

$$\begin{aligned} N_{AG}(a, b) - L_p(a, b) &= \frac{(x^2 - 1) + 4x \tanh^{-1}\left(\frac{x-1}{x+1}\right) - x^{p+1} + 1}{4(x-1)} - \frac{x^{p+1} + 1}{x^p + 1} \\ &= \frac{x}{x-1} \left[\tanh^{-1}\left(\frac{x-1}{x+1}\right) - \frac{(x-1)(3x^{p+1} - x^p - x + 3)}{4x(x^p + 1)} \right]. \end{aligned} \tag{2.1}$$

Let

$$F(x) = \tanh^{-1}\left(\frac{x-1}{x+1}\right) - \frac{(x-1)(3x^{p+1} - x^p - x + 3)}{4x(x^p + 1)}. \tag{2.2}$$

Then simple computations lead to

$$F(1) = 0, \tag{2.3}$$

$$F'(x) = -\frac{x-1}{4x^2(x^p+1)^2} f(x), \tag{2.4}$$

where

$$f(x) = 3x^{2p+1} + x^{2p} + 2(1+2p)x^{p+1} - 2(1+2p)x^p - x - 3. \tag{2.5}$$

We divide the proof into four cases.

Case 1.1 $p = -1/3$. Then (2.5) becomes

$$f(x) = -\frac{(x^{1/3} - 1)^3}{3x^{3/2}} \left(3x^{2/3} + 7x^{1/3} + 3 \right). \tag{2.6}$$

Therefore,

$$N_{AG}(a, b) > L_{-1/3}(a, b)$$

follows easily from (2.1)–(2.4) and (2.6).

Case 1.2 $p > -1/3$. Let $x > 0$ and $x \rightarrow 0$, then making use of (1.1) and (1.3) together with the Taylor expansion we have

$$\begin{aligned} & N_{AG}(1, 1+x) - L_p(1, 1+x) \tag{2.7} \\ &= \frac{2+x}{4} \left[1 + \frac{4(1+x)}{x(2+x)} \tanh^{-1} \left(\frac{x}{2+x} \right) \right] - \frac{1+(1+x)^{p+1}}{1+(1+x)^p} \\ &= -\frac{1+3p}{12}x^2 + o(x^2). \end{aligned}$$

Equation (2.7) implies that there exists small enough $\delta_1 = \delta_1(p) > 0$ such that $N_{AG}(1, 1+x) < L_p(1, 1+x)$ for $x \in (0, \delta_1)$.

Case 1.3 $p = 0$. Then (2.5) leads to

$$f(x) = 4(x-1) > 0 \tag{2.8}$$

for $x > 1$.

Therefore,

$$N_{AG}(a, b) < L_0(a, b)$$

follows from (2.1)–(2.4) and (2.8).

Case 1.4 $p < 0$. Then from (1.1) and (1.3) we clearly see that

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{N_{AG}(x, 1)}{L_p(x, 1)} &= \lim_{x \rightarrow +\infty} \left[\left(\frac{x+1}{4} + \frac{x}{x-1} \tanh^{-1} \left(\frac{x-1}{x+1} \right) \right) \frac{x^p+1}{x^{p+1}+1} \right] \tag{2.9} \\ &= \lim_{x \rightarrow +\infty} \left[\left(\frac{x+1}{4} + \frac{x \log x}{2(x-1)} \right) \frac{x^p+1}{x^{p+1}+1} \right] = +\infty. \end{aligned}$$

Equation (2.9) implies that there exists large enough $X_1 = X_1(p) > 1$ such that $N_{AG}(x, 1) > L_p(x, 1)$ for $x \in (X_1, \infty)$. \square

THEOREM 2.2. *The double inequality*

$$L_{\alpha_2}(a, b) < N_{GA}(a, b) < L_{\beta_2}(a, b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq -1/6$ and $\beta_2 \geq 0$.

Proof. Without loss of generality, we assume that $x = \sqrt{a} > 1$ and $b = 1$. Let $p \in \mathbb{R}$, then from (1.1) and (1.4) we have

$$\begin{aligned} & N_{GA}(a, b) - L_p(a, b) \tag{2.10} \\ &= \frac{(x^2+1)^2}{4(x^2-1)} \left[\arcsin \left(\frac{x^2-1}{x^2+1} \right) - \frac{2(x^2-1)(2x^{2p+2} - x^{2p+1} - x + 2)}{(x^2+1)^2(x^{2p}+1)} \right]. \end{aligned}$$

Let

$$G(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right) - \frac{2(x^2-1)(2x^{2p+2}-x^{2p+1}-x+2)}{(x^2+1)^2(x^{2p}+1)}. \quad (2.11)$$

Then simple computations lead to

$$G(1) = 0, \quad (2.12)$$

$$G'(x) = -\frac{8(x-1)}{x(x^2+1)^3(x^{2p}+1)^2}g(x), \quad (2.13)$$

where

$$g(x) = 3x^{4p+3} + x^{4p+2} + px^{2p+5} + px^{2p+4} + 2x^{2p+3} - 2x^{2p+2} - px^{2p+1} - px^{2p} - x^3 - 3x^2. \quad (2.14)$$

We divide the proof into four cases.

Case 2.1 $p = -1/6$. Then (2.14) leads to

$$g(x) = -\frac{(x^{1/3}+1)^2(x^{1/3}-1)^3}{6x^{1/3}} \times \left(x^{10/3} + x^3 + 3x^{8/3} + 4x^{7/3} + 7x^2 + 15x^{5/3} + 7x^{4/3} + 4x + 3x^{2/3} + x^{1/3} + 1\right) < 0 \quad (2.15)$$

for $x > 1$.

It follows from (2.10)–(2.13) and (2.15) that

$$N_{GA}(a, b) > L_{-1/6}(a, b).$$

Case 2.2 $p > -1/6$. Let $x > 0$ and $x \rightarrow 0$, then making use of (1.1) and (1.4) together with the Taylor expansion we get

$$\begin{aligned} & N_{GA}(1, 1+x) - L_p(1, 1+x) \\ &= \frac{2+x}{4} \left[\frac{2\sqrt{x+1}}{2+x} + \frac{2+x}{x} \arcsin\left(\frac{x}{2+x}\right) \right] - \frac{1+(1+x)^{p+1}}{1+(1+x)^p} \\ &= -\frac{1+6p}{24}x^2 + o(x^2). \end{aligned} \quad (2.16)$$

Equation (2.16) implies that there exists small enough $\delta_2 = \delta_2(p) > 0$ such that $N_{GA}(1, 1+x) < L_p(1, 1+x)$ for $x \in (0, \delta_2)$.

Case 2.3 $p = 0$. Then (2.14) becomes

$$g(x) = 4x^2(x-1). \quad (2.17)$$

Therefore,

$$N_{GA}(a, b) < L_0(a, b)$$

follows easily from (2.10)–(2.13) and (2.17).

Case 2.4 $p < 0$. Then (1.1) and (1.4) lead to

$$\begin{aligned} & \lim_{x \rightarrow +\infty} \frac{N_{GA}(x, 1)}{L_p(x, 1)} \tag{2.18} \\ &= \lim_{x \rightarrow +\infty} \left[\left(\frac{\sqrt{x}}{2} + \frac{(x+1)^2}{4(x-1)} \arcsin \left(\frac{x-1}{x+1} \right) \right) \frac{1+x^p}{1+x^{p+1}} \right] = +\infty. \end{aligned}$$

Equation (2.18) implies that there exists large enough $X_2 = X_2(p) > 1$ such that $N_{GA}(x, 1) > L_p(x, 1)$ for $x \in (X_2, \infty)$. \square

THEOREM 2.3. *The double inequality*

$$L_{\alpha_3}(a, b) < N_{QA}(a, b) < L_{\beta_3}(a, b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq 0$ and $\beta_3 \geq 1/6$.

Proof. Without loss of generality, we assume that $a = x > 1$ and $b = 1$. Let $p \in \mathbb{R}$, then (1.1) and (1.6) lead to

$$\begin{aligned} & N_{QA}(a, b) - L_p(a, b) \tag{2.19} \\ &= \frac{(x+1)^2}{4(x-1)} \left[\sinh^{-1} \left(\frac{x-1}{x+1} \right) - \frac{(x-1) \left(4(x^{p+1} + 1) - \sqrt{2(x^2+1)}(x^p + 1) \right)}{(x+1)^2(x^p + 1)} \right]. \end{aligned}$$

Let

$$H(x) = \sinh^{-1} \left(\frac{x-1}{x+1} \right) - \frac{(x-1) \left(4(x^{p+1} + 1) - \sqrt{2(x^2+1)}(x^p + 1) \right)}{(x+1)^2(x^p + 1)}. \tag{2.20}$$

Then simple computations lead to

$$H(1) = 0, \tag{2.21}$$

$$H'(x) = \frac{4}{x(x+1)^3(x^p+1)^2} h(x), \tag{2.22}$$

where

$$\begin{aligned} h(x) &= x\sqrt{2(x^2+1)}(x^p+1)^2 - 3x^{2p} + x^{2p+1} - px^{p+3} \tag{2.23} \\ &\quad - (2-p)x^{p+2} - (2-p)x^{p+1} - px^p + x^2 - 3x. \end{aligned}$$

We divide the proof into four cases.

Case 3.1 $p = 0$. Then (2.23) leads to

$$h(x) = 4x \left[\sqrt{2(x^2+1)} - (x+1) \right] > 0 \tag{2.24}$$

for $x > 1$.

It follows from (2.19)–(2.22) and (2.24) that

$$N_{QA}(a, b) > L_0(a, b).$$

Case 3.2 $p > 0$. Then (1.1) and (1.6) lead to

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{L_p(x, 1)}{N_{QA}(x, 1)} &= \lim_{x \rightarrow +\infty} \frac{4(x^{p+1} + 1)}{\left[\sqrt{2(x^2 + 1)} + \frac{(x+1)^2}{x-1} \sinh^{-1} \left(\frac{x-1}{x+1} \right) \right] (x^p + 1)} \quad (2.25) \\ &= \frac{4}{\sqrt{2} + \log(1 + \sqrt{2})} > 1. \end{aligned}$$

Inequality (2.25) implies that there exists large enough $X_3 = X_3(p) > 1$ such that $N_{QA}(x, 1) < L_p(x, 1)$ for $x \in (X_3, \infty)$.

Case 3.3 $p = 1/6$. Then (2.23) leads to

$$h(x) = \left(x^{1/6} + 1\right)^2 \left[x\sqrt{2(x^2 + 1)} - \frac{1}{6}h_1(x) \right], \quad (2.26)$$

where

$$\begin{aligned} h_1(x) &= x^{8/3} - 2x^{15/6} + 3x^{7/3} - 4x^{13/6} + 5x^2 + 12x^{11/6} - 18x^{5/3} + 18x^{3/2} \quad (2.27) \\ &\quad - 18x^{4/3} + 18x^{7/6} - 18x + 12x^{5/6} + 5x^{2/3} - 4x^{1/2} + 3x^{1/3} - 2x^{1/6} + 1. \end{aligned}$$

It follows from (2.27) that

$$\begin{aligned} h_1(x) &= x^{7/3} \left(x^{1/6} - 1\right)^2 + 2x^2 \left(x^{1/6} - 1\right)^2 + 3x^2 + 12x^{11/6} - 18x^{5/3} \quad (2.28) \\ &\quad + 18x^{4/3} \left(x^{1/6} - 1\right) + 18x \left(x^{1/6} - 1\right) + 12x^{5/6} + x^{2/3} \\ &\quad + 4x^{1/2} \left(x^{1/6} - 1\right) + x^{1/3} + 2x^{1/6} \left(x^{1/6} - 1\right) + 1 \\ &> 3x^2 + 12x^{11/6} - 18x^{5/3} + 12x^{5/6} > 15x^{11/6} - 18x^{5/3} + 3x^{5/6} \\ &= 3x^{5/6} \left(x^{1/6} - 1\right) \left(5x^{5/6} - x^{2/3} - x^{1/2} - x^{1/3} - x^{1/6} - 1\right) > 0 \end{aligned}$$

and

$$\begin{aligned} &x\sqrt{2(x^2 + 1)} - \frac{1}{6}x^{1/6}h_1(x) \quad (2.29) \\ &= \frac{\left[x\sqrt{2(x^2 + 1)} \right]^2 - \left[\frac{1}{6}x^{1/6}h_1(x) \right]^2}{x\sqrt{2(x^2 + 1)} + \frac{1}{6}x^{1/6}h_1(x)} \\ &= -\frac{1}{36}x^{1/3} \left(x^{1/6} - 1\right)^4 \left(x^{2/3} + x^{1/3} + 1\right) \left(x^4 + 3x^{11/3} + 6x^{10/3} + 36x^{19/6} + 33x^3 \right. \\ &\quad \left. + 96x^{17/6} + 84x^{8/3} + 180x^{5/2} + 411x^{7/3} + 420x^{13/6} + 532x^2 + 420x^{11/6} \right. \\ &\quad \left. + 411x^{5/3} + 180x^{3/2} + 84x^{4/3} + 96x^{7/6} + 33x + 36x^{5/6} + 6x^{2/3} + 3x^{1/3} + 1\right) < 0 \end{aligned}$$

for $x > 1$.

From (2.22), (2.26), (2.28) and (2.29) we clearly see that $H(x)$ is strictly decreasing on $(1, \infty)$. Therefore,

$$N_{QA}(a, b) < L_{1/6}(a, b)$$

follows from (2.19)–(2.21) and the monotonicity of $H(x)$.

Case 3.4 $p < 1/6$. Let $x > 0$ and $x \rightarrow 0$, then making use of (1.1) and (1.6) together with the Taylor expansion we have

$$\begin{aligned} & N_{QA}(1, 1+x) - L_p(1, 1+x) \tag{2.30} \\ &= \left(\frac{1}{2} + \frac{x}{4}\right) \left[\sqrt{1 + \left(\frac{x}{2+x}\right)^2} + \frac{2+x}{x} \sinh^{-1}\left(\frac{x}{2+x}\right) \right] - \frac{1 + (1+x)^{p+1}}{1 + (1+x)^p} \\ &= \frac{1-6p}{24}x^2 + o(x^2). \end{aligned}$$

Equation (2.30) implies that there exists small enough $\delta_3 = \delta_3(p) > 0$ such that $N_{QA}(1, 1+x) > L_p(1, 1+x)$ for $x \in (0, \delta_3)$. \square

THEOREM 2.4. *The double inequality*

$$L_{\alpha_4}(a, b) < N_{AQ}(a, b) < L_{\beta_4}(a, b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_4 \leq 0$ and $\beta_4 \geq 1/3$.

Proof. Without loss of generality, we assume that $x = a > 1$ and $b = 1$. Let $p \in \mathbb{R}$, then (1.1) and (1.5) lead to

$$N_{AQ}(a, b) - L_p(a, b) = \frac{x^2 + 1}{2(x-1)} \left[\arctan\left(\frac{x-1}{x+1}\right) - \frac{(x-1)(3x^{p+1} - x^p - x + 3)}{2(x^2 + 1)(x^p + 1)} \right]. \tag{2.31}$$

Let

$$J(x) = \arctan\left(\frac{x-1}{x+1}\right) - \frac{(x-1)(3x^{p+1} - x^p - x + 3)}{2(x^2 + 1)(x^p + 1)}. \tag{2.32}$$

Then simple computations lead to

$$J(1) = 0, \tag{2.33}$$

$$J'(x) = -\frac{x-1}{x(x^2 + 1)^2(x^p + 1)^2} J_1(x), \tag{2.34}$$

where

$$J_1(x) = x^{2p+2} + 3x^{2p+1} + 2px^{p+3} - 2(p+1)x^{p+2} + 2(p+1)x^{p+1} - 2px^p - 3x^2 - x. \tag{2.35}$$

We divide the proof into four cases.

Case 4.1 $p = 0$. Then (2.35) leads to

$$J_1(x) = -4(x - 1) < 0 \tag{2.36}$$

for $x > 1$.

It follows from (2.31)–(2.34) and (2.36) that

$$N_{AQ}(a, b) > L_0(a, b).$$

Case 4.2 $p > 0$. Then from (1.1) and (1.5) we have

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{L_p(x, 1)}{N_{AQ}(x, 1)} &= \lim_{x \rightarrow +\infty} \frac{4(x^{p+1} + 1)}{\left[(x + 1) + \frac{2(x^2 + 1)}{x - 1} \arctan\left(\frac{x - 1}{x + 1}\right) \right] (x^p + 1)} \\ &= \frac{8}{\pi + 2} > 1. \end{aligned} \tag{2.37}$$

Inequality (2.37) implies that there exists large enough $X_4 = X_4(p) > 1$ such that $N_{AQ}(x, 1) < L_p(x, 1)$ for $x \in (X_4, \infty)$.

Case 4.3 $p = 1/3$. Then (2.35) leads to

$$J_1(x) = \frac{1}{3}x^{1/3} \left(x^{1/3} - 1\right)^3 \left(2x^2 + 6x^{5/3} + 15x^{4/3} + 21x + 15x^{2/3} + 6x^{1/3} + 2\right) > 0 \tag{2.38}$$

for $x > 1$.

Therefore,

$$N_{AQ}(a, b) < L_{1/3}(a, b)$$

follows easily from (2.31)–(2.34) and (2.38).

Case 4.4 $p < 1/3$. Let $x > 0$ and $x \rightarrow 0$, then making use of (1.1) and (1.5) together with the Taylor expansion we have

$$\begin{aligned} &N_{AQ}(1, 1 + x) - L_p(1, 1 + x) \\ &= \left(\frac{1}{2} + \frac{x}{4}\right) \left[1 + \frac{2(x^2 + 2x + 2)}{x(x + 2)} \arctan\left(\frac{x}{x + 2}\right)\right] - \frac{(x + 1)^{p+1} + 1}{(x + 1)^p + 1} \\ &= \frac{1 - 3p}{12}x^2 + o(x^2). \end{aligned} \tag{2.39}$$

Equation (2.39) implies that there exists small enough $\delta_4 = \delta_4(p) > 0$ such that $N_{AQ}(1, 1 + x) > L_p(1, 1 + x)$ for $x \in (0, \delta_4)$. \square

3. Applications

In this section, we will establish several sharp inequalities involving the hyperbolic, trigonometric and inverse trigonometric functions by use of Theorems 2.1–2.4.

From (1.2) we clearly see that

$$N_{AG}(a, b) = \frac{1}{2} \left[A(a, b) + \frac{G^2(a, b)}{L(a, b)} \right], \quad N_{GA}(a, b) = \frac{1}{2} \left[G(a, b) + \frac{A^2(a, b)}{P(a, b)} \right], \tag{3.1}$$

$$N_{QA}(a, b) = \frac{1}{2} \left[Q(a, b) + \frac{A^2(a, b)}{M(a, b)} \right], \quad N_{AQ}(a, b) = \frac{1}{2} \left[A(a, b) + \frac{Q^2(a, b)}{T(a, b)} \right]. \quad (3.2)$$

Let $a > b$ and $x = \tanh^{-1} \left(\frac{a-b}{a+b} \right) = \frac{1}{2} \log \frac{a}{b} \in (0, \infty)$. Then simple computations lead to

$$\frac{A(a, b)}{G(a, b)} = \cosh(x), \quad \frac{G(a, b)}{L(a, b)} = \frac{x}{\sinh(x)}, \quad \frac{L_p(a, b)}{G(a, b)} = \frac{\cosh[(p+1)x]}{\cosh(px)}, \quad (3.3)$$

$$\frac{A(a, b)}{P(a, b)} = \tanh(x) \arcsin[\coth(x)], \quad \frac{L_p(a, b)}{A(a, b)} = \frac{\cosh[(p+1)x]}{\cosh(x) \cosh(px)}, \quad (3.4)$$

$$\frac{Q(a, b)}{A(a, b)} = \frac{\cosh^{1/2}(2x)}{\cosh(x)}, \quad \frac{A(a, b)}{M(a, b)} = \coth(x) \log \left[\frac{\sinh(x) + \cosh^{1/2}(2x)}{\cosh(x)} \right], \quad (3.5)$$

$$\frac{Q(a, b)}{T(a, b)} = \frac{\cosh^{1/2}(2x)}{\sinh(x)} \arctan[\tanh(x)], \quad \frac{L_p(a, b)}{Q(a, b)} = \frac{\cosh[(p+1)x]}{\cosh^{1/2}(2x) \cosh(px)}. \quad (3.6)$$

Theorems 2.1–2.4 and (3.1)–(3.6) lead to Theorem 3.1.

THEOREM 3.1. *The double inequalities*

$$\begin{aligned} \frac{2 \cosh[(\alpha_1 + 1)x]}{\cosh(\alpha_1 x)} &< \cosh(x) + \frac{x}{\sinh(x)} < \frac{2 \cosh[(\beta_1 + 1)x]}{\cosh(\beta_1 x)}, \\ \frac{2 \cosh[(\alpha_2 + 1)x]}{\cosh(\alpha_2 x)} &< 1 + \sinh(x) \arcsin[\coth(x)] < \frac{2 \cosh[(\beta_2 + 1)x]}{\cosh(\beta_2 x)}, \\ \frac{2 \cosh[(\alpha_3 + 1)x]}{\cosh(\alpha_3 x)} &< \cosh^{1/2}(2x) + \frac{\cosh^2(x)}{\sinh(x)} \log \left[\frac{\sinh(x) + \cosh^{1/2}(2x)}{\cosh(x)} \right] \\ &< \frac{2 \cosh[(\beta_3 + 1)x]}{\cosh(\beta_3 x)}, \\ \frac{2 \cosh[(\alpha_4 + 1)x]}{\cosh(\alpha_4 x)} &< \cosh(x) + \frac{\cosh(2x)}{\sinh(x)} \arctan[\tanh(x)] < \frac{2 \cosh[(\beta_4 + 1)x]}{\cosh(\beta_4 x)} \end{aligned}$$

hold for all $x > 0$ if and only if $\alpha_1 \leq -1/3$, $\beta_1 \geq 0$, $\alpha_2 \leq -1/6$, $\beta_2 \geq 0$, $\alpha_3 \leq 0$, $\beta_3 \geq 1/6$, $\alpha_4 \leq 0$ and $\beta_4 \geq 1/3$.

Let $a > b$ and $x = \arcsin \left(\frac{a-b}{a+b} \right) \in (0, \pi/2)$. Then it is not difficult to verify that

$$\frac{A(a, b)}{G(a, b)} = \sec(x), \quad \frac{G(a, b)}{L(a, b)} = \cot(x) \log[\sec(x) + \tan(x)], \quad \frac{A(a, b)}{P(a, b)} = \frac{x}{\sin(x)}, \quad (3.7)$$

$$\frac{L_p(a, b)}{G(a, b)} = \sec(x) L_p[1 + \sin(x), 1 - \sin(x)], \quad \frac{L_p(a, b)}{A(a, b)} = L_p[1 + \sin(x), 1 - \sin(x)], \quad (3.8)$$

$$\frac{Q(a, b)}{A(a, b)} = \sqrt{1 + \sin^2(x)}, \quad \frac{A(a, b)}{M(a, b)} = \csc(x) \log \left[\sin(x) + \sqrt{1 + \sin^2(x)} \right], \quad (3.9)$$

$$\frac{Q(a,b)}{T(a,b)} = \frac{\sqrt{1 + \sin^2(x)}}{\sin(x)} \arctan[\sin(x)], \quad \frac{L_p(a,b)}{Q(a,b)} = \frac{L_p[1 + \sin(x), 1 - \sin(x)]}{\sqrt{1 + \sin^2(x)}}. \quad (3.10)$$

From Theorems 2.1–2.4, (3.1), (3.2) and (3.7)–(3.10) we get Theorem 3.2 immediately.

THEOREM 3.2. *The double inequalities*

$$\begin{aligned} 2L_{\alpha_1}[1 + \sin(x), 1 - \sin(x)] &< 1 + \frac{\cos^2(x)}{\sin(x)} \log[\sec(x) + \tan(x)] \\ &< 2L_{\beta_1}[1 + \sin(x), 1 - \sin(x)], \\ 2L_{\alpha_2}[1 + \sin(x), 1 - \sin(x)] &< \cos(x) + \frac{x}{\sin(x)} < 2L_{\beta_2}[1 + \sin(x), 1 - \sin(x)], \\ 2L_{\alpha_3}[1 + \sin(x), 1 - \sin(x)] &< \sqrt{1 + \sin^2(x)} + \frac{\log \left[\sin(x) + \sqrt{1 + \sin^2(x)} \right]}{\sin(x)} \\ &< 2L_{\beta_3}[1 + \sin(x), 1 - \sin(x)], \\ 2L_{\alpha_4}[1 + \sin(x), 1 - \sin(x)] &< 1 + [\sin(x) + \csc(x)] \arctan[\sin(x)] \\ &< 2L_{\beta_4}[1 + \sin(x), 1 - \sin(x)] \end{aligned}$$

hold for all $x \in (0, \pi/2)$ if and only if $\alpha_1 \leq -1/3$, $\beta_1 \geq 0$, $\alpha_2 \leq -1/6$, $\beta_2 \geq 0$, $\alpha_3 \leq 0$, $\beta_3 \geq 1/6$, $\alpha_4 \leq 0$ and $\beta_4 \geq 1/3$.

Let $a > b$ and $x = \arctan((a - b)/(a + b)) \in (0, \pi/4)$. Then simple computations lead to

$$\frac{A(a,b)}{G(a,b)} = \frac{\cos(x)}{\cos^{1/2}(2x)}, \quad \frac{G(a,b)}{L(a,b)} = \frac{\cos^{1/2}(2x)}{2\sin(x)} \log \left[\frac{1 + \tan(x)}{1 - \tan(x)} \right], \quad (3.11)$$

$$\frac{L_p(a,b)}{G(a,b)} = \frac{L_p[\cos(x) + \sin(x), \cos(x) - \sin(x)]}{\cos^{1/2}(2x)}, \quad \frac{A(a,b)}{P(a,b)} = \cot(x) \arcsin[\tan(x)], \quad (3.12)$$

$$\frac{L_p(a,b)}{A(a,b)} = \frac{L_p[\cos(x) + \sin(x), \cos(x) - \sin(x)]}{\cos(x)}, \quad \frac{A(a,b)}{Q(a,b)} = \cos(x), \quad (3.13)$$

$$\frac{A(a,b)}{M(a,b)} = \cot(x) \log[\sec(x) + \tan(x)], \quad \frac{Q(a,b)}{T(a,b)} = \frac{x}{\sin(x)}, \quad (3.14)$$

$$\frac{L_p(a,b)}{Q(a,b)} = L_p[\cos(x) + \sin(x), \cos(x) - \sin(x)]. \quad (3.15)$$

Theorems 2.1–2.4, (3.1), (3.2) and (3.11)–(3.15) lead to Theorem 3.3.

THEOREM 3.3. *The double inequalities*

$$\begin{aligned} 2L_{\alpha_1}[\cos(x) + \sin(x), \cos(x) - \sin(x)] &< \cos(x) + \frac{\cos(2x)}{2\sin(x)} \log \left[\frac{1 + \tan(x)}{1 - \tan(x)} \right] \\ &< 2L_{\beta_1}[\cos(x) + \sin(x), \cos(x) - \sin(x)], \end{aligned}$$

$$\begin{aligned}
 2L_{\alpha_2}[\cos(x) + \sin(x), \cos(x) - \sin(x)] &< \cos^{1/2}(2x) + \frac{\cos^2(x)}{\sin(x)} \arcsin[\tan(x)] \\
 &< 2L_{\beta_2}[\cos(x) + \sin(x), \cos(x) - \sin(x)], \\
 2L_{\alpha_3}[\cos(x) + \sin(x), \cos(x) - \sin(x)] &< 1 + \frac{\cos^2(x)}{\sin(x)} \log[\sec(x) + \tan(x)] \\
 &< 2L_{\beta_3}[\cos(x) + \sin(x), \cos(x) - \sin(x)], \\
 2L_{\alpha_4}[\cos(x) + \sin(x), \cos(x) - \sin(x)] &< \cos(x) + \frac{x}{\sin(x)} \\
 &< 2L_{\beta_4}[\cos(x) + \sin(x), \cos(x) - \sin(x)]
 \end{aligned}$$

hold for all $x \in (0, \pi/4)$ if and only if $\alpha_1 \leq -1/3$, $\beta_1 \geq 0$, $\alpha_2 \leq -1/6$, $\beta_2 \geq 0$, $\alpha_3 \leq 0$, $\beta_3 \geq 1/6$, $\alpha_4 \leq 0$ and $\beta_4 \geq 1/3$.

Let $a > b$ and $x = \sinh^{-1}((a - b)/(a + b)) \in (0, \log(1 + \sqrt{2}))$. Then it is not difficult to verify that

$$\frac{A(a, b)}{G(a, b)} = \frac{1}{\sqrt{1 - \sinh^2(x)}}, \quad \frac{G(a, b)}{L(a, b)} = \frac{\sqrt{1 - \sinh^2(x)}}{2 \sinh(x)} \log \left[\frac{1 + \sinh(x)}{1 - \sinh(x)} \right], \quad (3.16)$$

$$\frac{L_p(a, b)}{G(a, b)} = \frac{L_p[1 + \sinh(x), 1 - \sinh(x)]}{\sqrt{1 - \sinh^2(x)}}, \quad \frac{A(a, b)}{P(a, b)} = \frac{\arcsin[\sinh(x)]}{\sinh(x)}, \quad (3.17)$$

$$\frac{L_p(a, b)}{A(a, b)} = L_p[1 + \sinh(x), 1 - \sinh(x)], \quad \frac{Q(a, b)}{A(a, b)} = \cosh(x), \quad \frac{A(a, b)}{M(a, b)} = \frac{x}{\sinh(x)}, \quad (3.18)$$

$$\frac{Q(a, b)}{T(a, b)} = \frac{\cosh(x)}{\sinh(x)} \arctan[\sinh(x)], \quad \frac{L_p(a, b)}{Q(a, b)} = \frac{L_p[1 + \sinh(x), 1 - \sinh(x)]}{\cosh(x)}. \quad (3.19)$$

From Theorems 2.1–2.4, (3.1), (3.2) and (3.16)–(3.19) we get Theorem 3.4 immediately.

THEOREM 3.4. *The double inequalities*

$$\begin{aligned}
 2L_{\alpha_1}[1 + \sinh(x), 1 - \sinh(x)] &< 1 + \frac{1 - \sinh^2(x)}{2 \sinh(x)} \log \left[\frac{1 + \sinh(x)}{1 - \sinh(x)} \right] \\
 &< 2L_{\beta_1}[1 + \sinh(x), 1 - \sinh(x)], \\
 2L_{\alpha_2}[1 + \sinh(x), 1 - \sinh(x)] &< \sqrt{1 - \sinh^2(x)} + \frac{\arcsin[\sinh(x)]}{\sinh(x)} \\
 &< 2L_{\beta_2}[1 + \sinh(x), 1 - \sinh(x)], \\
 2L_{\alpha_3}[1 + \sinh(x), 1 - \sinh(x)] &< \cosh(x) + \frac{x}{\sinh(x)} < 2L_{\beta_3}[1 + \sinh(x), 1 - \sinh(x)], \\
 2L_{\alpha_4}[1 + \sinh(x), 1 - \sinh(x)] &< 1 + \frac{\cosh^2(x)}{\sinh(x)} \arctan[\sinh(x)] \\
 &< 2L_{\beta_4}[1 + \sinh(x), 1 - \sinh(x)]
 \end{aligned}$$

hold for all $x \in (0, \log(1 + \sqrt{2}))$ if and only if $\alpha_1 \leq -1/3$, $\beta_1 \geq 0$, $\alpha_2 \leq -1/6$, $\beta_2 \geq 0$, $\alpha_3 \leq 0$, $\beta_3 \geq 1/6$, $\alpha_4 \leq 0$ and $\beta_4 \geq 1/3$.

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