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CORRIGENDUM TO: "ON A BETA FUNCTION INEQUALITY"

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Abstract. The purpose of this note is to correct an error in an earlier paper by the author: On a beta function inequality, J. Math. Inequal.

A correction needs to be made concerning the right side of Theorem, particularly (3.2) of the paper mentioned in the title, since the proof was based on Lemma 2.5, but the proof of the Lemma was incorrect. The cases of equality are also corrected. We need the following three Lemmas. The first one is due to H. Alzer [1, p. 382, Theorem 7].

LEMMA 1. *Let $n \geqslant 0$ be an integer and let $x > 0$ and $s \in (0,1)$ be real numbers. Then we have*

$$\frac{1-s}{x+s+n}+(1-s)\sum_{i=0}^{n-1}\frac{1}{(x+i+1)(x+i+s)}<\Psi(x+1)-\Psi(x+s).$$

LEMMA 2. *Let $1/2 \leqslant x \leqslant 1$ and $0 < y < 1$. Then we have*

$$\begin{aligned}p(x,y) := {} & x^4y^2+8x^3y^2+17x^2y^2+6xy^2-6y^2+2x^5y+19x^4y\\ & +61x^3y+74x^2y+11xy-25y+x^6+11x^5+46x^4\\ & +89x^3+70x^2-4x-25>0\end{aligned}$$

Proof. Taking y as fixed and building the partial derivative with respect to x, we have

$$\begin{aligned}\frac{\partial p(x,y)}{\partial x} = {} & 4x^3y^2+24x^2y^2+34xy^2+6y^2+10x^4y+76x^3y+183x^2y\\ & +148xy+11y+6x^5+55x^4+184x^3+267x^2+140x-4.\end{aligned}$$

It is clear, that $\partial p(x,y)/\partial x > 0$ for $x \geqslant 1/2$. Hence $p(x,y)$ is strictly increasing in x and we infer $p(x,y) > p(1/2,y) = 1/64\left(148y^2+504y+311\right) > 0$. $\square$

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LEMMA 3. *Consider the following transcendental equation:*

$$\Psi(x+1)-\Psi(2x+1)+\frac{x}{1+x^2}=0. \tag{1}$$

Then (1) has on $0<x\leqslant 1/2$*, only one zero which is* $x_1=0.4088\ldots$*. Moreover denote by* $h(x)$ *the left side of (1), then* $h'(x)$ *is a concave function on that interval.*

Proof. Building the first and third derivatives of h we obtain

$$h'(x)=\frac{1-x^2}{(1+x^2)^2}+\Psi'(1+x)-2\Psi'(1+2x),$$

and

$$h'''(x)=-\frac{6\left(x^4-6x^2+1\right)}{(1+x^2)^4}+\Psi'''(1+x)-8\Psi'''(1+2x)$$

respectively. We show, that $h'''(x)<0$. Now, after some calculations we find

$$\begin{aligned}h'''(x)&=-\frac{6\left(x^4-6x^2+1\right)}{(1+x^2)^4}+\Psi'''(1+x)-8\Psi'''(1+2x)\\&=-\frac{6\left(x^4-6x^2+1\right)}{(1+x^2)^4}-\sum_{k=0}^{\infty}\frac{6q(x,k)}{(1+x+k)^4(1+2x+k)^4},\end{aligned}$$

where

$$\begin{aligned}q(x,k):=&-8x^4+\left(24k^2+48k\right)x^2+24x^2+\left(24k^3+72k^2+72k\right)x\\&+24x+7k^4+28k^3+42k^2+28k+7>0,\end{aligned} \tag{2}$$

for all $0<x\leqslant 1/2$ and $k\geqslant 0$. Taking in (2), $k=0$, we conclude

$$\begin{aligned}h'''(x)&<-\frac{6\left(x^4-6x^2+1\right)}{(1+x^2)^4}-\frac{6\left(-8x^4+28x^2+24x+7\right)}{(1+x)^4(1+2x)^4}\\&=-\frac{24r(x)}{(1+x)^4(1+2x)^4(1+x^2)^4},\end{aligned}$$

where

$$\begin{aligned}r(x):=&\,2x^{12}+24x^{11}+36x^{10}-48x^9-274x^8-447x^7-369x^6\\&-141x^5+20x^4+51x^3+27x^2+9x+2.\end{aligned}$$

Since $r(1/2)=12481/2048>0$, $r(1)=-1108<0$ and $r(3)=3420092>0$ we obtain according Descartes' rule of signs that $r(x)>0$ on $0<x\leqslant 1/2$. Hence $h'''(x)<0$ and therefore $h'(x)$ is concave. To get the only zero of (1) observe, that $h'(0)=1-\pi^2/6=-0.6449\ldots<0$ and $h'(1/5)=0.1042\ldots>0$, so there exists an x_T, such that for $0<x<x_T<x<1/5$, we get $h'(x)<0$, $h'(x_T)=0$ and $h'(x)>0$ respectively.

A little calculation yields $h'(1/2) = \pi^2/6 - 38/25 = 0.1249... > 0$ therefore $h'(x) > 0$ holds also on $[1/5, 1/2]$. Since $h(x)$ is continuous and strictly increasing in $[1/5, 1/2]$, moreover $h(1/5) = -0.0353... < 0$ and $h(1/2) = 0.0137... > 0$, hence there exists a unique $1/5 < x_1 < 1/2$, for which $h(x_1) = 0$. A more precise calculation via MAPLE V Release 10.0 has given $x_1 = 0.4088...(< 1/2)$. □

THEOREM. *For all real numbers* $x, y \in (0,1]$, *we have*

$$\frac{1}{xy}(x+y-xy) \leqslant B(x,y) \leqslant \frac{1}{xy}\frac{x+y}{1+xy} \tag{3}$$

with equality if and only if $x = 1$ *and* $0 < y \leqslant 1$, *or* $y = 1$ *and* $0 < x \leqslant 1$.

Proof. We only prove the right side of (3). Consider the function G defined by

$$G(x,y) := \log \frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+1)} - \log \frac{1}{1+xy}.$$

After applying partial differentiation we get

$$\frac{\partial G(x,y)}{\partial x} = \Psi(x+1) - \Psi(x+y+1) + \frac{y}{1+xy},$$

$$\frac{\partial^2 G(x,y)}{\partial x^2} = \Psi'(x+1) - \Psi'(x+y+1) - \frac{y^2}{(1+xy)^2}$$

and

$$\frac{\partial^2 G(x,y)}{\partial x \partial y} = -\Psi'(x+y+1) + \frac{1}{(1+xy)^2}$$

respectively.

There are three cases to investigate.

Case 1. $1/2 \leqslant x < 1$ and $0 < y < 1$.

We show that $\partial G(x,y)/\partial x > 0$. Applying Lemma 1, with $n = 3$, to obtain

$$\begin{aligned}
\frac{\partial G(x,y)}{\partial x} &= [\Psi(x+1) - \Psi(x+y)] - \frac{1}{x+y} + \frac{y}{1+xy} > \frac{1-y}{x+y+3} \\
&\quad + (1-y)\left[\frac{1}{(x+1)(x+y)} + \frac{1}{(x+2)(x+y+1)} + \frac{1}{(x+3)(x+y+2)}\right] \\
&\quad + \frac{y^2-1}{(x+y)(1+xy)} \\
&= \frac{y(1-y)\,p(x,y)}{(x+1)(x+2)(x+3)(x+y+2)(1+xy)} > 0.
\end{aligned}$$

Here we use

$$\begin{aligned}
p(x,y) := {} & x^4y^2 + 8x^3y^2 + 17x^2y^2 + 6xy^2 - 6y^2 + 2x^5y + 19x^4y \\
& + 61x^3y + 74x^2y + 11xy - 25y + x^6 + 11x^5 \\
& + 46x^4 + 89x^3 + 70x^2 - 4x - 25
\end{aligned}$$

and Lemma 2. Therefore $\partial G(x,y)/\partial x > 0$ and $G(x,y)$ is strictly increasing in x. Hence we infer $G(1/2,y) < G(x,y) < G(1,y) = 0$, which gives (3).

Case 2. $0 < x < 1$ and $1/2 \leqslant y < 1$.

According to the symmetry of (3) in x and y, changing x to y, we get by a similar argument that $\partial G(x,y)/\partial y > 0$ which implies that $G(x,y)$ is strictly increasing in y. Hence we obtain $G(x,1/2) < G(x,y) < G(x,1) = 0$, and (3) follows.

Case 3. $0 \leqslant x \leqslant 1/2$ and $0 \leqslant y \leqslant 1/2$.

It is easy to see that on the boundary of the given square we have $G(x,0) = 0$, $G(1/2,y) < 0$, $G(x,1/2) < 0$ and $G(0,y) = 0$. Since $G(x,y)$ is continuous on the closed square it takes its maximum and also its minimum value. Denote by $P(x_0,y_0)$, the point within the square where the function takes its maximum value and suppose that there we have $G(x_0,y_0) > 0$. A sufficient condition for that maximum value leads to the following system of equations:

$$\frac{\partial G(x,y)}{\partial x} = \Psi(x+1) - \Psi(x+y+1) + \frac{y}{1+xy} = 0, \tag{4}$$

and

$$\frac{\partial G(x,y)}{\partial y} = \Psi(y+1) - \Psi(x+y+1) + \frac{x}{1+xy} = 0 \tag{5}$$

respectively.

Subtracting (5) from (4) gives

$$\Psi(x+1) - \Psi(y+1) + (y-x)\frac{1}{1+xy} = (y-x)\left[\sum_{k=1}^{\infty}\frac{1}{(x+k)(y+k)} + \frac{1}{1+xy}\right] = 0,$$

which holds if and only if $x = y$. Now, we asking for the solution of the equation

$$\Psi(x+1) - \Psi(2x+1) + \frac{x}{1+x^2} = 0, \tag{6}$$

on $(0,1/2]$. In view of Lemma 3 we only get the one zero: $x_1 = 0.4088\ldots$. Let us consider now the following function

$$\Delta(x,y) = \frac{\partial^2 G}{\partial x^2}\frac{\partial^2 G}{\partial y^2} - \left(\frac{\partial^2 G}{\partial x \partial y}\right)^2.$$

Now, we have to discuss the sign of $\Delta(x,y)$ for the special solution of (6). If $x_1 = y_1 = 0.4088\ldots$ then we get $\Delta(x_1,y_1) = 0.02756\ldots > 0$, and on the other hand

$$\left[\frac{\partial^2 G(x,y)}{\partial x^2}\right]_{x=x_1,y=y_1} = 0.1661\ldots > 0,$$

so we claim that at $x_1 = y_1 = 0.4088\ldots$ the function G attains its minimum value which is $G(x_1,y_1) = -0.1973\ldots < 0$. Therefore there is no $P(x_0,y_0)$, where G has a maximum value with $G(x_0,y_0) > 0$. This completes the proof of the theorem. □

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