Journal of
Mathematical
Inequalities
Volume 10, Number 3 (2016), 603–622 doi:10.7153/jmi-10-49

REVERSED HARDY INEQUALITY FOR C–MONOTONE FUNCTIONS

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(*Communicated by N. Elezović*)

Abstract. In this paper, we will give general Hardy and reversed Hardy type inequalities for a generalized class of monotone functions. Moreover we will give n-exponential convexity, exponential convexity and related results for some functionals obtained from the differences of these inequalities. At the end we will give mean value theorems and Cauchy means for these functionals.

1. Introduction

The classical Hardy inequality for $f \geqslant 0$ and $p > 1$ is given as

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx\right)^{1/p} \leqslant \frac{p}{p-1}\left(\int_0^\infty f^p(x)dx\right)^{1/p}.$$

When $f \geqslant 0$ is a decreasing function, then the reversed Hardy inequality, given by P.F.Renaud (1986) in [7],

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \geqslant \frac{p}{p-1}\int_0^\infty f^p(x)dx \tag{1}$$

holds. In this paper we prove the reversed Hardy type inequalities for a more general class of C-monotone functions by considering the inequalities given by Pečarić, Perić and Persson in [5]. A function f is C-decreasing $\big(C$-increasing$\big)$, $C \geqslant 1$, if $f(x) \leqslant Cf(y)$ $\big(f(y) \leqslant Cf(x)\big)$ whenever $y \leqslant x$, $x, y \in (a,b)$. Moreover, by constructing some linear functionals and n-exponential convex functions related to the obtained inequalities, we give refinements of the reversed Hardy type inequalities. Also we will give mean value theorems and Cauchy means for these functionals.

Mathematics subject classification (2010): 26D15, 26A48.
Keywords and phrases: C-monotone functions, reversed Hardy inequality, exponentially convex functions, Cauchy means.

2. Main results

In this paper the terms positive, decreasing and increasing shall be interpreted as nonnegative, nonincreasing and nondecreasing, respectively. We shall consider positive real valued functions f,g defined on an interval (a,b), $-\infty \leqslant a < b \leqslant +\infty$. Moreover, the function denoted by g will be monotone throughout the paper and we assume that the function denoted by f is integrable with respect to the measure generated by g, i. e. that $\int_a^b f(x)dg(x) < +\infty$ for an increasing g and $\int_a^b f(x)d[-g(x)] < +\infty$ for a decreasing g.

We start by considering the following results given in [1] and [5].

THEOREM 2.1. *Let $\phi : [0,\infty) \to \mathbb{R}$ be a convex function differentiable on $(0,\infty)$ and such that $\phi(0) = 0$ and let $-\infty \leqslant a < b \leqslant \infty$.*

(a) If f is C-decreasing and g is increasing, differentiable and such that $g(a+0) = 0$, then

$$\phi\left(C\int_a^b f(x)dg(x)\right) \geqslant C\int_a^b \phi'\big(f(x)g(x)\big)f(x)dg(x). \tag{2}$$

(b) If f is C-increasing and g is increasing, differentiable and such that $g(a+0) = 0$, then

$$\phi\left(\frac{1}{C}\int_a^b f(x)dg(x)\right) \leqslant \frac{1}{C}\int_a^b \phi'\big(f(x)g(x)\big)f(x)dg(x). \tag{3}$$

(c) If f is C-increasing and g is decreasing, differentiable and such that $g(b-0) = 0$, then

$$\phi\left(C\int_a^b f(x)d[-g(x)]\right) \geqslant C\int_a^b \phi'\big(f(x)g(x)\big)f(x)d[-g(x)]. \tag{4}$$

(d) If f is C-decreasing and g is decreasing, differentiable and such that $g(b-0) = 0$, then

$$\phi\left(\frac{1}{C}\int_a^b f(x)d[-g(x)]\right) \leqslant \frac{1}{C}\int_a^b \phi'\big(f(x)g(x)\big)f(x)d[-g(x)]. \tag{5}$$

(e) If the condition "ϕ is convex" is replaced by "ϕ is concave", then all the inequalities (2)–(5) hold in reversed direction.

Now we will state our main result, which we will obtain from the above inequalities.

THEOREM 2.2. *Let $\phi : [0,\infty) \to \mathbb{R}$ be a convex and differentiable function such that $\phi(0) = 0$ and let $-\infty \leqslant a < b \leqslant \infty$. Let $k : (a,b) \to [0,\infty)$ be a positive integrable function, $K_1(x) = \int_x^b k(t)dt$ and $K_2(x) = \int_a^x k(t)dt$.*

(a) If f is C-decreasing and g is increasing, differentiable and such that $g(a+0) = 0$, then

$$\int_a^b k(x)\phi\left(C\int_a^x f(t)dg(t)\right)dx \geqslant C\int_a^b K_1(x)\phi'\big(f(x)g(x)\big)f(x)dg(x). \tag{6}$$

(b) If f is C-increasing and g is increasing, differentiable and such that $g(a+0) = 0$, then

$$\int_a^b k(x)\phi\left(\frac{1}{C}\int_a^x f(t)dg(t)\right)dx \leqslant \frac{1}{C}\int_a^b K_1(x)\phi'\big(f(x)g(x)\big)f(x)dg(x). \tag{7}$$

(c) If f is C-increasing and g is decreasing, differentiable and such that $g(b-0) = 0$, then

$$\int_a^b k(x)\phi\left(C\int_x^b f(t)d[-g(t)]\right)dx \geqslant C\int_a^b K_2(x)\phi'\big(f(x)g(x)\big)f(x)d[-g(x)]. \tag{8}$$

(d) If f is C-decreasing and g is decreasing, differentiable and such that $g(b-0) = 0$, then

$$\int_a^b k(x)\phi\left(\frac{1}{C}\int_x^b f(t)d[-g(t)]\right)dx \leqslant \frac{1}{C}\int_a^b K_2(x)\phi'\big(f(x)g(x)\big)f(x)d[-g(x)]. \tag{9}$$

(e) If the condition "ϕ is convex" is replaced by "ϕ is concave", then all the inequalities (6)–(9) hold in reversed direction.

Proof.

(a) Under the given conditions, we have the following inequality by Theorem 2.1

$$\phi\left(C\int_a^t f(x)dg(x)\right) \geqslant C\int_a^t \phi'\big(f(x)g(x)\big)f(x)dg(x).$$

Multiplying the above inequality with a positive function k, then integrating from a to b and applying Fubini's theorem on the integral on the R.H.S., we get

$$\int_a^b k(t)\phi\left(C\int_a^t f(x)dg(x)\right)dt \geqslant C\int_a^b k(t)\int_a^t \phi'\big(f(x)g(x)\big)f(x)dg(x)dt.$$

Denote $L = \phi'(0)$. We have

$$\begin{aligned}&\int_a^b\int_a^t k(t)\phi'\big(f(x)g(x)\big)f(x)dg(x)dt\\ &= \int_a^b\int_a^t k(t)\Big(\phi'\big(f(x)g(x)\big) - L\Big)f(x)dg(x)dt + L\int_a^b\int_a^t k(t)f(x)dg(x)dt.\end{aligned} \tag{10}$$

Since the functions under the integrals on the R.H.S. are positive, we can apply Fubini's theorem. Furthermore, since the second integral on the R.H.S. is finite, we can change the order of integration in the integral on the L.H.S. Therefore,

$$\int_a^b k(t)\phi\left(C\int_a^t f(x)dg(x)\right)dt \geqslant C\int_a^b \phi'\big(f(x)g(x)\big)f(x)\left(\int_x^b k(t)dt\right)dg(x),$$

i.e. (6) holds.
Similarly, we can prove the inequality given in (b).

(c) Under the given conditions, we have the following inequality by Theorem 2.1

$$\phi\left(C\int_t^b f(x)d[-g(x)]\right) \geqslant C\int_t^b \phi'\big(f(x)g(x)\big)f(x)d[-g(x)].$$

Multiplying the above inequality with a positive function k, then integrating from a to b and changing the order of integration in the integral on the R.H.S., we get

$$\int_a^b k(t)\phi\left(C\int_t^b f(x)d[-g(x)]\right)dt \geqslant C\int_a^b k(t)\int_t^b \phi'\big(f(x)g(x)\big)f(x)d[-g(x)]dt$$
$$= C\int_a^b \phi'\big(f(x)g(x)\big)f(x)\left(\int_a^x k(t)dt\right)d[-g(x)]$$

i.e. (8) holds.
Similarly, we can prove the inequality given in (d). □

REMARK 2.3. If the function ϕ in Theorem 2.2 is monotone (i. e., ϕ' is of the same sign everywhere), then we can apply Fubini's theorem directly to the integral on the L.H.S of (10). In that case we do not need integrability of the function k and differentiability of ϕ at 0.

REMARK 2.4. The function ϕ in Theorems 2.1 and 2.2 can be restricted to a compact interval $[0,c]$ if the functions f and g satisfy certain additional conditions. For example, if f and g satisfy the assumptions in part (a) of Theorems 2.1 and 2.2 and, additionally, $\int_a^b f(x)dg(x) \leqslant c/C$, then

$$0 \leqslant f(x)g(x) = \int_a^x f(x)dg(t) \leqslant C\int_a^x f(t)dg(t) \leqslant C\int_a^b f(t)dg(t) \leqslant c.$$

Furthermore, $C\int_a^x f(t)dg(t) \leqslant C\int_a^b f(t)dg(t) \leqslant c$, so the expression on the L.H.S. and R.H.S. of (2) and (6) are well defined for a function ϕ defined on the interval $[0,c]$.

Similarly, if f and g satisfy the assumptions in part (b) of Theorems 2.1 and 2.2 and, additionally, $f(b-0)g(b-0) \leqslant c/C$, then $f(x)g(x) \leqslant Cf(b-0)g(b-0) \leqslant c$ for every $x \in (a,b)$ and

$$\frac{1}{C}\int_a^x f(t)dg(t) \leqslant \frac{1}{C}\int_a^b f(t)dg(t) \leqslant \int_a^b f(b-0)dg(t) = f(b-0)g(b-0) \leqslant c,$$

so the expression on the L.H.S. and R.H.S. of (3) and (7) are well defined for a function ϕ defined on the interval $[0,c]$. An analogous argument shows that the same holds for functions f and g satisfying the assumptions in part (c) (respectively, part (d)) of Theorems 2.1 and 2.2 and, additionally, $\int_a^b f(x)d[-g(x)] \leqslant c/C$ (respectively, $f(a+0)g(a+0) \leqslant c/C$).

From the proofs of Theorem 2.1 (see [5]) and Theorem 2.2 it is clear that they are still valid in this case (i. e. for ϕ defined on $[0,c]$), as are the corollaries that we will derive from these theorems.

COROLLARY 2.5. *Let* $-\infty \leqslant a < b \leqslant \infty$, $p \geqslant 1$, $k:(a,b) \to [0,\infty)$, $K_1(x) = \int_x^b k(t)dt$ *and* $K_2(x) = \int_a^x k(t)dt$.

(a) If f *is* C*-decreasing and* g *is increasing, differentiable and such that* $g(a+0) = 0$, *then*

$$\int_a^b k(x)\left(\int_a^x f(t)d[g(t)]\right)^p dx \geqslant C^{1-p}\int_a^b K_1(x)f^p(x)d[g^p(x)]. \tag{11}$$

(b) If f *is* C*-increasing and* g *is increasing, differentiable and such that* $g(a+0) = 0$, *then*

$$\int_a^b k(x)\left(\int_a^x f(t)d[g(t)]\right)^p dx \leqslant C^{p-1}\int_a^b K_1(x)f^p(x)d[g^p(x)]. \tag{12}$$

(c) If f *is* C*-increasing and* g *is decreasing, differentiable and such that* $g(b-0) = 0$, *then*

$$\int_a^b k(x)\left(\int_x^b f(t)d[-g(t)]\right)^p dx \geqslant C^{1-p}\int_a^b K_2(x)f^p(x)d[-g^p(x)]. \tag{13}$$

(d) If f *is* C*-decreasing and* g *is decreasing, differentiable and such that* $g(b-0) = 0$, *then*

$$\int_a^b k(x)\left(\int_x^b f(t)d[-g(t)]\right)^p dx \leqslant C^{p-1}\int_a^b K_2(x)f^p(x)d[-g^p(x)]. \tag{14}$$

(e) If the condition "$p \geqslant 1$*" is replaced by "*$0 < p \leqslant 1$*", then all the inequalities* (11)–(14) *hold in reversed direction.*

Proof. Applying Theorem 2.2 (a)–(d) for $\phi(x) = x^p$ and taking into account Remark 2.3 we get (11)–(14). □

COROLLARY 2.6. *Let* $-\infty < a < b \leqslant \infty$ *and* $p > 1$.

(a) If f *is* C*-decreasing, then*

$$\int_a^b \frac{1}{(x-a)^p}\left(\int_a^x f(t)dt\right)^p dx \geqslant \frac{pC^{1-p}}{p-1}\int_a^b \left[1-\left(\frac{x-a}{b-a}\right)^{p-1}\right] f^p(x)dx. \tag{15}$$

(b) If f is C-increasing then

$$\int_a^b \frac{1}{(x-a)^p}\left(\int_a^x f(t)dt\right)^p dx \leqslant \frac{pC^{p-1}}{(p-1)}\int_a^b \left[1-\left(\frac{x-a}{b-a}\right)^{p-1}\right] f^p(x)dx. \quad (16)$$

(c) If the condition " $p>1$ " is replaced by " $0<p<1$ ", then inequalities (15) and (16) hold in reversed direction.

Proof. Take $k(x)=(x-a)^{-p}$ and $g(x)=x-a$ in (11) and (12). By considering Remark 2.3 we get (15) and (16), respectively. □

REMARK 2.7. If we take $a=0$, $b=\infty$ and $C=1$ in (15), then we get the reversed Hardy inequality (1).

As a special case, we consider C-monotone functions with respect to the power functions (see [1],[5]). For $C_1, C_2 \geqslant 1$, $-\infty < \alpha_1 \leqslant \alpha_2 < \infty$, we say that $f \in Q^{\alpha_1}(C_1)$ if $f(x)x^{-\alpha_1}$ is C_1-increasing and $f \in Q_{\alpha_2}(C_2)$ if $f(x)x^{-\alpha_2}$ is C_2-decreasing.

THEOREM 2.8. *Let $p \geqslant 1$, $k:(a,b)\to[0,\infty)$, $K_1(x)=\int_x^b k(t)dt$ and $K_2(x)=\int_a^x k(t)dt$.*

(a) If $f \in Q^{\alpha_1}(C)$, $b=\infty$ and $\alpha>\alpha_1$, then the following inequality holds

$$\int_a^\infty k(x)\left(\int_x^\infty f(t)t^{-\alpha}\frac{dt}{t}\right)^p dx \geqslant p\big[C(\alpha-\alpha_1)\big]^{1-p}\int_a^\infty K_2(x)f^p(x)x^{-p\alpha}\frac{dx}{x}. \quad (17)$$

(b) If $f \in Q_{\alpha_2}(C)$, $a=0$ and $\alpha_2>\alpha$, then the following inequality holds

$$\int_0^b k(x)\left(\int_0^x f(t)t^{-\alpha}\frac{dt}{t}\right)^p dx \geqslant p\big[C(\alpha_2-\alpha)\big]^{1-p}\int_0^b K_1(x)f^p(x)x^{-p\alpha}\frac{dx}{x}. \quad (18)$$

Proof.

(a) Since $f \in Q^{\alpha_1}(C)$, by making substitutions $f \to f(x)x^{-\alpha_1}$ and taking $g(x)=x^{\alpha_1-\alpha}$ in (13), we get (17).

(b) Since $f \in Q_{\alpha_2}(C)$, by making substitutions $f \to f(x)x^{-\alpha_2}$ and taking $g(x)=x^{\alpha_2-\alpha}$ in (11), we get (18). □

3. Associated linear functionals and exponential convexity

The differences of the R.H.S. and L.H.S. of the inequalities from the previous section are linear with respect to the convex function φ. We will use this property to construct new families of exponentially convex functions and to derive some related results.

For the sake of simplicity and to avoid many notions, we introduce the following definitions:

(M_1) Under the assumptions of Theorem 2.1(a), we define linear functional as

$$\Omega_1(\phi) = \phi\left(C\int_a^b f(x)dg(x)\right) - C\int_a^b \phi'\big(f(x)g(x)\big)f(x)dg(x).$$

(M_2) Under the assumptions of Theorem 2.1(b), we define linear functional as

$$\Omega_2(\phi) = \frac{1}{C}\int_a^b \phi'\big(f(x)g(x)\big)f(x)dg(x) - \phi\left(\frac{1}{C}\int_a^b f(x)dg(x)\right).$$

(M_3) Under the assumptions of Theorem 2.1(c), we define linear functional as

$$\Omega_3(\phi) = \phi\left(C\int_a^b f(x)d[-g(x)]\right) - C\int_a^b \phi'\big(f(x)g(x)\big)f(x)d[-g(x)].$$

(M_4) Under the assumptions of Theorem 2.1(d), we define linear functional as

$$\Omega_4(\phi) = \frac{1}{C}\int_a^b \phi'\big(f(x)g(x)\big)f(x)d[-g(x)] - \phi\left(\frac{1}{C}\int_a^b f(x)d[-g(x)]\right).$$

(M_5) Under the assumptions of Theorem 2.2(a), we define linear functional as

$$\Omega_5(\phi) = \int_a^b k(x)\phi\left(C\int_a^x f(t)dg(t)\right)dx - C\int_a^b K_1(x)\phi'\big(f(x)g(x)\big)f(x)dg(x).$$

(M_6) Under the assumptions of Theorem 2.2(b), we define linear functional as

$$\Omega_6(\phi) = \frac{1}{C}\int_a^b K_1(x)\phi'\big(f(x)g(x)\big)f(x)dg(x) - \int_a^b k(x)\phi\left(\frac{1}{C}\int_a^x f(t)dg(t)\right)dx.$$

(M_7) Under the assumptions of Theorem 2.2(c), we define linear functional as

$$\Omega_7(\phi) = \int_a^b k(x)\phi\left(C\int_x^b f(t)d[-g(t)]\right)dx - C\int_a^b K_2(x)\phi'\big(f(x)g(x)\big)f(x)d[-g(x)].$$

(M_8) Under the assumptions of Theorem 2.2(d), we define linear functional as

$$\Omega_8(\phi) = \frac{1}{C}\int_a^b K_2(x)\phi'\big(f(x)g(x)\big)f(x)d[-g(x)] - \int_a^b k(x)\phi\left(\frac{1}{C}\int_x^b f(t)d[-g(t)]\right)dx.$$

REMARK 3.1. Under the assumptions of Theorem 2.1 and Theorem 2.2 a convex function ϕ and the linear functionals Ω_k for $k = 1,\dots,8$ satisfy $\Omega_k(\phi) \geqslant 0$.

Now we are ready to investigate the properties of functionals as defined above, regarding n-exponential and exponential convexity.

We start this part of the section by giving some definitions and notions which are used frequently in the results. Throughout this section I is an interval in $\mathbb{R}$. The following results for n-exponentially convex functions are cited from [4].

DEFINITION 1. A function $f : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j f\left(\frac{x_i + x_j}{2}\right) \geqslant 0$$

holds for all choices $\xi_i \in \mathbb{R}$ and every $x_i \in I$, $i = 1, \ldots, n$.

A function $f : I \longrightarrow \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

REMARK 3.2. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are, in fact, non-negative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leqslant n$.

By using some linear algebra and the definition of a positive semi-definite matrix, we have the following proposition.

PROPOSITION 3.3. *If f is n-exponentially convex in the Jensen sense then for any $x_i \in I$, $i = 1, \ldots, n$, the matrix*

$$\left[f\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k}$$

is positive semi-definite for all $k \in \mathbb{N}$, $k \leqslant n$. In particular,

$$\det\left[f\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k} \geqslant 0$$

for all $k \in \mathbb{N}$, $k \leqslant n$.

DEFINITION 2. A function $f : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$. Moreover, a function $f : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous on I.

REMARK 3.4. A function $f : I \to \mathbb{R}$ is log-convex in the Jensen sense, i. e.

$$f\left(\frac{x_1 + x_2}{2}\right)^2 \leqslant f(x_1) f(x_2), \quad \text{for all } x_1, x_2 \in I, \tag{19}$$

if and only if

$$\xi_1^2 f(x_1) + 2\xi_1 \xi_2 f\left(\frac{x_1 + x_2}{2}\right) + \xi_2^2 f(x_2) \geqslant 0$$

holds for every $\xi_1,\xi_2 \in \mathbb{R}$ and $x_1,x_2 \in I$, i. e, if and only if f is 2-exponentially convex in the Jensen sensse. By induction from (19) we have

$$f\left(\frac{1}{2^k}x_1+\left(1-\frac{1}{2^k}\right)x_2\right) \leqslant f(x_1)^{\frac{1}{2^k}} f(x_2)^{1-\frac{1}{2^k}}.$$

Therefore, if f is continuous and $f(x_1)=0$ for some $x_1 \in I$, then from the last inequality and non-negativity of f (see Remark 3.2) we get $f(x_2)=\lim_{k\to\infty} f(\frac{1}{2^k}x_1+(1-\frac{1}{2^k})x_2)=0$ for all $x_2 \in I$. Hence, a 2-exponentially convex function is either identically equal to zero or it is strictly positive and log-convex.

The following lemma is equivalent to the definition of convex functions [6, page 2].

LEMMA 3.5. *A function $f: I \to \mathbb{R}$ is convex if and only if the inequality*

$$(x_3-x_2)f(x_1)+(x_1-x_3)f(x_2)+(x_2-x_1)f(x_3) \geqslant 0$$

holds for all $x_1,x_2,x_3 \in I$ such that $x_1<x_2<x_3$.

We will also need the following result (see e.g. [6]).

LEMMA 3.6. *If Φ is a convex function on an interval I and if $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid:*

$$\frac{\Phi(x_2)-\Phi(x_1)}{x_2-x_1} \leqslant \frac{\Phi(y_2)-\Phi(y_1)}{y_2-y_1}. \tag{20}$$

If the function Φ is concave then the sign of the above inequality is reversed.

Divided differences are found to be very handy and interesting when we have to operate with different functions having different degree of smoothness. Let $f: I \to \mathbb{R}$ be a function, I an interval in $\mathbb{R}$. Then for distinct points $u_i \in I$, $i=0,1,2$, the divided differences of the first and second order are defined as follows:

$$[u_i;f]=f(u_i) \quad (i=0,1,2),$$

$$[u_i,u_{i+1};f]=\frac{f(u_{i+1})-f(u_i)}{u_{i+1}-u_i} \quad (i=0,1),$$

$$[u_0,u_1,u_2;f]=\frac{[u_1,u_2;f]-[u_0,u_1;f]}{u_2-u_0}. \tag{21}$$

The values of the divided differences are independent of the order of the points u_0,u_1,u_2 and may be extended to include the cases when some or all points are equal, that is

$$[u_0,u_0;f]=\lim_{u_1\to u_0}[u_0,u_1;f]=f'(u_0),$$

provided that f' exists.

Now, passing through the limit $u_1 \to u_0$ and replacing u_2 by u in (21), we have [6, p. 16]

$$[u_0,u_0,u;f] = \lim_{u_1\to u_0}[u_0,u_1,u;f] = \frac{f(u)-f(u_0)-f'(u_0)(u-u_0)}{(u-u_0)^2}, \ u \neq u_0,$$

provided that f' exists. Also passing to the limit $u_i \to u$ $(i=0,1,2)$ in (21), we have

$$[u,u,u;f] = \lim_{u_i\to u}[u_0,u_1,u_2;f] = \frac{f''(u)}{2},$$

provided that f'' exists.

REMARK 3.7. One can note that if for all distinct $u_0,u_1 \in I$, $[u_0,u_1;f] \geqslant 0$ then f is increasing on I and if for all distinct $u_0,u_1,u_2 \in I$, $[u_0,u_1,u_2;f] \geqslant 0$ then f is convex on I.

In order to obtain our main results regarding the exponential convexity, we define several families of functions with certain properties. Let $J \subseteq \mathbb{R}$ be an interval and let

$$\mathbf{E}_n = \{\phi_t : [0,\infty) \to \mathbb{R} : t \in J, t \mapsto [u_0,u_1,u_2;\phi_t] \text{ is } n\text{-exponentially convex on } J \\ \text{in the Jensen sense for every } u_0 \neq u_1 \neq u_2 \neq u_0 \in [0,\infty)\}$$

for $n \in \mathbb{N}$ and $\mathbf{E}_\infty$ be defined analogously using exponentially convex functions in the Jensen sense instead of n-exponentially convex functions in the Jensen sense.

THEOREM 3.8. *Let Ω_k be linear functionals defined as in (M_k) for $k=1,...,8$ associated with a family $\mathbf{E}_n$. Then $t \mapsto \Omega_k(\phi_t)$ is an n-exponentially convex function in the Jensen sense on J. If the function $t \mapsto \Omega_k(\phi_t)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. We prove n-exponential convexity in the Jensen sense of the function $t \mapsto \Omega_k(\phi_t)$, for $k=1,...,8$. For $\xi_i \in \mathbb{R}$ and $t_i \in J$, $i=1,...,n$, and the family of functions $\mathbf{E}_n$, define the function

$$h(u) = \sum_{i,j=1}^{n} \xi_i \xi_j \phi_{\frac{t_i+t_j}{2}}(u). \tag{22}$$

We have

$$[u_0,u_1,u_2;h] = \sum_{i,j=1}^{n} \xi_i \xi_j [u_0,u_1,u_2;\phi_{\frac{t_i+t_j}{2}}].$$

Since the function $t \mapsto [u_0,u_1,u_2;\phi_t]$ is n-exponentially convex in the Jensen sense on J, the right-hand side of the above expression is non-negative which implies that $h(u)$ is convex on I (see Remark 3.7).

Hence, taking into account the assumption (M_k) with Remark 3.1, we have

$$\Omega_k(h) \geqslant 0, \text{ for } k=1,...,8,$$

that is,

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Omega_k \left(\phi_{\frac{t_i+t_j}{2}} \right) \geqslant 0.$$

Therefore, we conclude that the functions $t \mapsto \Omega_k(\phi_t)$, $k = 1,...,8$, are n-exponentially convex in the Jensen sense on J.

If the function $t \mapsto \Omega_k(\phi_t)$ is also continuous on J, then $t \mapsto \Omega_k(\phi_t)$ is n-exponentially convex by definition for $k = 1,...,8$. □

The following corollary is an immediate consequence of the above theorem.

COROLLARY 3.9. *Let Ω_k be linear functionals defined as in (M_k) for $k = 1,...,8$ associated with a family $\mathbf{E}_\infty$. Then $t \mapsto \Omega_k(\phi_t)$ is an exponentially convex function in the Jensen sense on J. If $t \mapsto \Omega_k(\phi_t)$ is continuous on J then it is exponentially convex on J.*

Proof. Follows from the previous theorem. □

COROLLARY 3.10. *Let Ω_k be linear functionals defined as in (M_k) for $k = 1,...,8$ associated with a family $\mathbf{E}_2$. Then the following statements hold:*

(i) If the function $t \mapsto \Omega_k(\phi_t)$ is continuous on J then, for $r,s,t \in J$ such that $r < s < t$, we have

$$(\Omega_k(\phi_s))^{t-r} \leqslant (\Omega_k(\phi_r))^{t-s} (\Omega_k(\phi_t))^{s-r}. \tag{23}$$

(ii) If the function $t \mapsto \Omega_k(\phi_t)$ is strictly positive and differentiable on J, then for all $t,r,u,v \in J$ such that $t \leqslant u$, $r \leqslant v$, we have

$$\mathfrak{B}(t,r;\Omega_k,\mathbf{E}_2) \leqslant \mathfrak{B}(u,v;\Omega_k,\mathbf{E}_2),\ k = 1,...,4,$$

where

$$\mathfrak{B}(t,r;\Omega_k,\mathbf{E}_2) = \begin{cases} \left(\frac{\Omega_k(\phi_t)}{\Omega_k(\phi_r)} \right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{\frac{d}{dt}(\Omega_k(\phi_t))}{\Omega_k(\phi_t)} \right), & t = r. \end{cases} \tag{24}$$

Proof. (i) By Theorem 3.8 the mapping $t \mapsto \Omega_k(\phi_t)$ is 2-exponentially convex. Hence, by Remark 3.4, this mapping is either identically equal to zero, in which case inequality (23) has zero on both sides, or it is strictly positive and log-convex. Therefore, for $r,s,t \in J$ such that $r < s < t$ with $f(t) = \log \Omega_k(\phi_t)$ in Lemma 3.5 gives

$$(t-s)\log \Omega_k(\phi_r) + (r-t)\log \Omega_k(\phi_s) + (s-r)\log \Omega_k(\phi_t) \geqslant 0.$$

This is equivalent to inequality (23).

(ii) By (i), the function $t \mapsto \Phi_k(f_t)$ is log-convex on J, which means that the function $t \mapsto \log \Phi_k(f_t)$ is convex on J. Hence, by using Lemma 3.6 with $t \leqslant u$, $r \leqslant v$, $t \neq r$, $u \neq v$, we obtain

$$\frac{\log \Omega_k(\phi_t) - \log \Omega_k(\phi_r)}{t-r} \leqslant \frac{\log \Omega_k(\phi_u) - \log \Omega_k(\phi_v)}{u-v}, \tag{25}$$

that is,

$$\mathfrak{B}(t,r;\Omega_k,\mathbf{E}_2) \leqslant \mathfrak{B}(u,v;\Omega_k,\mathbf{E}_2).$$

Finally, if $t = r \leqslant u$, by taking the limit $\lim_{r \longrightarrow t}$, we have

$$\mathfrak{B}(t,t;\Omega_k,\mathbf{E}_2) \leqslant \mathfrak{B}(u,v;\Omega_k,\mathbf{E}_2).$$

Other possible cases are treated similarly. □

REMARK 3.11. The results given in Theorem 3.8 (respectively, Corollary 3.9; Corollary 3.10) hold when two of the points $u_0, u_1, u_2 \in [0,\infty)$ coincide, that is to say when the family $\mathbf{E}_n$ (respectively, $\mathbf{E}_\infty$; $\mathbf{E}_2$) is replaced with a family $(\phi_t)_{t\in J}$ of differentiable functions ϕ_t such that for every $u_0 \neq u_1$ the function $t \mapsto [u_0,u_0,u_1;\phi_t]$ is n-exponentially convex (respectively, exponentially convex; 2-exponentially convex) in the Jensen sense. Moreover, the above results also hold when all three points coincide, i.e. for a family of twice differentiable functions ϕ_t such that the mapping $t \mapsto [u_0,u_0,u_0;\phi_t] = \phi_t''(u_0)/2$ satisfies analogous properties. These results can be proved easily as before by using the extension of the divided differences to the case when some or all of the points u_0,u_1,u_2 are equal.

In particular, if the functions ϕ_t are twice differentiable, then the families $\mathbf{E}_n$ can be defined as

$$\mathbf{E}_n = \{\phi_t : [0,\infty) \to \mathbb{R} : t \in J, t \mapsto \frac{d^2}{dx^2}\phi_t(x) \text{ is } n\text{-exponentially convex on } J \text{ in the Jensen sense for every } x \in [0,\infty)\}.$$

In order to obtain refinements of the reversed Hardy inequality for C-monotone functions, we consider the following example of our interest and apply it to the functionals Ω_k, $k = 5,\ldots,8$. Let a family of functions $\phi_p : [0,\infty) \to \mathbb{R}$, $p > 0$, be defined by

$$\phi_p(x) = \begin{cases} \frac{x^p}{p(p-1)}, & p>0,\ p\neq 1 \\ x\log x, & p=1, \end{cases} \tag{26}$$

with $0\log 0 = 0$. Then $\phi_p''(x) = x^{p-2}$, so ϕ_p is convex. The following Corollary is an immediate consequence of Corollary 3.10.

COROLLARY 3.12. *Let a,b,f,g,k,K_1 and K_2 be as in Corollary 2.5. Then, for $p<q<r\,(p,q,r\in\mathbb{R}^+\backslash\{1\})$ the following inequalities hold:*

$$(a)\ \left[\frac{\int_a^b k(x)(C\int_a^x f(t)dg(t))^q dx - C\int_a^b K_1(x)f^q(x)dg^q(x)}{q(q-1)}\right]^{r-p} \leqslant \left[\frac{\int_a^b k(x)(C\int_a^x f(t)dg(t))^p dx - C\int_a^b K_1(x)f^p(x)dg^p(x)}{p(p-1)}\right]^{r-q} \times \left[\frac{\int_a^b k(x)(C\int_a^x f(t)dg(t))^r dx - C\int_a^b K_1(x)f^r(x)dg^r(x)}{r(r-1)}\right]^{q-p}, \quad (27)$$

$$(b)\ \left[\frac{\frac{1}{C}\int_a^b K_1(x)f^q(x)dg^q(x) - \frac{1}{C^q}\int_a^b k(x)(C\int_a^x f(t)dg(t))^q dx}{q(q-1)}\right]^{r-p} \leqslant \left[\frac{\frac{1}{C}\int_a^b K_1(x)f^p(x)dg^p(x) - \frac{1}{C^p}\int_a^b k(x)(C\int_a^x f(t)dg(t))^p dx}{p(p-1)}\right]^{r-q} \times \left[\frac{\frac{1}{C}\int_a^b K_1(x)f^r(x)dg^r(x) - \frac{1}{C^r}\int_a^b k(x)(C\int_a^x f(t)dg(t))^r dx}{r(r-1)}\right]^{q-p}, \quad (28)$$

$$(c)\ \left[\frac{\int_a^b k(x)(C\int_x^b f(t)d[-g(t)])^q dx - C\int_a^b K_2(x)f^q(x)d[-g^q(x)]}{q(q-1)}\right]^{r-p} \leqslant \left[\frac{\int_a^b k(x)(C\int_x^b f(t)d[-g(t)])^p dx - C\int_a^b K_2(x)f^p(x)d[-g^p(x)]}{p(p-1)}\right]^{r-q} \times \left[\frac{\int_a^b k(x)(C\int_x^b xf(t)d[-g(t)])^r dx - C\int_a^b K_2(x)f^r(x)d[-g^r(x)]}{r(r-1)}\right]^{q-p}, \quad (29)$$

$$(d)\ \left[\frac{\frac{1}{C}\int_a^b K_2(x)f^q(x)d[-g^q(x)] - \frac{1}{C^q}\int_a^b k(x)(C\int_x^b f(t)d[-g(t)])^q dx}{q(q-1)}\right]^{r-p} \leqslant \left[\frac{\frac{1}{C}\int_a^b K_2(x)f^p(x)d[-g^p(x)] - \frac{1}{C^p}\int_a^b k(x)(C\int_x^b f(t)d[-g(t)])^p dx}{p(p-1)}\right]^{r-q} \times \left[\frac{\frac{1}{C}\int_a^b K_2(x)f^r(x)d[-g^r(x)] - \frac{1}{C^r}\int_a^b k(x)(C\int_a^x f(t)d[-g(t)])^r dx}{r(r-1)}\right]^{q-p}. \quad (30)$$

Proof. Notice that $\phi_p''(x)=x^{p-2}=e^{(p-2)\log x}$ and the mapping $p\mapsto e^{(p-2)\log x}$ is

n-exponentially convex for every $n \in \mathbb{N}$ since

$$\sum_{i,j=1}^{n} \xi_i \xi_j e^{\left(\frac{p_i+p_j}{2}-2\right)\log x} = \left(\sum_{i=1}^{n} \xi_i e^{\frac{p_i-2}{2}\log x}\right)^2 \geqslant 0.$$

Therefore, by Theorem 3.8 and Remark 3.11, the mapping $p \mapsto \Omega_k(\phi_p)$ is n-exponentially convex in the Jensen sense. In, particular, it is 2-exponentially convex and it is straightforward to check that it is continuous, so inequalities (27)–(30) follow from Corollary 3.10(i). □

COROLLARY 3.13. *Let a,b,k,K_1 and K_2 be as in Theorem 2.8. Then, for $p < q < r\,(p,q,r \in \mathbb{R}^+\backslash\{1\})$, the following holds:*
(a) If $f \in Q^{\alpha_1}(C)$, $b = \infty$ and $\alpha > \alpha_1$, then

$$\left[\frac{\left[C(\alpha-\alpha_1)\right]^q \int_a^\infty k(x)(\int_x^\infty f(t)t^{-\alpha}\frac{dt}{t})^q dx - qC(\alpha-\alpha_1)\int_a^\infty K_2(x)f^q(x)x^{-q\alpha}\frac{dx}{x}}{q(q-1)}\right]^{r-p}$$
$$\leqslant \left[\frac{\left[C(\alpha-\alpha_1)\right]^p \int_a^\infty k(x)(\int_x^\infty f(t)t^{-\alpha}\frac{dt}{t})^p dx - pC(\alpha-\alpha_1)\int_a^\infty K_2(x)f^p(x)x^{-p\alpha}\frac{dx}{x}}{p(p-1)}\right]^{r-q}$$
$$\times \left[\frac{\left[C(\alpha-\alpha_1)\right]^r \int_a^\infty k(x)(\int_x^\infty f(t)t^{-\alpha}\frac{dt}{t})^r dx - rC(\alpha-\alpha_1)\int_a^\infty K_2(x)f^r(x)x^{-r\alpha}\frac{dx}{x}}{r(r-1)}\right]^{q-p}.$$

(b) If $f \in Q_{\alpha_2}(C)$, $a = 0$ and $\alpha_2 > \alpha$, then

$$\left[\frac{\left[C(\alpha_2-\alpha)\right]^q \int_0^b k(x)(\int_0^x f(t)t^{-\alpha}\frac{dt}{t})^q dx - qC(\alpha_2-\alpha)\int_0^b K_1(x)f^q(x)x^{-q\alpha}\frac{dx}{x}}{q(q-1)}\right]^{r-p}$$
$$\leqslant \left[\frac{\left[C(\alpha_2-\alpha)\right]^p \int_0^b k(x)(\int_0^x f(t)t^{-\alpha}\frac{dt}{t})^p dx - pC(\alpha_2-\alpha)\int_0^b K_1(x)f^p(x)x^{-p\alpha}\frac{dx}{x}}{p(p-1)}\right]^{r-q}$$
$$\times \left[\frac{\left[C(\alpha_2-\alpha)\right]^r \int_0^b k(x)(\int_0^x f(t)t^{-\alpha}\frac{dt}{t})^r dx - rC(\alpha_2-\alpha)\int_0^b K_1(x)f^r(x)x^{-r\alpha}\frac{dx}{x}}{r(r-1)}\right]^{q-p}.$$

Proof. (a) It follows from Corollary 3.12(c) by making substitutions $f \to f(t)t^{-\alpha_1}$ and taking $g(t) = t^{\alpha_1-\alpha}$ in (29).

(b) It follows from Corollary 3.12(a) by making substitutions $f \to f(t)t^{-\alpha_2}$ and taking $g(t) = t^{\alpha_2-\alpha}$ in (27). □

4. Mean value results

In this section we assume that the assumptions of Remark 2.4 are satisfied, so $\Omega_k(\phi)$ are well defined for $\phi \in C^2[0,c]$. We will first state and prove a Lagrange type mean value theorem for the linear functionals Ω_k, $k = 1,...,8$ defined by $(M_1)-(M_8)$.

THEOREM 4.1. *Let* Ω_k, $k=1,\dots,8$ *be the linear functionals defined by* $(M_1)-(M_8)$ *and* $\phi \in C^2[0,c]$, $c>0$, *such that* $\phi(0)=0$. *Then there exists* $\xi_k \in [0,c]$ *such that*

$$\Omega_k(\phi)=\frac{\phi''(\xi_k)}{2}\Omega_k(x^2). \tag{31}$$

Proof. Fix $k=1,\dots,8$. Since ϕ'' is continuous on $[0,c]$, it attains its maximum and minimum value on $[0,c]$. Let us consider

$$m=\min_{x\in[0,c]}\{\phi''(x)\} \quad \text{and} \quad M=\max_{x\in[0,c]}\{\phi''(x)\}.$$

Let us consider functions $F_1,F_2:[0,c]\to\mathbb{R}$ defined by

$$F_1(x)=M\frac{x^2}{2}-\phi(x) \quad \text{and} \quad F_2(x)=\phi(x)-m\frac{x^2}{2}.$$

Then

$$F_1''(x)=M-\phi''(x)\geqslant 0 \quad \text{and} \quad F_2''(x)=\phi''(x)-m\geqslant 0,$$

so F_1,F_2 are convex functions and it holds $F_1(0)=F_2(0)=0$. Hence, from Theorem 2.1 and 2.2 with F_1 and F_2 respectively, we have

$$\Omega_k(\phi)\leqslant\frac{M}{2}\Omega_k(x^2) \quad \text{and} \quad \Omega_k(\phi)\geqslant\frac{m}{2}\Omega_k(x^2),$$

i.e., by combining these two inequalities, we have

$$\frac{m}{2}\Omega_k(x^2)\leqslant\Omega_k(\phi)\leqslant\frac{M}{2}\Omega_k(x^2).$$

If $\Omega_k(x^2)=0$ then $\Omega_k(\phi)=0$ and (31) holds for all $\xi_k\in[0,c]$. Otherwise

$$m\leqslant\frac{2\Omega_k(\phi)}{\Omega_k(x^2)}\leqslant M.$$

Since $\phi''(x)$ is continuous, there exists $\xi_k\in[0,c]$ such that (31) holds and the proof is complete. □

Next, we will state and prove a Cauchy type mean value theorem for the linear functionals Ω_k.

THEOREM 4.2. *Let* Ω_k, $k=1,\dots,8$ *be linear functionals defined by* $(M_1)-(M_8)$ *and* $\phi,\psi\in C^2[0,c]$, $c>0$, *be such that* $\Omega_k(\psi)\neq 0$ *and* $\phi(0)=\psi(0)=0$. *Then there exists* $\xi_k\in[0,c]$ *such that either the following identity*

$$\frac{\Omega_k(\phi)}{\Omega_k(\psi)}=\frac{\phi''(\xi_k)}{\psi''(\xi_k)} \tag{32}$$

holds or $\phi''(\xi_k)=\psi''(\xi_k)=0$.

Proof. Fix $1 \leqslant k \leqslant 8$ and let $L \in C^2[0,c]$ be given by

$$L = v_1\phi - v_2\psi,$$

where $v_1 = \Omega_k(\psi)$ and $v_2 = \Omega_k(\phi)$. Now, using Theorem 4.1 for the function L, we have

$$\left(v_1\frac{\phi''(\xi_k)}{2} - v_2\frac{\psi''(\xi_k)}{2}\right)\Omega_k(x^2) = 0. \tag{33}$$

Since $\Omega_k(x^2) \neq 0$ (otherwise we have a contradiction with $\Omega_k(\psi) \neq 0$ by Theorem 4.1), either $\phi''(\xi_k) = \psi''(\xi_k) = 0$ or (33) yields (32). □

Theorem 4.2 can be used in construction of Cauchy means. Suppose that ϕ''/ψ'' has inverse. Then (32) gives

$$\xi_k = \left(\frac{\phi''}{\psi''}\right)^{-1}\left(\frac{\Omega_k(\phi)}{\Omega_k(\psi)}\right), \tag{34}$$

where $\xi_k \in [0,c]$. We conclude that the expression on the R.H.S. of the above expression is a Cauchy type mean of the interval $[0,c]$. For the family of functions ϕ_p given by (26) and $r,l \in \mathbb{R}^+$, the mapping $\phi_l''(x)/\phi_r''(x) = x^{l-r}$ has an inverse and we denote the Cauchy means

$$M_{l,r}^k = \left(\frac{\Omega_k(\phi_l)}{\Omega_k(\phi_r)}\right)^{\frac{1}{l-r}}, r \neq l. \tag{35}$$

Since $\Omega_k(\phi_p)$ and the Cauchy means $M_{l,r}^k$ for $k = 1,\dots,4$ for the class of functions defined in (26) were given explicitly in [1], we will give them here only for $k = 5,\dots,8$. But, before doing this, we will introduce some notations for our convenience. Let us denote

$$H_p(a,b,K,f,g) = \left(\int_a^b K(x)f^p(x)d[g^p(x)]\right)^{1/p},$$

$$\widetilde{H}_p(a,b,K,f,g) = \left(\int_a^b K(x)f^p(x)d[-g^p(x)]\right)^{1/p},$$

$$R_l^n(K,f,g) = \int_a^b K(x)\left(\frac{1}{l} + \ln\left(f(x)g(x)\right)\right)^n f^l(x)d\left[g^l(x)\right]$$

and

$$\widetilde{R}_l^n(K,f,g) = \int_a^b K(x)\left(\frac{1}{l} + \ln\left(f(x)g(x)\right)\right)^n f^l(x)d\left[-g^l(x)\right]$$

We will first give the expressions for $\Omega_k(\phi_p)$, $k = 5,\dots,8$.

$$\Omega_5(\phi_p) = \begin{cases} \frac{C^p}{p(p-1)}\int_a^b k(x)H_1^p(a,x,1,f,g)dx - CH_p^p(a,b,K_1,f,g), & p>0,\ p\neq 1 \\ C\int_a^b k(x)H_1^1(a,x,1,f,g)\ln[CH_1^1(a,x,1,f,g)]dx - CR_1^1(K_1,f,g), & p=1. \end{cases}$$

$$\Omega_6(\phi_p) = \begin{cases} \dfrac{\frac{1}{C}\int_a^b H_p^p(a,b,K_1,f,g) - \frac{1}{C^p}\int_a^b k(x)H_1^p(a,x,1,f,g)dx}{p(p-1)}, & p>0,\ p\neq 1 \\ \frac{1}{C}R_1^1(K_1,f,g) - \frac{1}{C}\int_a^b (k(x)H_1^1(a,x,1,f,g)\ln[CH_1^1(a,x,1,f,g)])dx, & p=1. \end{cases}$$

$$\Omega_7(\phi_p) = \begin{cases} \frac{C^p \int_a^b k(x)\widetilde{H}_1^p(x,b,1,f,g)dx - C\widetilde{H}_p^p(a,b,K_2,f,g)}{p(p-1)}, & p>0,\ p\neq 1 \\ C\int_a^b (k(x)\widetilde{H}_1^1(x,b,1,f,g)\ln[C\widetilde{H}_1^1(x,b,1,f,g)])dx - C\widetilde{R}_1^1(K_2,f,g), & p=1. \end{cases}$$

$$\Omega_8(\phi_p) = \begin{cases} \frac{\frac{1}{C}\int_a^b \widetilde{H}_p^p(a,b,K_2,f,g) - \frac{1}{C^p}\int_a^b k(x)\widetilde{H}_1^p(x,b,1,f,g)dx}{p(p-1)}, & p>0,\ p\neq 1 \\ \frac{1}{C}\widetilde{R}_1^1(K_2,f,g) - \frac{1}{C}\int_a^b (k(x)\widetilde{H}_1^1(x,b,1,f,g)\ln[C\widetilde{H}_1^1(x,b,1,f,g)])dx, & p=1. \end{cases}$$

These expressions for $\Omega_k(\phi_p)$ inserted in (35) give the Cauchy means for $l\neq r$. Also, we have continuous extensions of the Cauchy means in other cases. Therefore, by limit, we have the following

$$M_{r,r}^5 = \begin{cases} \exp\left(\frac{1-2r}{r(r-1)} + \frac{\int_a^b [k(x)C^r H_1^r(a,x,1,h,g)\ln(CH_1^1(a,x,1,h,g))]dx - CR_r^1(K_1,h,g)}{(\int_a^b k(x)C^r H_1^r(a,x,1,h,g)dx - CH_r^r(a,b,K_1,h,g))}\right), & r\neq 1, \\ \exp\left(-1 + \frac{\int_a^b [k(x)CH_1^1(a,x,1,h,g)(\ln(CH_1^1(a,x,1,h,g)))^2]dx + CH_1^1(a,b,K_1,h,g) - CR_1^2(K_1,h,g)}{2(\int_a^b [CH_1^1(a,x,1,h,g)\ln(CH_1^1(a,x,1,h,g))]dx - CR_1^1(K_1,h,g))}\right), & \\ & r=1. \end{cases}$$

$$M_{r,r}^6 = \begin{cases} \exp\left(\frac{1-2r}{r(r-1)} + \frac{\frac{1}{C}R_r^1(K_1,h,g) - \int_a^b [k(x)\frac{1}{C^r}H_1^r(a,x,1,h,g)\ln(\frac{1}{C}H_1^1(a,x,1,h,g))]dx}{(\frac{1}{C}H_r^r(a,b,K_1,h,g) - \int_a^b k(x)\frac{1}{C^r}H_1^r(a,x,1,h,g)dx)}\right), & r\neq 1, \\ \exp\left(-1 + \frac{-\frac{1}{C}H_1^1(a,b,K_1,h,g) + \frac{1}{C}R_1^2(K_1,h,g) - \int_a^b [k(x)\frac{1}{C}H_1^1(a,x,1,h,g)(\ln(\frac{1}{C}H_1^1(a,x,1,h,g)))^2]dx}{2(\frac{1}{C}R_1^1(K_1,h,g) - \int_a^b [\frac{1}{C}H_1^1(a,x,1,h,g)\ln(\frac{1}{C}H_1^1(a,x,1,h,g))]dx)}\right), & \\ & r=1. \end{cases}$$

$$M_{r,r}^7 = \begin{cases} \exp\left(\frac{1-2r}{r(r-1)} + \frac{\int_a^b [k(x)C^r \widetilde{H}_1^r(x,b,1,h,g)\ln(C\widetilde{H}_1^1(x,b,1,h,g))]dx - C\widetilde{R}_r^1(K_2,h,g)}{(\int_a^b k(x)C^r \widetilde{H}_1^r(x,b,1,h,g)dx - C\widetilde{H}_r^r(a,b,K_2,h,g))}\right), & r\neq 1, \\ \exp\left(-1 + \frac{\int_a^b [k(x)C\widetilde{H}_1^1(x,b,1,h,g)(\ln(C\widetilde{H}_1^1(x,b,1,h,g)))^2]dx + C\widetilde{H}_1^1(a,b,K_2,h,g) - C\widetilde{R}_1^2(K_2,h,g)}{2(\int_a^b [C\widetilde{H}_1^1(x,b,1,h,g)\ln(C\widetilde{H}_1^1(x,b,1,h,g))]dx - C\widetilde{R}_1^1(K_2,h,g))}\right), & \\ & r=1. \end{cases}$$

$$M_{r,r}^8 = \begin{cases} \exp\left(\frac{1-2r}{r(r-1)} + \frac{\frac{1}{C}\widetilde{R}_r^1(K_2,h,g) - \int_a^b [k(x)\frac{1}{C^r}\widetilde{H}_1^r(x,b,1,h,g)\ln(\frac{1}{C}\widetilde{H}_1^1(x,b,1,h,g))]dx}{(\frac{1}{C}\widetilde{H}_r^r(a,b,K_2,h,g) - \int_a^b k(x)\frac{1}{C^r}\widetilde{H}_1^r(x,b,1,h,g)dx)}\right), & r\neq 1, \\ \exp\left(-1 + \frac{-\frac{1}{C}\widetilde{H}_1^1(a,b,K_2,h,g) + \frac{1}{C}\widetilde{R}_1^2(K_2,h,g) - \int_a^b [k(x)\frac{1}{C}\widetilde{H}_1^1(x,b,1,h,g)(\ln(\frac{1}{C}\widetilde{H}_1^1(x,b,1,h,g)))^2]dx}{2(\frac{1}{C}\widetilde{R}_1^1(K_2,h,g) - \int_a^b [\frac{1}{C}\widetilde{H}_1^1(x,b,1,h,g)\ln(\frac{1}{C}\widetilde{H}_1^1(x,b,1,h,g))]dx)}\right), & \\ & r=1. \end{cases}$$

By Corollary 3.10(ii), the means $M_{l,r}^k$ given by (35) are monotonic, i.e. for $r,l,u,v\in\mathbb{R}^+$ such that $l\leqslant v$, $r\leqslant u$ we have

$$M_{l,r}^k \leqslant M_{v,u}^k, \qquad k=5,\dots,8.$$

The monotonicity of the Cauchy means for $k=1,\dots,4$ was proven in [1].

5. Further Examples

In the earlier sections we applied Theorem 3.8 to the family of functions ϕ_p given by (26) and constructed exponentially convex functions. By using properties of exponentially convex functions, we refined the reverse Hardy inequality and constructed Cauchy means. In this section, we apply Theorem 3.8 to other families of convex functions to get other exponentially convex functions and Cauchy means.

EXAMPLE 5.1. Consider the family of functions

$$\Upsilon_1 = \{\lambda_l : [0,\infty) \to \mathbb{R} : l \in \mathbb{R}\}$$

defined by

$$\lambda_l(x) = \begin{cases} \frac{e^{lx}-1}{l^2}, & l \neq 0, \\ \frac{x^2}{2}, & l = 0. \end{cases}$$

Notice that $\lambda_l(0) = 0$ and the mapping $l \mapsto \frac{d^2\lambda_l}{dx^2}(x) = e^{lx}$ is exponentially convex (see the proof of Corollary 3.12). By Corollary 3.9 and Remark 3.11 the mapping $l \mapsto \Omega_k(\lambda_l)$, $k = 1,...,8$, is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous so it is exponentially convex.

For this family of functions, $\mathfrak{B}(l,r;\Omega_k,\Upsilon_1)$, $k = 1,...,8$, from (24) is equal to

$$\mathfrak{B}(l,r;\Omega_k,\Upsilon_1) = \begin{cases} \left(\frac{\Omega_k(\lambda_l)}{\Omega_k(\lambda_r)}\right)^{\frac{1}{l-r}}, & l \neq r, \\ \exp\left(\frac{\Omega_k(id\cdot\lambda_l)}{\Omega_k(\lambda_l)} - \frac{2}{l}\right), & l = r \neq 0, \\ \exp\left(\frac{\Omega_k(id\cdot\lambda_0))}{3\Omega_k(\lambda_0))}\right), & l = r = 0, \end{cases}$$

where id is the identity function. Also, by Corollary 3.10, it is monotonic in the parameters l and r. Applying Theorem 4.2 for $\phi = \lambda_l$ and $\psi = \lambda_r$ we see that there exists ξ_k, $k = 1,...,8$, such that

$$e^{(l-r)\xi_k} = \frac{\Omega_k(\lambda_l)}{\Omega_k(\lambda_r)}.$$

Therefore

$$M^k_{l,r}(\Upsilon_1) = \log \mathfrak{B}(l,r;\Omega_k,\Upsilon_1)$$

is a Cauchy mean.

EXAMPLE 5.2. Consider the family of functions

$$\Upsilon_2 = \{\omega_l : [0,\infty) \to \mathbb{R} : l > 0\}$$

defined by

$$\omega_l(x) = \begin{cases} \frac{l^{-x}-1}{\log^2 l}, & l \neq 1, \\ \frac{x^2}{2}, & l = 1. \end{cases}$$

Notice that $\omega_l(0) = 0$ and the mapping $l \mapsto \frac{d^2\omega_l}{dx^2}(x) = l^{-x} = \frac{1}{\Gamma(x)}\int_0^\infty e^{-lt}t^{x-1}\,dt$ is the Laplace transform of a non-negative function (see [8]). For $\xi_i \in \mathbb{R}$ and $l_i > 0$ we have

$$\sum_{i,j=1}^{n} \xi_i\xi_j \left(\frac{l_i+l_j}{2}\right)^{-x} = \frac{1}{\Gamma(x)}\int_0^\infty \left(\sum_{i=1}^{n} \xi_i e^{\frac{-l_i}{2}t}\right)^2 t^{x-1}dt \geqslant 0,$$

so the mapping $l \mapsto \frac{d^2\omega_l}{dx^2}(x)$ is exponentially convex on $(0,\infty)$ (see [3]). By Corollary 3.9 and Remark 3.11 the mapping $l \mapsto \Omega_k(\omega_l)$, $k=1,\dots,8$, is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous, so it is exponentially convex.

For this family of functions, $\mathfrak{B}(l,r;\Omega_k,\Upsilon_2)$, $k=1,\dots,8$, from (24) is equal to

$$\mathfrak{B}(l,r;\Omega_k,\Upsilon_2) = \begin{cases} \left(\frac{\Omega_k(\omega_l)}{\Omega_k(\omega_r)}\right)^{\frac{1}{l-r}}, & l \neq r, \\ \exp\left(-\frac{\Omega_k(id\cdot\omega_l)}{l\Omega_k(\omega_l)} - \frac{2}{l\log l}\right), & l = r \neq 1, \\ \exp\left(-\frac{\Omega_k(id\cdot\omega_1))}{3\Omega_k(\omega_1))}\right), & l = r = 1, \end{cases}$$

where id is the identity function. Also, by Corollary 3.10, it is monotonic in the parameters l and r. Applying Theorem 4.2 for $\phi = \omega_l$ and $\psi = \omega_r$ we see that there exists ξ_k, $k=1,\dots,8$, such that

$$\left(\frac{l}{r}\right)^{-\xi_k} = \frac{\Omega_k(\omega_l)}{\Omega_k(\omega_r)}.$$

Therefore

$$M^k_{l,r}(\Upsilon_2) = -L(l,r)\log\mathfrak{B}(l,r;\Omega_k,\Upsilon_2)$$

is a Cauchy mean, where $L(l,r)$ is the logarithmic mean defined by

$$L(l,r) = \begin{cases} \frac{l-r}{\log l - \log r}, & l \neq r, \\ l, & l = r. \end{cases}$$

EXAMPLE 5.3. Consider the family of functions

$$\Upsilon_3 = \{\mu_l : [0,\infty) \to \mathbb{R} : l > 0\}$$

defined by

$$\mu_l(x) = \frac{e^{-x\sqrt{l}} - 1}{l}.$$

Notice that $\mu_l(0) = 0$ and the mapping $l \mapsto \frac{d^2\mu_l}{dx^2}(x) = e^{-x\sqrt{l}} = \frac{x}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-lt}e^{-x^2/4t}}{t\sqrt{t}}\,dt$ is also the Laplace transform of a non-negative function (see [8]). Analogously as in Example 5.2, we can show that the mapping $l \mapsto \frac{d^2\mu_l}{dx^2}(x)$ is exponentially convex in the Jensen sense and, by Corollary 3.9 and Remark 3.11, the mapping $l \mapsto \Omega_k(\mu_l)$, $k=1,\dots,8$, is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous so it is exponentially convex.

For this family of functions, $\mathfrak{B}(l,r;\Omega_k,\Upsilon_3)$, $k=1,\dots,8$, from (24) is equal to

$$\mathfrak{B}(l,r;\Omega_k,\Upsilon_3) = \begin{cases} \left(\frac{\Omega_k(\mu_l)}{\Omega_k(\mu_r)}\right)^{\frac{1}{l-r}}, & l \neq r, \\ \exp\left(-\frac{\Omega_k(id\cdot\mu_l)}{2\sqrt{l}\Omega_k(\mu_l)} - \frac{1}{l}\right), & l = r, \end{cases}$$

where id is the identity function. Also, by Corollary 3.10, it is monotonic in the parameters l and r. Applying Theorem 4.2 for $\phi = \mu_l$ and $\psi = \mu_r$ we see that there exists ξ_k, $k = 1, \ldots, 8$, such that

$$e^{-\xi_k(\sqrt{l}-\sqrt{r})} = \frac{\Omega_k(\mu_l)}{\Omega_k(\mu_r)}.$$

Therefore

$$M^k_{l,r}(\Upsilon_3) = -(\sqrt{l}+\sqrt{r})\log \mathfrak{B}(l,r;\Omega_k,\Upsilon_3)$$

is a Cauchy mean.

Acknowledgements. The reseacrch of the 1st author has been fully supported by H.E.C. Pakistan under the start up research project under IPFP. The research of 2nd and 3rd author has been fully supported-supported in part by Croatian Science Foundation under the project 5435.

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(Received October 27, 2014)

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