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WEIGHTED KERNEL OPERATORS IN $L^{p(x)}(\mathbb{R}_+)$ SPACES

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Abstract. Necessary and sufficient conditions on a weight v governing the boundedness/ compactness of the weighted kernel operator $K_v f(x) = v(x) \int_0^x k(x,t)f(t)dt$ from the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}_+)$ into another one $L^{q(\cdot)}(\mathbb{R}_+)$ is established under the local log–Hölder continuity condition and the decay condition at infinity on exponents. The distance between K_v and the class of compact integral operators acting from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(\mathbb{R}_+)$ (measure of non–compactness) is also estimated from above and below.

1. Introduction

The paper is devoted to the criteria for the kernel operator

$$K_v f(x) = v(x) \int_0^x k(x,t)f(t)dt, \ \ x > 0,$$

to be bounded/compact in variable exponent Lebesgue spaces when exponents of spaces satisfy the local log– Hölder continuity condition and decay condition at infinity. This operator involves, for example, one–sided potentials such as the weighted Riemann-Liouville transform with variable parameter. In the case when the operator K_v is not compact, we establish two–sided estimates of the measure of non–compactness for this operator in terms of the weight v and kernel k. The paper can be considered as a continuation of the research carried out in the paper [10], where the same problems were studied under the local log–Hölder continuity condition on exponents provided that they are constants outside some large interval.

The space $L^{p(\cdot)}$ is a special case of the Musielak-Orlicz space (see [18], [19]). Historically, the first systematic study of modular spaces is due to H. Nakano [20].

Variable exponent Lebesgue and Sobolev spaces arise e.g., in the study of mathematical problems related to applications to mechanics of the continuum medium (see [24], [5]). The list of those references, where mapping properties of operators of Harmonic Analysis in $L^{p(x)}$ spaces were studied is quite long. For those properties we refer e.g., to the monographs [24], [5], the survey paper [9] and references therein.

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The main statements of this paper generalize also appropriate results of [15], where the similar problems were studied for K_ν in the classical Lebesgue spaces (Lebesgue spaces with constant exponents).

The paper consists of five sections. Section 2 gives well–known results about $L^{p(\cdot)}$ spaces. Section 3 is devoted to the boundedness criteria for the operator K_ν, while Section 4 is devoted to the compactness problem in $L^{p(\cdot)}$ spaces. In Section 5 we derive two–sided estimates of the measure of non-compactness for K_ν acting in variable exponent Lebesgue spaces.

Throughout the paper constants (often different constants in the same series of inequalities) will mainly be denoted by c or C; under the symbol $p'(x)$ we mean the function $\frac{p(x)}{p(x)-1}$, $1 < p(x) < \infty$. The symbol χ_E means the characteristic function of a set E, in particular, $\chi_{(a,b)}$ is the characteristic function of an interval (a,b).

2. Preliminaries

Let E be a measurable set in $\mathbb{R}$ with positive measure. We denote:

$$p_-(E) := \inf_E p, \qquad p_+(E) := \sup_E p$$

for a measurable function p on E. By $\mathscr{P}(E)$ we denote the class of measurable function p for which $1 < p_-(E) \leqslant p_+(E) < \infty$. We say that a measurable function f on E belongs to $L^{p(\cdot)}(E)$ (or to $L^{p(x)}(E)$) if

$$S_{p(\cdot)}(f) = \int_E |f(x)|^{p(x)} dx < \infty.$$

It is a Banach space with respect to the norm (see e.g., [9], [12], [26], [27])

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\{\lambda > 0 : S_{p(\cdot)}(f/\lambda) \leqslant 1\}.$$

In the sequel we will denote by $\mathbb{Z}$ and $\mathbb{Z}_-$ the set of all integers and the set of non-positive integers respectively.

To prove the main results we need some known statements:

PROPOSITION A. ([12], [26], [27]) *Let E be a measurable subset of $\mathbb{R}$. Suppose that $p \in \mathscr{P}(E)$. Then*

(i) $$\|f\|^{p_+(E)}_{L^{p(\cdot)}(E)} \leqslant S_{p(\cdot)}(f\chi_E) \leqslant \|f\|^{p_-(E)}_{L^{p(\cdot)}(E)}, \ \ \|f\|_{L^{p(\cdot)}(E)} \leqslant 1;$$

$$\|f\|^{p_-(E)}_{L^{p(\cdot)}(E)} \leqslant S_{p(\cdot)}(f\chi_E) \leqslant \|f\|^{p_+(E)}_{L^{p(\cdot)}(E)}, \ \ \|f\|_{L^{p(\cdot)}(E)} \geqslant 1;$$

(ii) *Hölder's inequality*

$$\left|\int_E f(x)g(x)dx\right| \leqslant \left(\frac{1}{p_-(E)} + \frac{1}{(p_+(E))'}\right) \|f\|_{L^{p(\cdot)}(E)} \|g\|_{L^{p'(\cdot)}(E)}$$

holds, where $f \in L^{p(\cdot)}(E)$, $g \in L^{p'(\cdot)}(E)$.

PROPOSITION B. ([12], [26], [27]) *Let* $1 \leqslant r(x) \leqslant p(x)$, $x \in E$. *Then the following inequality*

$$\|f\|_{L^{r(\cdot)}(E)} \leqslant (|E|+1)\|f\|_{L^{p(\cdot)}(E)}$$

holds.

DEFINITION 2.1. We say that p satisfies the weak Lipschitz (log-Hölder continuity) condition on E $(p \in \mathscr{P}^{log}(E))$, if there is a positive constant A such that for all x and y in E with $0 < |x-y| < 1/2$, the inequality

$$|p(x)-p(y)| \leqslant A/(-\ln|x-y|)$$

holds.

DEFINITION 2.2. Let E be an unbounded set. We say that p satisfies the decay condition on E at infinity $(p \in \mathscr{P}_\infty(E))$, if there are constants $A_\infty \geqslant 0$ and $p_\infty \in (1,\infty)$ such that for all x in E the inequality

$$|p(x)-p_\infty| \leqslant \frac{A_\infty}{\ln(e+|x|)}$$

holds.

In the sequel we will use the notation: $\mathscr{P}^{log}(E) \cap \mathscr{P}_\infty(E) =: \mathscr{P}^{log}_\infty(E)$.

It is known (see [4]) that if $p \in \mathscr{P}^{log}$, then the Hardy–Littlewood maximal operator M is bounded in $L^{p(x)}$ space defined on a bounded domain, while the condition $p \in \mathscr{P}^{log}_\infty$ implies the boundedness of M in $L^{p(x)}$ space on unbounded domain. The latter result was derived in [3].

LEMMA A. ([4]) *Let* I_0 *be an interval in* $\mathbb{R}$. *Then* $p \in \mathscr{P}^{log}(I_0)$ *if and only if there exists a positive constant* C *such that*

$$|J|^{p_-(J)-p_+(J)} \leqslant C$$

for all intervals $J \subseteq I_0$ *with* $|J| > 0$.

REMARK 2.1. If $p \in \mathscr{P}^{log}_\infty(\mathbb{R}_+)$, then following conditions are satisfied at 0 and ∞:

$$|p(x)-p(0)| \leqslant \frac{A_0}{|ln|x||} \qquad |x| \leqslant 1, \tag{2.1}$$

$$|p(x)-p_\infty| \leqslant \frac{A_\infty}{ln|x|} \qquad |x| > 1. \tag{2.2}$$

REMARK 2.2. Let $I=\mathbb{R}_+$. It is known that $\|\chi_{(0,r)}\|_{L^{p(\cdot)}(I)} \approx r^{1/p(0)}$ as $r\to 0$ if $p(x)$ satisfies the local log-Hölder continuity condition, and $\|\chi_{(0,r)}\|_{L^{p(\cdot)}(I)} \approx r^{1/p_\infty}$ as $r\to\infty$, if $p\in \mathscr{P}_\infty^{\log}(I)$.

LEMMA B. *Let D be a constant greater than 1 and $p\in \mathscr{P}_\infty^{log}(\mathbb{R}_+)$. Then*

$$\frac{1}{c_0} r^{\frac{1}{p(0)}} \leqslant \|\chi_{(r,Dr)}\|_{L^{p(\cdot)}} \leqslant c_0 r^{\frac{1}{p(0)}} \qquad for \qquad 0<r\leqslant 1 \tag{2.3}$$

and

$$\frac{1}{c_\infty} r^{\frac{1}{p_\infty}} \leqslant \|\chi_{(r,Dr)}\|_{L^{p(\cdot)}} \leqslant c_\infty r^{\frac{1}{p_\infty}} \qquad for \qquad r\geqslant 1 \tag{2.4}$$

holds, where $c_0\geqslant 1$ and $c_\infty \geqslant 1$ depend on D, but do not depend on r.

Proof. We follow the proof of Lemma 4.6 in [25]. We prove only (2.4). The proof for (2.3) is similar. Recall that $\int_{\mathbb{R}_+} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leqslant 1 \Leftrightarrow \|f\|_{L^{p(\cdot)}} \leqslant \lambda$ for $\lambda>0$; $\int_{\mathbb{R}_+} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \geqslant 1 \Leftrightarrow \|f\|_{L^{p(\cdot)}} \geqslant \lambda$ for $\lambda>0$. Therefore the right–hand side inequality of (2.4) holds if and only if

$$\int_r^{Dr} \frac{dx}{(c_\infty r^{\frac{1}{p_\infty}})^{p(x)}} \leqslant 1\text{dot is removed from here} \tag{2.5}$$

holds.

The left–hand side of (2.5) is estimated as follows

$$\begin{aligned}\int_r^{Dr} \frac{dx}{(c_\infty r^{\frac{1}{p_\infty}})^{p(x)}} &\leqslant \frac{1}{c_\infty^{p_-}} \int_r^{Dr} \frac{dx}{(\frac{x}{D})^{\frac{p(x)}{p_\infty}}} \\ &\leqslant \frac{D^{\frac{p_+}{p_\infty}}}{c_\infty^{p_-}} \int_r^{Dr} \frac{dx}{x^{\frac{p(x)}{p_\infty}}}.\end{aligned}$$

By (2.2) we have $e^{\frac{-A_\infty}{p_\infty}} x \leqslant x^{\frac{p(x)}{p_\infty}} \leqslant e^{\frac{A_\infty}{p_\infty}} x$ for $x\geqslant 1$.
Therefore,

$$\int_r^{Dr} \frac{dx}{(c_\infty r^{\frac{1}{p_\infty}})^{p(x)}} \leqslant \frac{e^{\frac{A_\infty}{p_\infty}} D^{\frac{p_+}{p_\infty}}}{c_\infty^{p_-}} \int_r^{Dr} \frac{dx}{x} = \frac{e^{\frac{A_\infty}{p_\infty}} D^{\frac{p_+}{p_\infty}}}{c_\infty^{p_-}} lnD.$$

Hence, by choosing $c_\infty^{p_-} = e^{\frac{A_\infty}{p_\infty}} D^{\frac{p_+}{p_\infty}} lnD$ we prove the right–hand side of inequality (2.4). The proof for the left–hand side of (2.4) is similar.

In the sequel the following notation will be used:

$$E_n := [2^n, 2^{n+1}); \quad I_n := [2^{n-1}, 2^{n+1}).$$

For the next statements we refer to [11] and [1].

PROPOSITION C. *Let p and q be measurable functions on $I := (a,b)$ $(-\infty < a < b \leqslant +\infty)$ satisfying the condition $1 < p_-(I) \leqslant p(x) \leqslant q(x) < q_+(I) < \infty$, $x \in I$. Let $p, q \in \mathscr{P}_\infty^{log}(I)$. Then there is a positive constant c depending only on p and q such that for all $f \in L^{p(\cdot)}(I)$, $g \in L^{q'(\cdot)}(I)$ and all sequences of intervals $S_k := [x_k, x_{k+1})$, where $[x_k, x_{k+1})$ are disjoint intervals satisfying the condition $\cup_k [x_k, x_{k+1}) = I$, the inequality*

$$\sum_k \|f\chi_{S_k}\|_{L^{p(\cdot)}(I)} \|g\chi_{S_k}\|_{L^{q'(\cdot)}(I)} \leqslant c\|f\|_{L^{p(\cdot)}(I)} \|g\|_{L^{q'(\cdot)}(I)}$$

holds.

In the next statement the intervals S_k are replaced by $I_k^{a,b}$, where

$$I_k^{a,b} := \Big[a + \frac{b-a}{2^{k+1}}, a + \frac{b-a}{2^{k-1}}\Big), \ k \in \mathbb{N},$$

for $b < \infty$;

$$I_k^{a,\infty} := \big[a + 2^{k-1}, a + 2^{k+1}\big), \quad k \in \mathbb{Z}.$$

PROPOSITION D. *Let p and q be measurable functions on $I := (a,b)$ $(-\infty < a < b \leqslant +\infty)$ satisfying the condition $1 < p_-(I) \leqslant p(x) \leqslant q(x) < q_+(I) < \infty$, $x \in I$. Let $p, q \in \mathscr{P}_\infty^{log}(I)$. Then there is a positive constant c depending only on p and q such that for all $f \in L^{p(\cdot)}(I)$, $g \in L^{q'(\cdot)}(I)$ and all intervals $I_k^{a,b}$, the inequality*

$$\sum_k \|f\chi_{I_k^{a,b}}\|_{L^{p(\cdot)}(I)} \|g\chi_{I_k^{a,b}}\|_{L^{q'(\cdot)}(I)} \leqslant c\|f\|_{L^{p(\cdot)}(I)} \|g\|_{L^{q'(\cdot)}(I)}$$

holds.

Proof. The proof in the case of $I = (0,1)$ can be found in [1]. For simplicity let us assume that $I = \mathbb{R}_+$. In this case $a = 0$, $b = \infty$ and consequently, $I_k^{0,\infty} = I_k$. Now the proof follows in same manner as in [11] Proposition 3.4, since the map $g := I \to (-1/2, 1/2)$ defined by $g(x) = \frac{\arctan x}{\pi}$ keeps the property $\sum_k \chi_{g(I_k)}(x) \leqslant 2$. Details are omitted.

Let v and w be a.e. positive measurable function on $\mathbb{R}_+$ and let

$$(H_{v,w}f)(x) = v(x) \int_0^x f(t)w(t)dt, \qquad x \in \mathbb{R}_+.$$

THEOREM A. *Let* $I = \mathbb{R}_+$ *and* $1 < p_-(I) \leqslant p(x) \leqslant q(x) \leqslant q_+(I) < \infty$. *Suppose that* $p, q \in \mathscr{P}_\infty^{log}(I)$. *Then* $H_{v,w}$ *is bounded from* $L^{p(\cdot)}(I)$ *to* $L^{q(\cdot)}(I)$ *if and only if*

$$D_\infty := \sup_{t>0} D_\infty(t) = \sup_{t>0} \|\chi_{(t,\infty)}(\cdot)v(\cdot)\|_{L^{q(\cdot)}(I)} \|\chi_{(0,t)}(\cdot)w(\cdot)\|_{L^{p'(\cdot)}(I)} < \infty.$$

Proof. Sufficiency. Let $f \geqslant 0$, and $\int\limits_0^\infty f(t)w(t)dt = \infty$. We construct a sequence $\{x_k\}$ so that

$$\int\limits_0^{x_k} fw = \int\limits_{x_k}^{x_{k+1}} fw = 2^k.$$

It is easy to check that $[0,\infty) = \cup_k [x_k, x_{k+1})$. Let g be a function satisfying the condition, $\|g\|_{L^{q'(\cdot)}(\mathbb{R}_+)} \leqslant 1$. By applying Hölder's inequality for variable exponent Lebesgue spaces and Proposition D we have that

$$\begin{aligned}
\int\limits_0^\infty (H_{v,w}f)g &\leqslant \sum_k \Bigg(\int\limits_{x_k}^{x_{k+1}} gv \Bigg) \Bigg(\int\limits_0^{x_{k+1}} fw \Bigg) \\
&= 4 \sum_k \Bigg(\int\limits_{x_k}^{x_{k+1}} gv \Bigg) \Bigg(\int\limits_{x_{k-1}}^{x_k} fw \Bigg) \\
&\leqslant 4 \sum_k \|\chi_{(x_k,x_{k+1})}(\cdot)g(\cdot)\|_{L^{q'(\cdot)}(I)} \|\chi_{(x_k,x_{k+1})}(\cdot)v(\cdot)\|_{L^{q(\cdot)}(I)} \\
&\quad \times \|\chi_{(x_{k-1},x_k)}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)} \|\chi_{(x_{k-1},x_k)}(\cdot)w(\cdot)\|_{L^{p'(\cdot)}(I)} \\
&\leqslant 4D_\infty \sum_k \|\chi_{(x_k,x_{k+1})}(\cdot)g(\cdot)\|_{L^{q'(\cdot)}(I)} \|\chi_{(x_{k-1},x_k)}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)} \\
&\leqslant 4D_\infty \|f(\cdot)\|_{L^{p(\cdot)}(I)} \|g(\cdot)\|_{L^{q'(\cdot)}(I)}.
\end{aligned}$$

Taking now the supremum with respect to g gives sufficiency.
Necessity follows by the standard way taking the test function f supported in $(0,t)$ with $\|f\|_{L^{p(\cdot)}} \leqslant 1$.

We refer for the two–weight criteria for the Hardy transform in the classical Lebesgue spaces e.g. to [8], [14], [17], [13].

REMARK 2.3. If w is constant and $p \in \mathscr{P}_\infty^{log}(I)$, then $D_\infty < \infty$ is equivalent to the condition:

$$\bar{D}_\infty := \sup_{n\in\mathbb{Z}} \|\chi_{E_n}(\cdot)v(\cdot)\|_{L^{q(\cdot)}(I)} \|\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)} < \infty.$$

The norm $\|\chi_{(0,2^n)}\|_{L^{p'(\cdot)}(I)}$ can be replaced by $\|\chi_{E_n}(\cdot)\|_{L^{p'(\cdot)}(I)}$. This follows from Lemma B and Remark 2.2. The fact that $D_\infty < \infty$ implies $\bar{D}_\infty < \infty$ is obvious.

Conversely, let $\bar{D}_\infty < \infty$. Let us now take $t \in I$. Then $t \in [2^m, 2^{m+1})$ for some $m \in \mathbb{Z}$. Consequently,

$$D_\infty(t) \leqslant \sum_{n=m}^{\infty} \|\chi_{E_n}(x)v(x)\|_{L^{q(x)}(I)} \|\chi_{(0,2^{m+1})(\cdot)}\|_{L^{p'(\cdot)}(I)}$$
$$\leqslant \bar{D}_\infty \Big(\sum_{n=m}^{\infty} \|\chi_{(0,2^n)}(\cdot)\|^{-1}_{L^{p'(\cdot)}(I)} \Big) \|\chi_{(0,2^{m+1})}(\cdot)\|_{L^{p'(\cdot)}(I)}.$$

Hence,

$$D_\infty(t) \leqslant \begin{cases} \bar{D}_\infty [(\sum\limits_{n=m}^{0} 2^{-n/p'(0)}) 2^{m/p'(0)} + (\sum\limits_{n=0}^{\infty} 2^{-n/(p_\infty)'}) 2^{m/(p_\infty)'}] \leqslant c_1(p)\bar{D}_\infty & \text{if } m < 0, \\ \bar{D}_\infty (\sum\limits_{n=m}^{\infty} 2^{-n/(p_\infty)'}) 2^{m/(p_\infty)'} \leqslant c_2(p)\bar{D}_\infty & \text{if } m \geqslant 0. \end{cases}$$

where $c_1(p)$ and $c_2(p)$ are constants depending only on p. Finally, $D_\infty < c\bar{D}_\infty$.

THEOREM B. ([6]) *Let $p(x)$ and $q(x)$ be measurable functions on an interval $I \subseteq R_+$. Suppose that $1 < p_-(I) \leqslant p_+(I) < \infty$ and $1 < q_-(I) \leqslant q_+(I) < \infty$. If*

$$\Big\| \|k(x,y)\|_{L^{p'(y)}(I)} \Big\|_{L^{q(x)}(I)} < \infty,$$

where k is a non-negative kernel, then the operator

$$Kf(x) = \int_I k(x,y) f(y) dy$$

is compact from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.

DEFINITION 2.3. Let $I := (0,a)$, $0 < a \leqslant \infty$. We say that a kernel $k : \{(x,y) : 0 < y < x < a\} \to (0,\infty)$ belongs to the class $V(I)$ $(k \in V(I))$ if there exists a constant c_1 such that for all x,y,t with $0 < y < t < x < a$, the inequality

$$k(x,y) \leqslant c_1 k(x,t)$$

holds.

DEFINITION 2.4. Let r be a measurable function on $I = (0,a)$, $0 < a \leqslant \infty$ with values in $(1,+\infty)$. We say that a kernel k belongs to the class $V_{r(\cdot)}(I)$ if there exists a positive constant c_2 such that for a.e. $x \in (0,a)$, the inequality

$$\|\chi_{(\frac{x}{2},x)}(\cdot) k(x,\cdot)\|_{L^{r(\cdot)}(I)} \leqslant c_2 \|\chi_{(\frac{x}{2},x)}\|_{L^{r(\cdot)}(I)} k\Big(x, \frac{x}{2}\Big)$$

is fulfilled.

These conditions on a kernel k were introduced by the first named author in the paper [15] for the constant p.

REMARK 2.4. Using Lemmas A and B we have $\|\chi_{(\frac{x}{2},x)}\|_{L^{r(\cdot)}} \approx x^{1/r(0)} \approx x^{1/r(x)}$ near zero. Similarly by Lemma B we see that $\|\chi_{(\frac{x}{2},x)}\|_{L^{r(\cdot)}} \approx x^{1/r_\infty}$ near infinity.

EXAMPLE 2.1. *Let* $I := \mathbb{R}_+$. *Let* α *be a measurable function on* I *satisfying the condition* $0 < \alpha_-(I) \leqslant \alpha_+(I) \leqslant 1$. *Let* $r \in \mathscr{P}_\infty^{log}(I)$. *Suppose that* r *be non-increasing on* (a,∞) *for some large* $a > 0$. *Then* $k(x,t) = (x-t)^{\alpha(x)-1} \in V(I) \cap V_{r(\cdot)}(I)$ *when* $(\alpha r')_+(I) > 1$.

Indeed, first it is easy to check that $k \in V(I)$. Further to prove that $k \in V_{r(\cdot)}(I)$ we need to show

$$I(x) := \|(x-\cdot)^{\alpha(x)-1}\chi_{(x/2,x)}(\cdot)\|_{L^{r(\cdot)}} \leqslant c\|\chi_{(x/2,x)}(\cdot)\|_{L^{r(\cdot)}} x^{\alpha(x)-1}, \tag{2.6}$$

where the constant c does not depend on x. Since $r \in \mathscr{P}_\infty^{\log}(I)$, by Lemma A for $x-t<1$, we have

$$(x-t)^{r(t)} \leqslant c_1(x-t)^{r(x)} \leqslant c_2(x-t)^{r(t)} \tag{2.7}$$

where c_1 and c_2 does not depend on x.
Since r is non-increasing, for $x-t \geqslant 1$, we have

$$(x-t)^{r(t)} \geqslant (x-t)^{r(x)}. \tag{2.8}$$

Consequently,

$$\begin{aligned} S(x) := \int_{x/2}^{x} (x-t)^{(\alpha(x)-1)r(t)}dt &= \int_{\{t:t\in(x/2,x),\,(x-t)<1\}} (\cdots) + \int_{\{t:t\in(x/2,x),\,(x-t)\geqslant 1\}} (\cdots) \\ &:= S_1(x) + S_2(x). \end{aligned}$$

First we estimate $S_1(x)$. Taking into account (2.7) we have the following pointwise estimate

$$\begin{aligned} S_1(x) &\leqslant \int_{\{t:t\in(x/2,x),\,(x-t)<1\}} (x-t)^{(\alpha(x)-1)r(x)}dt \\ &\leqslant \int_{x/2}^{x} (x-t)^{(\alpha(x)-1)r(x)}dt = cx^{(\alpha(x)-1)r(x)+1} \end{aligned}$$

By using (2.8) for $S_2(x)$, we have

$$\begin{aligned} S_2(x) &\leqslant \int_{\{t:t\in(x/2,x),\,(x-t)\geqslant 1\}} (x-t)^{(\alpha(x)-1)r(x)}dt \\ &\leqslant \int_{x/2}^{x} (x-t)^{(\alpha(x)-1)r(x)}dt = cx^{(\alpha(x)-1)r(x)+1}. \end{aligned}$$

Since $I(x) \geqslant d$ for some positive constant d, by Proposition A and Lemma B we have

$$\begin{aligned}
\frac{I(x)}{d} &\leqslant cS(x)^{1/r_-([x/2,x])} = cS(x)^{1/r(x)} \\
&= cx^{\alpha(x)-1+\frac{1}{r(x)}} \leqslant cx^{\alpha(x)-1+\frac{1}{r_\infty}} \\
&= c\|\chi_{(x/2,x)}(\cdot)\|_{L^{r(\cdot)}(I)} k(x/2,x).
\end{aligned}$$

Hence, we have estimate (2.6).

For other examples of kernels in the classical and variable exponent Lebesgue spaces we refer to the papers [15], [10].

3. Boundedness in $L^{p(x)}$ spaces

In this section we derive boundedness criteria for the operator K_v from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(\mathbb{R}_+)$.

Now we formulate and prove the main results of this section.

THEOREM 3.1. *Let $I := \mathbb{R}_+$ and let $1 < p_-(I) \leqslant p(x) \leqslant q(x) \leqslant q_+(I) < \infty$. Suppose that $k \in V(I) \cap V_{p'(\cdot)}(I)$. Further, assume that $p, q \in \mathscr{P}^{log}_\infty(I)$. Then the following statements are equivalent*

(i) $\quad \|K_v f\|_{L^{q(\cdot)}(I)} \leqslant c\|f\|_{L^{p(\cdot)}(I)}, \quad f \in L^{p(\cdot)}(I),$

(ii) $\quad \bar{C}_\infty := \sup\limits_{n\in\mathbb{Z}} \bar{C}_\infty(n) := \sup\limits_{n\in\mathbb{Z}} \left\|\chi_{E_n}(x)v(x)k(x,\tfrac{x}{2})\right\|_{L^{q(x)}(I)} \|\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)} < \infty,$

(iii) $\quad C_\infty := \sup\limits_{t>0} C_\infty(t) := \sup\limits_{t>0} \left\|\chi_{(t,\infty)}(x)v(x)k(x,\tfrac{x}{2})\right\|_{L^{q(x)}(I)} \|\chi_{(0,t)}(\cdot)\|_{L^{p'(\cdot)}(I)} < \infty.$

Moreover, $\|K_v\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} \approx C_\infty \approx \bar{C}_\infty$.

Proof. (iii)$\Rightarrow$(i): Suppose that $f \geqslant 0$.

$$\begin{aligned}
(K_v f)(x) &= v(x)\int_0^{x/2} k(x,t)f(t)dt + v(x)\int_{x/2}^{x} k(x,t)f(t)dt \\
&=: (K_v^{(1)} f)(x) + (K_v^{(2)} f)(x).
\end{aligned}$$

Hence,

$$\|(K_v f)(x)\|_{L^{q(x)}(I)} \leqslant \|(K_v^{(1)} f)(x)\|_{L^{q(x)}(I)} + \|(K_v^{(2)} f)(x)\|_{L^{q(x)}(I)} =: S^{(1)} + S^{(2)}.$$

It is easy to see that if $0 < t < x/2$, then $k(x,t) \leqslant c_1 k(x, \frac{x}{2})$. Hence, taking Theorem A into account we have that

$$S^{(1)} \leqslant c \left\| v(x)k(x, \frac{x}{2}) \left(\int_0^x f(t)dt \right) \right\|_{L^{q(x)}(I)} \leqslant cC_\infty \|f\|_{L^{p(\cdot)}(I)}.$$

Suppose now that $g \geqslant 0$, $\|g\|_{L^{q'(\cdot)}(I)} \leqslant 1$. Applying Hölder's inequality twice with respect to the pairs of exponents $(p(\cdot), p'(\cdot))$, $(q(\cdot), q'(\cdot))$ (see (ii) of Proposition A), Lemmas A, B, Proposition D and the condition $k \in V_{p'(\cdot)}(I)$ we find that

$$\begin{aligned}
&\int_0^\infty v(x) \left(\int_{x/2}^x k(x,t) f(t)dt \right) g(x)dx \\
&\leqslant c \sum_{n\in\mathbb{Z}} \int_{E_n} v(x) \|\chi_{(x/2,x)}(\cdot) f(\cdot)\|_{L^{p(\cdot)}(I)} \|\chi_{(x/2,x)}(\cdot) k(x,\cdot)\|_{L^{p'(\cdot)}(I)} g(x)dx \\
&\leqslant c \sum_{n\in\mathbb{Z}} \|\chi_{I_n}(\cdot) f(\cdot)\|_{L^{p(\cdot)}(I)} \int_{E_n} v(x) \|\chi_{(x/2,x)}(\cdot)\|_{L^{p'(\cdot)}(I)} k(x, \frac{x}{2}) g(x)dx \\
&\leqslant c \sum_{n\in\mathbb{Z}} \|\chi_{I_n}(\cdot) f(\cdot)\|_{L^{p(\cdot)}(I)} \|\chi_{I_n}(\cdot)\|_{L^{p'(\cdot)}(I)} \int_{E_n} v(x) k(x, \frac{x}{2}) g(x)dx \\
&\leqslant c \sum_{n\in\mathbb{Z}} \|\chi_{I_n}(\cdot) f(\cdot)\|_{L^{p(\cdot)}(I)} \|\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)} \left\| \chi_{E_n}(x) v(x) k(x, \frac{x}{2}) \right\|_{L^{q(x)}(I)} \\
&\qquad \times \|\chi_{E_n}(\cdot) g(\cdot)\|_{L^{q'(\cdot)}(I)} \\
&\leqslant cC_\infty \|f\|_{L^{p(\cdot)}(I)} \|g\|_{L^{q'(\cdot)}(I)} \leqslant cC_\infty \|f\|_{L^{p(\cdot)}(I)}.
\end{aligned}$$

Taking the supremum with respect to g and summarizing the estimates for $S^{(1)}$ and $S^{(2)}$ we have the desired result.

(i)$\Rightarrow$(ii): For necessity take the test function $f_n(x) = \chi_{(0,2^n)}(x)$. Then by Remark 2.2 we see that

$$\begin{aligned}
\|f_n\|_{L^{p(\cdot)}(I)} &\approx 2^{n/p(0)} \qquad n < 0, \\
\|f_n\|_{L^{p(\cdot)}(I)} &\approx 2^{n/p_\infty} \qquad n \geqslant 0.
\end{aligned}$$

Hence,

$$\|K_v f_n\|_{L^{q(\cdot)}(I)} \geqslant \|K_v^{(2)} f_n\|_{L^{q(\cdot)}(I)} \geqslant c2^n \|\chi_{E_{n-1}}(x) v(x) k(x, \frac{x}{2})\|_{L^{q(\cdot)}(I)}$$

Using the boundedness we have

$$\|\chi_{E_{n-1}}(x) v(x) k(x, \frac{x}{2})\|_{L^{q(\cdot)}(I)} 2^{n/p'(0)} < \infty \qquad \text{for} \qquad n < 0 \tag{3.1}$$

$$\|\chi_{E_{n-1}}(x) v(x) k(x, \frac{x}{2})\|_{L^{q(\cdot)}(I)} 2^{n/p'_\infty} < \infty \qquad \text{for} \qquad n \geqslant 0. \tag{3.2}$$

Combining (3.1) and (3.2) we have the required conclusion. The implication (ii)$\Rightarrow$(iii) can be proved in similar manner as in Remark 2.3; therefore we omit details.

4. Compactness

In this section we derive criteria for the compactness of K_v from $L^{p(\cdot)}$ to $L^{q(\cdot)}$. For the compactness problems in variable exponent Lebesgue spaces we refer e.g., to [1], [6], [7], [10], [22], [23], (see also [16] and references cited therein).

THEOREM 4.1. *Let $I=\mathbb{R}_+$. Suppose that $1<p_-(I)\leqslant p(x)\leqslant q(x)\leqslant q_+(I)<\infty$. Suppose also that $k\in V(I)\cap V_{p'(\cdot)}(I)$. Further, assume that $p,q\in\mathscr{P}^{log}_\infty(I)$. Then the following statements are equivalent:*

(i) K_v *is compact from* $L^{p(\cdot)}(I)$ *to* $L^{q(\cdot)}(I)$;

(ii) $$\bar{C}_\infty<\infty \text{ and } \lim_{n\to-\infty}\bar{C}_\infty(n)=\lim_{n\to\infty}\bar{C}_\infty(n)=0,$$
where $\bar{C}_\infty$ and $\bar{C}_\infty(n)$ are defined in Theorem 3.1.

(iii) $$C_\infty<\infty \text{ and } \lim_{d\to 0^+}C_d=\lim_{b\to+\infty}C_b=0,$$
where C_∞ is defined in Theorem 3.1 and

$$C_d:=\sup_{0<t<d}C_d(t):=\sup_{0<t<d}\left\|\chi_{(t,\infty)}(x)v(x)k(x,\frac{x}{2})\right\|_{L^{q(x)}(I)}\|\chi_{(0,t)}(\cdot)\|_{L^{p'(\cdot)}(I)};$$

$$C_b:=\sup_{t\geqslant b}C_b(t):=\sup_{t\geqslant b}\left\|\chi_{(t,\infty)}(x)v(x)k(x,\frac{x}{2})\right\|_{L^{q(x)}(I)}\|\chi_{(0,t)}(\cdot)\|_{L^{p'(\cdot)}(I)}.$$

Proof. First we show that the implication (iii)$\Rightarrow$(i) holds. We represent $K_vf=\sum_{n=1}^4 K_v^{(n)}f$, where

$$\begin{aligned}
K_v^{(1)}f(x)&=\chi_{(0,d)}(x)(K_v(\chi_{(0,d)}f)(x),\\
K_v^{(2)}f(x)&=\chi_{[d,b)}(x)K_v(\chi_{(0,b)}f)(x),\\
K_v^{(3)}f(x)&=\chi_{[b,\infty)}(x)K_v(\chi_{(0,b/2]}f)(x),\\
K_v^{(4)}f(x)&=\chi_{[b,\infty)}K_v(\chi_{(b/2,\infty)}f)(x),
\end{aligned}$$

where $0<d<1<b<\infty$. Now observe that

$$K_v^{(2)}f(x)=\int_I k^{(2)}(x,y)f(y)dy,$$

where $k^{(2)}(x,y) = v(x)\chi_{[d,b)}(x)k(x,y)$ when $0 < y < x < \infty$ and $k^{(2)}(x,y) = 0$ if $0 < x \leqslant y < \infty$. Consequently, since $k \in V(I) \cap V_{p'(\cdot)}(I)$, we have for $K_v^{(2)}$,

$$
\begin{aligned}
&\left\| \chi_{[d,b]}(x)v(x) \left\| k^{(2)}(x,y) \right\|_{L^{p'(y)}(I)} \right\|_{L^{q(x)}(I)} \\
&= \left\| \chi_{[d,b]}(x)v(x) \left\| \chi_{(0,x)}(y)k(x,y) \right\|_{L^{p'(y)}(I)} \right\|_{L^{q(x)}(I)} \\
&\leqslant \left\| \chi_{[d,b]}(x)v(x) \left\| \chi_{(0,x/2)}(y)k(x,y) \right\|_{L^{p'(y)}(I)} \right\|_{L^{q(x)}(I)} \\
&+ \left\| \chi_{[d,b]}(x)v(x) \left\| \chi_{[x/2,x)}(y)k(x,y) \right\|_{L^{p'(y)}(I)} \right\|_{L^{q(x)}(I)}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \left\| \chi_{[d,b]}(x)v(x)k(x,\frac{x}{2}) \right\|_{L^{q(x)}(I)} \left\| \chi_{(0,b/2)}(y) \right\|_{L^{p'(y)}(I)} \\
&+ \left\| \chi_{[d,b]}(x)v(x)k(x,\frac{x}{2}) \right\|_{L^{q(x)}(I)} \left\| \chi_{(d/2,b)}(y) \right\|_{L^{p'(y)}(I)} \\
&\leqslant 2 \left\| \chi_{[d,b]}(x)v(x)k(x,\frac{x}{2}) \right\|_{L^{q(x)}(I)} \left\| \chi_{(0,b)}(y) \right\|_{L^{p'(y)}(I)} =: J.
\end{aligned}
$$

It is easy to see that $J < \infty$ because $C_\infty < \infty$. Hence, by Theorem B we conclude that $K_v^{(2)}$ is compact. Similarly we can show that $K_v^{(3)}$ is compact. Applying now Theorem 3.1 for the interval $(0,d)$ (see also [10]) we find that

$$
\|K_v^{(1)}\|_{L^{p(\cdot)}(I) \to L^{q(\cdot)}(I)} = \|K_v\|_{L^{p(\cdot)}([0,d)) \to L^{q(\cdot)}([0,d))} \leqslant c \sup_{0<t<d} C_d(t)
$$

as $d \to 0^+$, where the positive constant c depends only on p, q. Further following the proof of Theorem 3.1 we have

$$
\begin{aligned}
&\left\| \; K_v^{(4)} f(x) \right\|_{L^{p(x)}([b,\infty)) \to L^{q(x)}([b,\infty))} \\
&\leqslant c \sup_{t \geqslant b} \|\chi_{(t,\infty)}(x)v(x)k(x,\frac{x}{2})\|_{L^{q(\cdot)}} \|\chi_{(0,t)}(\cdot)\|_{L^{p'(\cdot)}} = c \sup_{t \geqslant b} C_b(t).
\end{aligned}
$$

Further,

$$
\begin{aligned}
\|K_v - K_v^{(2)} - K_v^{(3)}\|_{L^{p(\cdot)}(I) \to L^{q(\cdot)}(I)} &\leqslant \|K_v^{(1)}\|_{L^{p(\cdot)}(I) \to L^{q(\cdot)}(I)} + \|K_v^{(4)}\|_{L^{p(\cdot)}(I) \to L^{q(\cdot)}(I)} \\
&\leqslant c \left(\sup_{0<t<d} C_d(t) + \sup_{t \geqslant b} C_b(t) \right)
\end{aligned}
$$

where the positive constant c depends only on p, q and α. Passing d to 0^+ and b to $+\infty$ we have that K_v is compact.

(i)$\Rightarrow$(ii): Suppose that $f_n(x) = 2^{-n/p(0)}\chi_{I_n}(x)$, $n \in \mathbb{Z}_-$ and $f_n(x) = 2^{-n/p_\infty}\chi_{I_n}(x)$, $n > 0$. Let us denote $p_n = p(0)$ for $n < 0$ and $p_n = p_\infty$ for $n \geqslant 0$. Hence by the condition $k \in V(I)$ and Proposition A, Lemmas A, B we have that

$$\begin{aligned}\left|\int_0^\infty f_n(x)\varphi(x)dx\right| &\leqslant c_p\|f_n(\cdot)\|_{L^{p(\cdot)}(I)}\|\varphi(\cdot)\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)}\\ &\leqslant c2^{-n/p_n}\|\chi_{I_n}(\cdot)\|_{L^{p(\cdot)}}\|\varphi(\cdot)\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)}\\ &\leqslant c\|\varphi(\cdot)\chi_{I_n}(\cdot)\|_{L^{p'(\cdot)}(I)} \to 0\end{aligned}$$

for all $\varphi \in L^{p'(x)}(I)$ as $n \to \pm\infty$. Hence, f_n converges weakly to 0 as $n \to \pm\infty$. Further, it is obvious that

$$\|K_v f_n\|_{L^{q(\cdot)}(I)} \geqslant c2^{n/p'(0)}\left\|\chi_{E_n}(x)v(x)k(x,\frac{x}{2})\right\|_{L^{q(x)}(I)}.$$

for $n \leqslant -1$,

$$\|K_v f_n\|_{L^{q(\cdot)}(I)} \geqslant c2^{n/p'_\infty}\left\|\chi_{E_n}(x)v(x)k(x,\frac{x}{2})\right\|_{L^{q(x)}(I)}.$$

for $n > 1$.

Finally we conclude that $\lim\limits_{n\to\pm\infty}\bar{C}_\infty(n) = 0$ because a compact operator maps weakly convergent sequence into strongly convergent one. The implication (ii)$\Rightarrow$(iii) follows from estimates similar to those given in Remark 2.3; therefore we omit details.

5. Measure of Non–compactness

This section deals with two-sided estimates of the distance between the operator K_v and the class of compact linear operators from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.
Let X and Y be Banach spaces. Suppose that $\mathscr{K}(X,Y)$ (resp. $F_R(X,Y)$) denotes the class of compact linear operators (resp. finite rank operators) acting from X to Y. Let

$$\|T\|_{\mathscr{K}(X,Y)} := \mathrm{dist}\{T,\mathscr{K}(X,Y)\}; \quad \overline{\alpha}(T) := \mathrm{dist}\{T,F_R(X,Y)\},$$

where T is a bounded linear operator from X to Y, $\mathrm{dist}\{T,\mathscr{K}(X,Y)\}$ and $\mathrm{dist}\{T,F_R(X,Y)\}$ denote the distance from T to $K(X,Y)$ and to $F_R(X,Y)$ respectively.

THEOREM C. [16, p. 80] *Let $I := (0,a)$, where $0 < a \leqslant \infty$. Let $q \in \mathscr{P}^{log}_\infty(I)$. Assume that X is a Banach space. Suppose that $1 < q_-(I) \leqslant q_+(I) < \infty$. Then*

$$\|T\|_{\mathscr{K}\left(X,L^{q(\cdot)}(I)\right)} = \overline{\alpha}(T),$$

where T is a bounded linear operator from X to $L^{q(\cdot)}(I)$.

THEOREM 5.1. *Let $I := \mathbb{R}_+$. Suppose that $1 < p_-(I) \leqslant p(x) \leqslant q(x) \leqslant q_+(I) < \infty$. Let $p,q \in \mathscr{P}^{log}_\infty(I)$ and let $\bar{C}_\infty < \infty$ (see Theorem 3.1 for the definition of $\bar{C}_\infty$). Then there exist two positive constants b_1 and b_2 depending only on p, q and the constants c_1 and c_2 defined in Definitions 2.3 and 2.4 respectively such that*

$$b_1 J \leqslant \|K_v\|_{\mathscr{K}(L^{p(\cdot)}(I),L^{q(\cdot)}(I))} \leqslant b_2 J, \tag{5.1}$$

where

$$J = \overline{\lim_{n\to\infty}}\, \bar{C}_\infty(n) + \overline{\lim_{n\to-\infty}}\, \bar{C}_\infty(n),$$

and $\bar{C}_\infty(n)$ is defined in Theorem 3.1.

Proof. The upper estimate follows immediately from the inequalities

$$\begin{aligned}\|K_v - K_v^{(2)} - K_v^{(3)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} &\leqslant \|K_v^{(1)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} + \|K_v^{(4)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)}\\ &\leqslant c[\sup_{\substack{i\leqslant m\\ i\in\mathbb{Z}}} \bar{C}_\infty(i) + \sup_{\substack{j\geqslant n\\ j\in\mathbb{Z}}} \bar{C}_\infty(j)]\end{aligned}$$

where $K_v^{(i)}$, $i = 1\cdots 4$ are defined in Theorem 4.1 assuming $d = 2^m$, $b = 2^n$, $m < 0$ and $n > 0$ (see the proof of Theorem 4.1 for the details) and the fact that $K_v^{(2)}$ and $K_v^{(3)}$ are compact according to Theorem B.
To get the lower estimate we take a positive number λ so that $\lambda > \|K_v\|_{\mathscr{K}(L^{p(\cdot)}(I),L^{q(\cdot)}(I))}$. Consequently, by Theorem C we have that $\lambda > \overline{\alpha}(K_v)$. Hence, there exist $g_1,\dots,g_N \in L^{q(\cdot)}(I)$ such that

$$\overline{\alpha}(K_v) \leqslant \|K_v - F\| < \lambda,$$

where $Ff(x) = \sum_{j=1}^N \alpha_j(f) g_j(x)$, α_j are linear bounded functionals in $L^{p(\cdot)}(I)$ and g_i are linearly independent. Further, there exist $\bar{g}_1,\dots,\bar{g}_N$ such that supports of $\bar{g}_i$ are in $[\sigma_i,\eta_i]$, $0 < \sigma_i < \eta_i < \infty$, and

$$\|K_v - F_0\| < \lambda,$$

where $F_0 f(x) = \sum_{j=1}^N \alpha_j(f)\bar{g}_j(x)$. Suppose that $\sigma = \min\{\sigma_j\}$, $\eta = \max\{\eta_j\}$. Then obviously, $\operatorname{supp} F_0 f \subset [\sigma,\eta]$. Let $f_n := \chi_{(2^{n-1},2^{n+1})}$. Then by applying the condition $k \in V(I)$ for a negative integer n chosen so that $2^{n+1} < \sigma$, we find that

$$\begin{aligned}\lambda\ &\|f_n\|_{L^{p(\cdot)}(I)} \geqslant \|\chi_{E_n}(x)(K_v f_n(x) - F_0 f_n(x))\|_{L^{q(x)}(I)}\\ &\geqslant \|\chi_{E_n}(x)(K_v f_n)(x)\|_{L^{q(x)}(I)}\\ &\geqslant \left\|\chi_{E_n}(x)v(x)\int_{x/2}^x k(x,y)f_n(y)dy\right\|_{L^{q(x)}(I)}\\ &\geqslant c_1\left\|\chi_{E_n}(x)v(x)xk(x,x/2)\right\|_{L^{q(x)}(I)}\\ &\geqslant c_1 2^n\cdot\left\|\chi_{E_n}(x)v(x)k(x,x/2)\right\|_{L^{q(x)}(I)}\end{aligned}$$

Further, by using the condition $p \in \mathscr{P}_\infty^{log}(I)$ the condition $k \in V(I)$, Lemma B we find that

$$\lambda \geqslant d_1 \left\| \chi_{E_n}(x) v(x) k(x, x/2) \right\|_{L^{q(x)}(I)} 2^{n/p'(0)},$$

where the positive constant d_1 depends only on p, q and the constant c_1 from Definition 2.3. Consequently, we have $\lambda \geqslant d_1 \overline{\lim}_{n \to -\infty} \bar{C}_\infty(n)$.

Similarly let $f_m := \chi_{(2^{m-1}, 2^{m+1})}$. Now choosing a positive integer m so that $2^{m+1} > \eta$ and using Lemma A we find that

$$\lambda \geqslant d_2 \left\| \chi_{E_m}(x) v(x) k(x, x/2) \right\|_{L^{q(x)}(I)} 2^{m/(p_\infty)'}.$$

where the positive constant d_2 depends only on p, q and the constant c_1 from Definition 2.3. Hence, we have $\lambda \geqslant d_2 \overline{\lim}_{m \to +\infty} \bar{C}_\infty(m)$.

Since λ is arbitrarily close to $\|K_v\|_{\mathscr{K}(L^{p(\cdot)}(I), L^{q(\cdot)}(I))}$, hence we conclude that the lower estimate of (5.1) holds.

Analogously follows the next statement, proof of which is omitted

THEOREM 5.2. *Let $I := \mathbb{R}_+$. Suppose that $1 < p_-(I) \leqslant p(x) \leqslant q(x) \leqslant q_+(I) < \infty$. Let $p, q \in \mathscr{P}_\infty^{log}(I)$ and let $C_\infty < \infty$ (see Theorem 3.1 for the definition of C_∞). Then there exist two positive constants e_1 and e_2 depending only on p, q and the constants c_1 and c_2 defined in Definitions 2.3 and 2.4 respectively such that*

$$e_1 U \leqslant \|K_v\|_{\mathscr{K}(L^{p(\cdot)}(I), L^{q(\cdot)}(I))} \leqslant e_2 U,$$

where,

$$U = \lim_{d \to 0^+} C_d + \lim_{b \to +\infty} C_b,$$

C_b and C_d are defined in Theorem 4.1.

For estimates of the measure of non-compactness of kernel operators with singularity in the classical Lebesgue spaces we refer e.g., to the monograph [16] and references cited therein.

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