Journal of Mathematical Inequalities
Volume 10, Number 3 (2016), 641–648 doi:10.7153/jmi-10-51

A GEOMETRIC INEQUALITY WITH APPLICATIONS

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(*Communicated by G. Leng*)

Abstract. In this paper, we present a new geometric inequality which involves an arbitrary point in the plane of a triangle. A simpler proof of a known inequality with one parameter is obtained by using our result. We also derive the famous Sondat fundamental triangle inequality from it.

1. Introduction

In the recent paper [10] the following geometric inequality with one parameter has been established.

For a point P in the plane of a triangle ABC with side lengths $BC = a$, $CA = b$ and $AB = c$, we denote by R_1, R_2, R_3 the distances of P from the vertices A, B, C and from the sides BC, CA, AB by r_1, r_2, r_3, respectively. Then

$$\frac{R_2^2+R_3^2+\lambda r_1^2}{a^2}+\frac{R_3^2+R_1^2+\lambda r_2^2}{b^2}+\frac{R_1^2+R_2^2+\lambda r_3^2}{c^2}\geqslant\frac{8+\lambda}{4},\tag{1}$$

where λ is a parameter such that $-2\leqslant\lambda\leqslant 2$. The equality condition of (1) is also given. If $\lambda=-2$, then the equality in (1) holds if and only if O is the circumcenter of ABC. If $\lambda=2$, then the equality holds if and only if P is the Lhuilier-Lemoine point of ABC. If $-2<\lambda<2$, then the equality holds if and only if $\triangle ABC$ is equilateral and P would be its center.

When $\lambda=2$ and $\lambda=-2$, (1) yields respectively,

$$\frac{R_2^2+R_3^2+2r_1^2}{a^2}+\frac{R_3^2+R_1^2+2r_2^2}{b^2}+\frac{R_1^2+R_2^2+2r_3^2}{c^2}\geqslant\frac{5}{2}\tag{2}$$

and

$$\frac{R_2^2+R_3^2-2r_1^2}{a^2}+\frac{R_3^2+R_1^2-2r_2^2}{b^2}+\frac{R_1^2+R_2^2-2r_3^2}{c^2}\geqslant\frac{3}{2}.\tag{3}$$

It is a pity that the proof of (1), given in [10] by the author, is rather complicated and aided by computer software Maple.

Our purpose of this note is to give an improvement of (3), which can be used to deduce inequality (1) rapidly. The main result is the following:

Mathematics subject classification (2010): 51M16.
Keywords and phrases: Triangle, point, the Sondat fundamental triangle inequality, Euler formula.

Paper JMI-10-51

THEOREM 1. *Let R be the circumradius of the triangle ABC and O be its circumcenter. Denote the distance between O and any point P in the plane by d. Then*

$$\frac{R_2^2+R_3^2-2r_1^2}{a^2}+\frac{R_3^2+R_1^2-2r_2^2}{b^2}+\frac{R_1^2+R_1^2-2r_3^2}{c^2} \geqslant \frac{3}{2}+\frac{d^2}{R^2}, \tag{4}$$

with equality holds if and only if the point P lies on the line OK, where K is the Lhuilier-Lemoine point of ABC.

Clearly, inequality (4) improves the ordinary inequality (3). It is interesting that the equality condition of (4) is very special (see Figure 1).

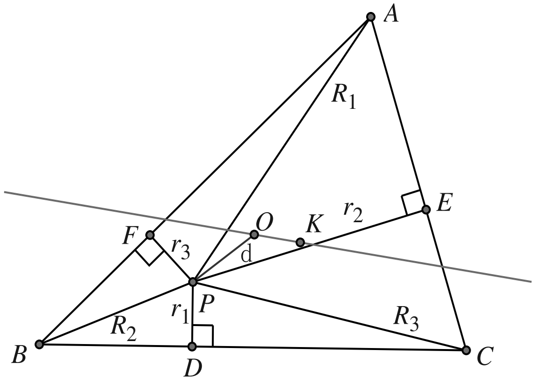

Figure 1: *Equality in* (4) *occurs iff P lies on the line OK.*

In the next section, we shall prove Theorem 1. In the third section, we shall use inequality (4) to give a simpler proof of inequality (1) and derive the Sondat fundamental triangle inequality.

2. Proof of Theorem 1

In order to prove our theorems, we bring up "directed distances" (for the definition, see e.g. [9]). In what follows, we denote the directed distances from the point P to the sides BC, CA, AB of $\triangle ABC$ by d_1, d_2, d_3, respectively. Denote the directed area of $\triangle ABC$ and the area of the pedal triangle DEF of P with respect to $\triangle ABC$ by S, S_p, respectively. For simplicity, we also denote cyclic sums over the triples $(a,b,c),(R_1,R_2,R_3)$ (r_1,r_2,r_3) and (d_1,d_2,d_3) by $\sum$, for instance

$$\sum ad_1 = ad_1+bd_2+cd_3,$$

$$\sum d_1\frac{b^2-c^2}{a} = d_1\frac{b^2-c^2}{a}+d_2\frac{c^2-a^2}{b}+d_3\frac{a^2-b^2}{c},$$

$$\sum \frac{R_2^2+R_3^2-2r_1^2}{a^2} = \frac{R_2^2+R_3^2-2r_1^2}{a^2}+\frac{R_3^2+R_1^2-2r_2^2}{b^2}+\frac{R_1^2+R_2^2-2r_3^2}{c^2}.$$

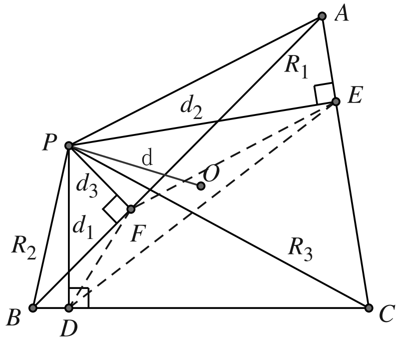

Figure 2: $d_1 = r_1 > 0,\ d_2 = r_2 > 0,\ d_3 = -r_3 < 0.$

Now, we are to prove Theorem 1.

Proof. Without loss of generality, we may assume that the triangle ABC has positive directed area $(S > 0)$. We firstly prove the following identity:

$$\sum \frac{R_2^2 + R_3^2}{a^2} = \frac{1}{4S^2} \sum (2a^2 + b^2 + c^2) {d_1}^2 + \frac{1}{4S^2} \sum \frac{(b^2 + c^2)(b^2 + c^2 - a^2) d_2 d_3}{bc}. \quad (5)$$

For any point P in the plane of $\triangle ABC$ (see Figure 2, where D, E and F are projections of P on the sides), we apply the law of cosines to $\triangle PEF$, then it is easy to obtain that

$$EF = \sqrt{{d_2}^2 + {d_3}^2 + 2 d_2 d_3 \cos A}, \quad (6)$$

where A denotes the angle $\angle BAC$ of $\triangle ABC$. Since PA is a diameter of the circumradius of $\triangle PEF$, therefore $EF = PA \sin A = R_1 \sin A$ and then the length of R_1 is given by

$$R_1 = \frac{\sqrt{{d_2}^2 + {d_3}^2 + 2 d_2 d_3 \cos A}}{\sin A}. \quad (7)$$

Thus, we have

$$\begin{aligned}
\sum \frac{R_2^2 + R_3^2}{a^2} &= \sum R_1^2 \left(\frac{1}{b^2} + \frac{1}{c^2} \right) \\
&= \sum \left({d_2}^2 + {d_3}^2 + 2 d_2 d_3 \cos A \right) \left(\frac{1}{b^2} + \frac{1}{c^2} \right) \frac{1}{\sin^2 A} \\
&= \frac{1}{4S^2} \sum (b^2 + c^2) \left({d_2}^2 + {d_3}^2 + 2 d_2 d_3 \cos A \right) \\
&= \frac{1}{4S^2} \sum (b^2 + c^2) \left({d_2}^2 + {d_3}^2 \right) + \frac{1}{2S^2} \sum (b^2 + c^2) d_2 d_3 \cos A \\
&= \frac{1}{4S^2} \sum (2a^2 + b^2 + c^2) {d_1}^2 + \frac{1}{4S^2} \sum \frac{(b^2 + c^2)(b^2 + c^2 - a^2) d_2 d_3}{bc},
\end{aligned}$$

where we used the area formula $2S = bc\sin A$ and the law of cosines in $\triangle ABC$. The identity (5) is proved.

Next, we further make use of (5) to prove the following identity:

$$\sum \frac{R_2^2 + R_3^2 - 2d_1^2}{a^2} + 4\sum \frac{d_2 d_3}{bc} - \frac{5}{2} = \frac{1}{8S^2}\left(\sum d_1 \frac{b^2 - c^2}{a}\right)^2. \tag{8}$$

By the area relation $\vec{S}_{\triangle PBC} + \vec{S}_{\triangle PCA} + \vec{S}_{\triangle PAB} = \vec{S}_{\triangle ABC}$, we get

$$\sum a d_1 = 2S. \tag{9}$$

In addition, by $\vec{S}_{\triangle DEF} = \vec{S}_{\triangle PEF} + \vec{S}_{\triangle PFD} + \vec{S}_{\triangle PDE}$ and $\vec{S}_{\triangle PEF} = \frac{1}{2} d_2 d_3 \sin A = \frac{S}{bc} d_2 d_3$ etc., we obtain the following identity:

$$\frac{S_p}{S} = \sum \frac{d_2 d_3}{bc}. \tag{10}$$

Using (5), (9), (10), and the equivalent form of Heron's formula will be as follows:

$$16S^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4, \tag{11}$$

we have that

$$\begin{aligned}
&\sum \frac{R_2^2 + R_3^2 - 2d_1^2}{a^2} + 4\sum \frac{d_2 d_3}{bc} - \frac{5}{2} \\
&= \sum \frac{R_2^2 + R_3^2}{a^2} - 2\sum \frac{{d_1}^2}{a^2} + 4\sum \frac{d_2 d_3}{bc} - \frac{5\left(\sum a d_1\right)^2}{8S^2} \\
&= \frac{1}{4S^2}\sum (2a^2 + b^2 + c^2){d_1}^2 + \frac{1}{4S^2}\sum \frac{(b^2 + c^2)(b^2 + c^2 - a^2) d_2 d_3}{bc} - 2\sum \frac{{d_1}^2}{a^2} \\
&\quad + 4\sum \frac{d_2 d_3}{bc} - \frac{5\left(\sum a^2 {d_1}^2 + 2\sum bc d_2 d_3\right)}{8S^2} \\
&= \frac{1}{4S^2}\sum \left(2a^2 + b^2 + c^2 - \frac{8S^2}{a^2} - \frac{5a^2}{2}\right){d_1}^2 \\
&\quad + \frac{1}{4S^2}\sum \left[\frac{(b^2 + c^2)(b^2 + c^2 - a^2)}{bc} + \frac{16S^2}{bc} - 5bc\right] d_2 d_3 \\
&= \frac{1}{4S^2}\sum \left(b^2 + c^2 - \frac{a^2}{2} - \frac{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}{2a^2}\right){d_1}^2 \\
&\quad + \frac{1}{4S^2}\sum \frac{(b^2 + c^2)(b^2 + c^2 - a^2) + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4 - 3b^2c^2}{bc} d_2 d_3 \\
&= \frac{1}{8S^2}\sum \frac{(b^2 - c^2)^2}{a^2}{d_1}^2 - \frac{1}{4S^2}\sum \frac{(a^2 - b^2)(a^2 - c^2)}{bc} d_2 d_3 \\
&= \frac{1}{8S^2}\left(\sum d_1 \frac{b^2 - c^2}{a}\right)^2.
\end{aligned}$$

Hence, identity (8) is proved.

By (8) and (10), we have

$$\sum \frac{R_2^2+R_3^2-2d_1^2}{a^2}+4\frac{S_p}{S}-\frac{5}{2}=\frac{1}{8S^2}\left(\sum d_1\frac{b^2-c^2}{a}\right)^2. \tag{12}$$

Note that the following well-known identity (see e.g. [4, 9.5]):

$$\frac{d^2}{R^2}=1-\frac{4S_p}{S}. \tag{13}$$

We obtain the following identity from (12):

$$\sum \frac{R_2^2+R_3^2-2d_1^2}{a^2}=\frac{3}{2}+\frac{d^2}{R^2}+\frac{1}{8S^2}\left(\sum d_1\frac{b^2-c^2}{a}\right)^2, \tag{14}$$

which implies that the following inequality holds:

$$\sum \frac{R_2^2+R_3^2-2d_1^2}{a^2}\geqslant\frac{3}{2}+\frac{d^2}{R^2}. \tag{15}$$

Since $d_1^2=r_1^2$, etc., inequality (4) follows from (15). Also, one sees that the equality in (4) holds if and only if

$$\sum d_1\frac{b^2-c^2}{a}=0, \tag{16}$$

which signifies that the point P lies on a line. Obviously, the trilinear coordinate $(a:b:c)$ of the Lhuilier-Lemoine point K satisfies equation (16). Also, it is easy to check that the identity $\sum(b^2+c^2-a^2)(b^2-c^2)=0$, which shows that the trilinear coordinate $(a(b^2+c^2-a^2):b(c^2+a^2-b^2):c(a^2+b^2-c^2))$ of the circumcenter O satisfies (16). Therefore, the equality of (4) holds if and only if P lies on the line OK. This completes the proof of Theorem 1. $\square$

3. Applications of Theorem 1

In this section, we provide two applications of Theorem 1.

Firstly, we use Theorem 1 to give a simpler proof of the parameterized inequality (1) as follows:

By (13), we know that inequality (4) of Theorem 1 is equivalent to

$$\sum \frac{R_2^2+R_3^2-2d_1^2}{a^2}+4\sum\frac{d_2d_3}{bc}-\frac{5}{2}\geqslant 0. \tag{17}$$

Hence, if the real number λ satisfies $\lambda\geqslant -2$, then

$$\sum \frac{R_2^2+R_3^2-2d_1^2}{a^2}+4\sum\frac{d_2d_3}{bc}-\frac{5}{2}+\frac{\lambda+2}{2}\sum\left(\frac{d_2}{b}-\frac{d_3}{c}\right)^2\geqslant 0,$$

which is equivalent to

$$\sum \frac{R_2^2+R_3^2+\lambda d_1^2}{a^2}+(2-\lambda)\sum \frac{d_2d_3}{bc} \geqslant \frac{5}{2}. \tag{18}$$

Further, using (10), (13) and noticing that $d_1^2 = r_1^2$ etc., we can obtain the following inequality:

$$\frac{R_2^2+R_3^2+\lambda r_1^2}{a^2}+\frac{R_3^2+R_1^2+\lambda r_2^2}{b^2}+\frac{R_1^2+R_2^2+\lambda r_3^2}{c^2} \geqslant \frac{8+\lambda}{4}+\frac{(2-\lambda)d^2}{4R^2}, \tag{19}$$

where $\lambda \geqslant -2$.

From the above inequality, one can see that inequality (1) holds for $-2 \leqslant \lambda \leqslant 2$ and the equality condition of (1) is easily obtained.

REMARK 1. Inequality (19) is an unified generalization of inequalities (1), (2), (3) and (4).

REMARK 2. By Theorem 1 and the area inequality $S \geqslant 4S_p$ following from (13), we can easily deduce the following interesting inequality:

$$\sum \frac{R_2^2+R_3^2-2r_1^2}{a^2}+\frac{S_p^2}{S^2} \geqslant \frac{25}{16}, \tag{20}$$

which is weaker than (4) but stronger than (3). Equality in (20) holds if and only if P is the circumcenter of $\triangle ABC$ or P lies on the line OK and $S_p : S = 4:15$.

In the following, we shall give another more interesting application of Theorem 1, namely, to use (4) to derive the Sondat fundamental triangle inequality (see [4, inequality 13.8]), stating that

$$s^4-2(2R^2+10Rr-r^2)s^2+r(4R+r)^3 \leqslant 0, \tag{21}$$

where s, R and r are the semi-perimeter, circumcenter and inradius of arbitrary triangle ABC, respectively. Equality in (21) holds if and only if $\triangle ABC$ is isosceles.

There are several proofs of the fundamental triangle inequality in the literature (see e.g., [1], [3], [5], [13]–[15]). However, the inequalities from which the fundamental triangle inequality can be deduced are rare. We now deduce (21) from (4) as follows:

In Theorem 1, we take the point P to be the incenter of $\triangle ABC$. Then, by Pythagoras theorem, we have $R_2^2 - r_1^2 = (s-b)^2$ and $R_3^2 - r_1^2 = (s-c)^2$. Also, the distance d in this setting is given by the well-known Euler formula, i.e.,

$$d^2 = R^2 - 2Rr. \tag{22}$$

Thus, it follows from (4) that

$$\sum \frac{(s-b)^2+(s-c)^2}{a^2} \geqslant \frac{5}{2}-\frac{2r}{R}. \tag{23}$$

Next, we calculate the left of (22) in terms of R, r and s. Since

$$\begin{aligned}
&\sum \frac{(s-b)^2+(s-c)^2}{a^2} \\
&= \frac{1}{a^2b^2c^2} \sum b^2c^2 \left[2s^2 - 2(b+c)s + b^2 + c^2\right] \\
&= \frac{1}{(abc)^2} \left[2s^2 \sum b^2c^2 - 2s \sum (2s-a)b^2c^2 + \sum b^2c^2(b^2+c^2)\right] \\
&= \frac{1}{(abc)^2} \left[-2s^2 \sum b^2c^2 + 2sabc \sum bc + \sum b^2c^2 \sum a^2 - (abc)^2\right].
\end{aligned}$$

Then, with the following known identities (see e.g. [12, pp. 52–55]):

$$abc = 4Rrs, \tag{24}$$

$$\sum bc = s^2 + 4Rr + s^2, \tag{25}$$

$$\sum a^2 = 2s^2 - 8Rr - 2r^2, \tag{26}$$

$$\sum b^2c^2 = s^4 - 2r(4R-r)s^2 + r^2(4R+r)^2, \tag{27}$$

we easily obtain the following identity:

$$\sum \frac{(s-b)^2+(s-c)^2}{a^2} = \frac{-s^4 + 2(12R^2 + 2Rr - r^2)s^2 - r(4R+r)^3}{8s^2R^2}. \tag{28}$$

Furthermore, one gets

$$\sum \frac{(s-b)^2+(s-c)^2}{a^2} + \frac{2r}{R} - \frac{5}{2} = \frac{-s^4 + 2(2R^2 + 10Rr - r^2)s^2 - r(4R+r)^3}{8s^2R^2}. \tag{29}$$

Finally, with inequality (23) and (29), we can conclude that inequality (21) holds.

According to the equality condition of (4), we know that the equality in (21) holds if and only if the incenter I, the circumcenter O and the Lhuilier-Lemoine point K of triangle ABC are collinear. Note that the trilinear corrdinates of the points I, O and K are $(1:1:1)$, $(b^2+c^2-a^2 : c^2+a^2-b^2 : a^2+b^2-c^2)$, and $(a:b:c)$, respectively. These three points are collinear if and only if

$$\begin{vmatrix} 1 & 1 & 1 \\ b^2+c^2-a^2 & c^2+a^2-b^2 & a^2+b^2-c^2 \\ a & b & c \end{vmatrix} = 0,$$

from which we easily obtain

$$2(b-c)(c-a)(a-b) = 0.$$

We therefore conclude that the equality in (21) holds if and only if $\triangle ABC$ is isosceles.

REMARK 3. If we take the point P to be the centroid of ABC in inequality (17), then the following inequality follows:

$$\sum \frac{2m_b^2 + 2m_c^2 - h_a^2}{a^2} + 2\sum \frac{h_b h_c}{bc} \geqslant \frac{45}{4}, \tag{30}$$

where m_a, m_b, m_c are the medians of $\triangle ABC$ and h_a, h_b, h_c are the altitudes. From (30), we can also obtain the fundamental triangle inequality (21) after some calculations.

The Sondat fundamental triangle inequality and its two consequences, i.e., Gerretsen inequalities $s^2 \geqslant 16Rr - 5r^2$ and $s^2 \leqslant 4R^2 + 4Rr + 3r^2$ (see e.g., [12, p. 45]) have wide applications in the field of triangle geometric inequalities (cf. [6]–[8], [11] and [17]). Some related results with historical comments on this inequality can be found in [1]–[3], [6] and [12].

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(Received February 18, 2015)

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