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JORDAN–VON NEUMANN TYPE CONSTANT AND FIXED POINTS FOR MULTIVALUED NONEXPANSIVE MAPPINGS

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Abstract. We give some sufficient conditions for the Domínguez-Lorenzo condition in terms of the Jordan-von Neumann type constant, and the coefficient of weak orthogonality. As a consequence, we obtain some sufficient conditions for normal structure and fixed point theorems for multivalued nonexpansive mappings. These fixed point theorems improve some previous results in the recently papers.

1. Introduction

Fixed point theory for multivalued mappings has many useful applications in applied sciences, such as game theory and mathematical economics. Thus, it is natural to extend the known fixed point results for single-valued mappings to the setting of multivalued mappings. In 1969 Nadler [11] extended the Banach contraction principle to multivalued contractive mappings in complete metric spaces. From then on, many researchers have studied the possibility of extending classical fixed point theorems for singlevalued nonexpansive mappings to the setting of multivalued nonexpansive mappings. For instance, the celebrated Kirk's theorem [10] which states the fixed point property (FPP)for singlevalued nonexpansive mappings in reflexive Banach spaces with normal structure yields to a very natural question: Do reflexive Banach spaces with normal structure have the FPP for multivalued nonexpansive mappings? Until now, the answer is unknown.

Since weak normal structure is implied by different geometric properties of Banach spaces, it is natural to study if those properties imply the FPP for multivalued mappings. S. Dhompongsa et al. [5,6] introduced the (DL)-condition and property (D) which imply the FPP for multivalued nonexpansive mappings. A possible approach to the above problem is to look for geometric conditions in a Banach space X which imply either the (DL)-condition or property (D). In this setting, the following results have been obtained:

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(i) A. Kaewkhao [9] proved that a Banach space X with

$$J(X) < 1 + \frac{1}{\mu(X)}$$

satisfies the (DL)-condition.

(ii) He also showed that the condition

$$C_{\mathrm{NJ}}(X) < 1 + \frac{1}{\mu(X)^2}$$

implies the (DL)-condition.

Recently, Yang et al get the following results (see [15]):

(i) Let X be a Banach space, if $C_{\mathrm{t}}^{'}(X) < \frac{1}{4}(1+2\omega(X)+5\omega^2(X))$, $t \in [1,+\infty)$, then X and X^* have normal structure.

(ii) Let X be a Banach space, if $C_{-\infty}(X) < 1 + \frac{1}{\mu(X)^2}$, then X have normal structure.

The constants $C_{\mathrm{t}}^{'}(X)$ and $C_{-\infty}(X)$ are the special case of the Jordan-von Neumann type constant is defined as

$$C_{\mathrm{t}}(X) = \sup\left\{ \frac{J_{X,t}^2(\tau)}{1+\tau^2} : 0 \leqslant \tau \leqslant 1 \right\}$$

Moreover, for $\tau \geqslant 0$ and $-\infty \leqslant t < \infty$, the James type constant$J_{X,t}(\tau)$ (see [14, 17]) is defined as

$$J_{X,t}(\tau) := \begin{cases} \sup\left\{ \left(\frac{\|x+\tau y\|^t + \|x-\tau y\|^t}{2} \right)^{\frac{1}{t}} : \|x\| = 1, \|y\| = 1 \right\}, -\infty < t < \infty \text{ and } t \neq 0 \\ \sup\{ \sqrt{\|x+\tau y\| \|x-\tau y\|} : \|x\| = 1, \|y\| = 1\},\ t = 0 \\ \sup\{ \min\{\|x+\tau y\|, \|x-\tau y\|\} \|x\| = 1, \|y\| = 1\},\ t = -\infty \end{cases}$$

In particular, taking $\tau = 1$ or $t = -\infty$ in the definition of $C_{\mathrm{t}}(X)$, we get the constants $C_{\mathrm{t}}^{'}(X) = \frac{J_{X,t}(1)^2}{2}$ and $C_{-\infty}(X) = \sup\left\{ \frac{J_{X,-\infty}(\tau)^2}{1+\tau^2} : 0 \leqslant \tau \leqslant 1 \right\}$. Similarly, we can get that $C_{\mathrm{NJ}}^{'}(X) = \frac{J_{X,2}(1)^2}{2}$, $C_{\mathrm{NJ}}(X) = C_2(X)$, $C_Z(X) = C_0(X)$, $J(X) = J_{X,-\infty}(1)$, $\rho_X(\tau) = J_{X,1}(\tau) - 1$, so Jordan-von Neumann type constant and James type constant contain some well known geometrical constants (see [1–4, 7–9, 12–20]). It is obvious that $C_{\mathrm{t}}^{'}(X) \leqslant C_{\mathrm{t}}(X)$, furthermore, there are some spaces for which the inequalities is strict (see the below Remark 3). The following inequalities can been found in [14], $\frac{J^2(X)}{2} \leqslant C_{\mathrm{t}}^{'}(X) \leqslant C_{NJ}^{'}(X)$ and $\frac{J^2(X)}{2} \leqslant C_{-\infty}(X) \leqslant C_Z(X) \leqslant C_{NJ}(X)$.

2. Preliminaries

Before going to the results, let us recall some concepts and results which will be used in the following sections. In [13], Sims defined the coefficient of weak orthogonality $\omega(X)$ as

$$\omega(X) = \sup\{\lambda : \lambda \liminf_{n\to\infty} \|x_n + x\| \leqslant \liminf_{n\to\infty} \|x_n - x\|\},$$

where the supremum is taken over all $x \in X$ and all weakly null sequences (x_n). It is proved that $\frac{1}{3} \leqslant \omega(X) \leqslant 1$. In [8], Jiménez-Melado and Llorens-Fuster introduced a coefficient $\mu(X)$ related to $\omega(X)$ as

$$\mu(X) = \inf\{\lambda : \limsup_{n\to\infty} \|x_n + x\| \leqslant \lambda \limsup_{n\to\infty} \|x_n - x\|\},$$

where the infimum is taken over all $x \in X$ and all weakly null sequences (x_n). It is obvious that $\frac{1}{\mu(X)} = \omega(X)$ from the definition, furthermore it is known that $\mu(X) = \mu(X^*)$ in reflexive Banach space ([8]).

In the sequel, we introduce some concepts and results on multivalued mapping. Let C be a nonempty subset of a Banach space X, we shall denote by $CB(X)$ the family of all nonempty closed bounded subsets of X and by $KC(X)$ the family of all nonempty compact convex subsets of X. A multivalued mapping $T : C \to CB(X)$ is said to be nonexpansive if

$$H(Tx, Ty) \leqslant \|x - y\|, x, y \in C$$

where $H(.,.)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A,B) := \max\{\sup_{x\in A} \inf_{y\in B} \|x - y\|, \sup_{y\in B} \inf_{x\in A} \|x - y\|\}, A, B \in CB(X).$$

Let $\{x_n\}$ be a bounded sequence in X, the asymptotic radius $r(C, \{x_n\})$ and the asymptotic center $A(C, \{x_n\})$ of $\{x_n\}$ in C are defined by

$$r(C, \{x_n\}) = \inf\{\limsup_{n} \|x_n - x\| : x \in C\}$$

and

$$A(C, \{x_n\}) = \{x \in C : \limsup_{n} \|x_n - x\| = r(C, \{x_n\})\},$$

respectively. It is known that $A(C, \{x_n\})$ is a nonempty weakly compact convex set whenever C is. The sequence $\{x_n\}$ is called regular with respect to C if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$, and $\{x_n\}$ is called asymptotically uniform with respect to C if $A(C, \{x_n\}) = A(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

If D is a bounded subset of X, the Chebyshev radius of D relative to C is defined by

$$r_C(D) = \inf_{x\in C} \sup_{y\in D} \|x - y\|.$$

The Domínguez-Lorenzo condition((DL)-condition in short)for a Banach space introduced in [5] is defined as follows: if there exists $\lambda \in [0,1)$ such that for every weakly compact convex subset C of X and for every bounded sequence $\{x_n\}$ in C which is regular with respect to C,

$$r_C(A(C,\{x_n\})) \leqslant \lambda r(C,\{x_n\}).$$

The next results show that property (DL)-condition is stronger than weak normal structure and also implies the existence of fixed points for multivalued nonexpansive mappings [6].

THEOREM 1. *Let X be a Banach space satisfying(DL)-condition, then X has weak normal structure.*

THEOREM 2. *Let C be a nonempty weakly compact convex subset of a Banach space X which satisfies (DL)-condition and $T: C \to KC(C)$ be a multivalued nonexpansive mapping, then T has a fixed point.*

3. Main results

THEOREM 3. *Let C be a weakly compact convex subset of a Banach space X and $\{x_n\}$ be a bounded sequence in C regular with respect to C, if $t > 0$, then*

$$r_C(A(C,\{x_n\})) \leqslant \left(\frac{(2[2C_t'(X)]^{\frac{t}{2}} - (1+\frac{1}{\mu(X)})^t)^{\frac{1}{t}}}{1+\frac{1}{\mu(X)}} \right) r(C,\{x_n\}).$$

Proof. Denote $r = r(C,\{x_n\})$ and $A = A(C,\{x_n\})$. We can assume $r > 0$. By passing to a subsequence if necessary, we can also assume that $\{x_n\}$ is weakly convergent to a point $x \in C$. Since $\{x_n\}$ is regular with respect to C, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$. Let $z \in A$, then we have

$$\limsup_n \|x_n - z\| = r.$$

Denote $\mu = \mu(X)$. By the definition of μ, we have

$$\begin{aligned}\limsup_n \|x_n - 2x + z\| &= \limsup_n \|(x_n - x) + (z - x)\| \\ &\leqslant \mu \limsup_n \|(x_n - x) - (z - x)\| = \mu r.\end{aligned}$$

Convexity of C implies that $\frac{2}{1+\mu}x + \frac{\mu-1}{1+\mu}z \in C$ and we obtain

$$\limsup_n \|x_n - \left(\frac{2}{1+\mu}x + \frac{\mu-1}{1+\mu}z \right) \| \geqslant r.$$

On the other hand, by the weak lower semicontinuity of the norm, we have

$$\liminf_n \left\| \left(1 - \frac{1}{\mu}\right)(x_n - x) - \left(1 + \frac{1}{\mu}\right)(z - x) \right\| \geqslant \left(1 + \frac{1}{\mu}\right)\|z - x\|.$$

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

(1) $\|x_N - z\| \leqslant r + \varepsilon$.

(2) $\|x_N - 2x + z\| \leqslant \mu(r + \varepsilon)$.

(3) $\|x_N - (\frac{2}{1+\mu}x + \frac{\mu-1}{1+\mu}z)\| \geqslant r - \varepsilon$.

(4) $\|(1 - \frac{1}{\mu})(x_N - x) - (1 + \frac{1}{\mu})(z - x)\| \geqslant (1 + \frac{1}{\mu})\|z - x\|(\frac{r-\varepsilon}{r})$.

Now, put $u = \frac{1}{r+\varepsilon}(x_N - z)$, $v = \frac{1}{\mu(r+\varepsilon)}(x_N - 2x + z)$. Using the above estimates we obtain $u, v \in B_X$ and

$$\begin{aligned}
\|u+v\| &= \left\|\frac{x_N - x}{r+\varepsilon} - \frac{z-x}{r+\varepsilon} + \frac{x_N - x}{\mu(r+\varepsilon)} + \frac{z-x}{\mu(r+\varepsilon)}\right\| \\
&= \left\|\left(\frac{1}{r+\varepsilon} + \frac{1}{\mu(r+\varepsilon)}\right)(x_N - x) - \left(\frac{1}{r+\varepsilon} - \frac{1}{\mu(r+\varepsilon)}\right)(z-x)\right\| \\
&= \frac{1}{r+\varepsilon}\left(1 + \frac{1}{\mu}\right)\left\|(x_N - x) - \left(\frac{1-\frac{1}{\mu}}{1+\frac{1}{\mu}}\right)(z-x)\right\| \\
&= \frac{1}{r+\varepsilon}\left(1 + \frac{1}{\mu}\right)\left\|x_N - (\frac{2}{1+\mu}x + \frac{\mu-1}{1+\mu}z)\right\| \\
&\geqslant \left(1 + \frac{1}{\mu}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right), \\
\|u-v\| &= \left\|\frac{x_N - x}{r+\varepsilon} - \frac{z-x}{r+\varepsilon} - \frac{(x_N - x)}{\mu(r+\varepsilon)} - \frac{(z-x)}{\mu(r+\varepsilon)}\right\| \\
&= \frac{1}{r+\varepsilon}\left\|\left(1 - \frac{1}{\mu}\right)(x_N - x) - \left(1 + \frac{1}{\mu}\right)(z-x)\right\| \\
&\geqslant \left(1 + \frac{1}{\mu}\right)\frac{\|z-x\|}{r}\left(\frac{r-\varepsilon}{r+\varepsilon}\right).
\end{aligned}$$

Thus

$$C_{\mathrm{t}}^{'}(X) \geqslant \frac{\left(\frac{\|u+v\|^t + \|u-v\|^t}{2}\right)^{\frac{2}{t}}}{2} \geqslant \frac{\left(\frac{(1+\frac{1}{\mu})^t\left(\frac{r-\varepsilon}{r+\varepsilon}\right)^t + (1+\frac{1}{\mu})^t\left(\frac{\|z-x\|}{r}\right)^t\left(\frac{r-\varepsilon}{r+\varepsilon}\right)^t}{2}\right)^{\frac{2}{t}}}{2}$$

Let $\varepsilon \to 0^+$, we obtain that $C_{\mathrm{t}}^{'}(X) \geqslant \frac{\left(\frac{(1+\frac{1}{\mu})^t + (1+\frac{1}{\mu})^t\left(\frac{\|z-x\|}{r}\right)^t}{2}\right)^{\frac{2}{t}}}{2}$. Since the above inequality is true for every $\varepsilon > 0$ and every $z \in A$, we obtain

$$\sup_{z\in A}\|x - z\| \leqslant \left(\frac{(2[2C_{\mathrm{t}}^{'}(X)]^{\frac{t}{2}} - (1 + \frac{1}{\mu})^t)^{\frac{1}{t}}}{1 + \frac{1}{\mu}}\right)r,$$

and therefore

$$r_C(A) \leqslant \left(\frac{(2[2C_t'(X)]^{\frac{t}{2}} - (1+\frac{1}{\mu})^t)^{\frac{1}{t}}}{1+\frac{1}{\mu}}\right) r. \quad \square$$

COROLLARY 1. *Let X a Banach space, if there exists a real number $t > 0$ for which $C_t'(X) < \frac{(1+\frac{1}{\mu(X)})^2}{2}$, then X and X^* satisfy the (DL)-condition.*

Proof. Since $C_t'(X) < \frac{(1+\frac{1}{\mu(X)})^2}{2}$, then X satisfy the (DL)-condition by Theorem 3. On the other hand,

$$C_t'(X) \geqslant \frac{1}{2}(1+\rho_X(1))^2 \geqslant \frac{1}{2}(1+\rho_{X^*}(1))^2 \geqslant \frac{1}{2}J^2(X^*)$$

then $J(X^*) < 1 + \frac{1}{\mu(X)}$. From the condition $C_t'(X) < \frac{(1+\frac{1}{\mu(X)})^2}{2}$ and $\mu(X) \geqslant 1$, so X is uniformly nonsquare, then $\mu(X) = \mu(X^*)$. Therefore, we get $J(X^*) < 1 + \frac{1}{\mu(X^*)}$, then X^* satisfy the (DL)-condition from the result of A. Kaewkhao [9, Corollary 3.2]. $\square$

COROLLARY 2. *Let X a Banach space,if there exists a real number $t > 0$ such that $C_t'(X) < \frac{(1+\frac{1}{\mu(X)})^2}{2}$, then X and X^* has normal structure.*

REMARK 1. The results in Corollary 1 not only give a sufficient condition for fixed point theorems of multivalued nonexpansive mappings of a Banach space, but also give a sufficient condition on its dual space. Moreover, since

$$2(1+\omega(X))^2 - 1 - 2\omega(X) - 5(\omega(X))^2 = (1-\omega(X))(3\omega(X)+1).$$

and $\frac{1}{3} \leqslant \omega(X) \leqslant 1$, Corollary 2 is a strict generalization of Theorem 3.4 in [15].

THEOREM 4. *Let C be a weakly compact convex subset of a Banach space X and $\{x_n\}$ be a bounded sequence in C regular with respect to C, then*

$$r_C(A(C,\{x_n\})) \leqslant \left(\sqrt{\frac{C_{-\infty}(X)\mu(X)^2}{\mu(X)^2+1}}\right) r(C,\{x_n\}).$$

Proof. Let $r, A, \{x_n\}, x, z$ and μ be as in the proof of the pervious theorem. Then we have

$$\limsup_n \|x_n - z\| = r,$$

and

$$\limsup_n \|x_n - 2x + z\| \leqslant \mu r.$$

Since $\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z \in C$ and by the definition of r, we obtain

$$\limsup_n \left\| x_n - \left(\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z\right)\right\| \geqslant r.$$

On the other hand, by the weak lower semicontinuity of the norm, we have

$$\liminf_n \|(\mu^2-1)(x_n-x)-(\mu^2+1)(z-x)\| \geqslant (\mu^2+1)\|z-x\|.$$

For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

(1) $\|x_N - z\| \leqslant r+\varepsilon$.

(2) $\|x_N - 2x + z\| \leqslant \mu(r+\varepsilon)$.

(3) $\left\|x_N - \left(\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z\right)\right\| \geqslant r-\varepsilon$.

(4) $\|(\mu^2-1)(x_N-x)-(\mu^2+1)(z-x)\| \geqslant (\mu^2+1)\|z-x\|(\frac{r-\varepsilon}{r})$.

Now, put $u = \mu^2(x_N - z)$, $v = (x_N - 2x + z)$, use the above estimates to obtain $\|u\| \leqslant \mu^2(r+\varepsilon)$, $\|v\| \leqslant \mu(r+\varepsilon)$ and so that

$$\begin{aligned}
\|u+v\| &= \|\mu^2((x_N-x)-(z-x))+(x_N-x)+(z-x)\| \\
&= (\mu^2+1)\left\|(x_N-x)-\frac{\mu^2-1}{\mu^2+1}(z-x)\right\| \\
&= (\mu^2+1)\left\|x_N-\left(\frac{2}{\mu^2+1}x+\frac{\mu^2-1}{\mu^2+1}z\right)\right\| \\
&\geqslant (\mu^2+1)(r-\varepsilon), \\
\|u-v\| &= \|\mu^2((x_N-x)-(z-x))-((x_N-x)+(z-x))\| \\
&= \|(\mu^2-1)(x_N-x)-(\mu^2+1)(z-x)\| \\
&\geqslant (\mu^2+1)\|z-x\|\left(\frac{r-\varepsilon}{r}\right).
\end{aligned}$$

By the definition of $C_{-\infty}(X)$ and since $\|z-x\| \leqslant r$, we get

$$\begin{aligned}
C_{-\infty}(X) &\geqslant \frac{\min\{\|u+v\|^2, \|u-v\|^2\}}{\|u\|^2+\|v\|^2} \\
&\geqslant \frac{\min\{(\mu^2+1)^2(r-\varepsilon)^2, (\mu^2+1)^2\|z-x\|^2\left(\frac{r-\varepsilon}{r}\right)^2\}}{\mu^4(r+\varepsilon)^2+\mu^2(r+\varepsilon)^2}.
\end{aligned}$$

Let $\varepsilon \to 0^+$, we obtain that $C_{-\infty}(X) \geqslant \frac{(\mu^2+1)^2\|z-x\|^2}{\mu^4 r^2+\mu^2 r^2}$, then

$$\|z-x\| \leqslant \left(\sqrt{\frac{C_{-\infty}(X)\mu^2}{\mu^2+1}}\right)r.$$

This holds for arbitrary $z \in A$, hence we have

$$r_C(A) \leqslant \left(\sqrt{\frac{C_{-\infty}(X)\mu^2}{\mu^2+1}}\right)r. \quad \square$$

COROLLARY 3. *Let C be a nonempty bounded closed convex subset of a Banach space X such that $C_{-\infty}(X) < 1 + \frac{1}{\mu(X)^2}$ and $T : C \to KC(C)$ be a nonexpansive mapping, then T has a fixed point.*

Proof. Since $C_{-\infty}(X) < 1 + \frac{1}{\mu(X)^2}$, then X satisfy the (DL)-condition by Theorem 7, so T has a fixed point by Theorem 2. $\square$

COROLLARY 4. *Let X a Banach space such that $C_{-\infty}(X) < 1 + \frac{1}{\mu(X)^2}$, then X has normal structure.*

Proof. Since $C_{-\infty}(X) < 1 + \frac{1}{\mu(X)^2}$, then X satisfy the (DL)-condition by Theorem 2, so X has normal structure by Theorem 1. $\square$

REMARK 2. Since the constants $C_Z(X)$ and $C_{\mathrm{NJ}}(X)$ are more than or equal to $C_{-\infty}(X)$, Corollary 3 is a generalization of Corollary 2.1 in [18] and Corollary 4.2 in [9]. Meanwhile, Corollary 3 generalized the Theorem 3.2 in [15].

REMARK 3. In the sequel, we give a example to show that our results are sharp. The Bynum space $l_{2,\infty}$, which is the space l_2 renormed according to $\|x\|_{2,\infty} = \max\{\|x^+\|, \|x^-\|\}$, where x^+ and x^- are the positive and the negative part of x, respectively, defined as $x^+(i) = \max\{x(i), 0\}$ and $x^- = x^+ - x$. We use the computation to conclude that the space $l_{2,\infty}$ is a limiting space for both Corollary 1, 2, 3, 4. It is well known that $J(l_{2,\infty}) = 1 + \frac{1}{\sqrt{2}}$, $C_{NJ}(l_{2,\infty}) = \frac{3}{2}$. Using the same method in [8], we can get that $C'_{NJ}(l_{2,\infty}) = \frac{3+2\sqrt{2}}{4}$. From the inequality $\frac{J^2(X)}{2} \leqslant C'_{\mathrm{t}}(X) \leqslant C'_{NJ}(X)$ and $C_{-\infty}(X) \leqslant C_{NJ}(X)$, we get that $C'_{\mathrm{t}}(l_{2,\infty}) = \frac{3+2\sqrt{2}}{4}(-\infty \leqslant t \leqslant 2)$ and $C_{-\infty}(l_{2,\infty}) \leqslant \frac{3}{2}$. Take $x = (-1, 1, 0, \ldots), y = (\frac{1}{2}, \frac{1}{2}, 0, \ldots) \in l_{2,\infty}$ and $\|x+y\| = \|x-y\| = \frac{3}{2}$, so $C_{-\infty}(X) \geqslant \frac{3}{2}$, then $C_{-\infty}(l_{2,\infty}) = \frac{3}{2}$. It is well known that $\mu(l_{2,\infty}) = \sqrt{2}$, so $C'_{\mathrm{t}}(l_{2,\infty}) = \frac{(1+\frac{1}{\mu(l_{2,\infty})})^2}{2}$ and $C_{-\infty}(l_{2,\infty}) = 1 + \frac{1}{\mu(l_{2,\infty})^2}$. However, $l_{2,\infty}$ lacks normal structure, we conclude that the results obtained in the paper are sharp.

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