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WEIGHTED NORM INEQUALITIES FOR TOEPLITZ TYPE OPERATOR ASSOCIATED TO SINGULAR INTEGRAL OPERATOR WITH NON–SMOOTH KERNEL

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Abstract. Let $T^{k,1}$ be singular integrals with non-smooth kernels, which are associated with an approximation to the identity or $\pm I$ (the identity operator), $T^{k,2}$ and $T^{k,4}$ are the linear operators, $T^{k,3} = \pm I$. Denote the Toeplitz type operator by

$$T_b = \sum_{k=1}^{m}(T^{k,1}M_bI_\alpha T^{k,2} + T^{k,3}I_\alpha M_bT^{k,4}),$$

where $M_bf(x) = b(x)f(x)$, and I_α is the fractional integral operator. In this paper, we establish the sharp maximal function estimates for T_b when b belongs to weighted Lipschitz function space. As an application, the boundedness of the operator on weighted Lebesgue space is obtained.

1. Introduction and results

As the development of the singular integral operators, their commutators have been well studied [1, 2, 3]. In [1], the authors proved that the commutators $[b,T]$, which were generated by Calderón-Zygmund singular integral operators and BMO functions, are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo (see [4]) obtained a similar result when Calderón-Zygmund singular integral operators were replaced by the fractional integral operators. Recently, some Toeplitz type operators associated to the singular integral operators are introduced, and the boundedness for the operators generated by singular integral operators and BMO functions and Lipschitz functions are obtained (see [5, 6]). In [7, 8], some singular integral operators with non-smooth kernel are introduced, and the boundedness for the operators and their commutators are obtained [9, 10].

DEFINITION 1.1. A family of operators A_t, $t > 0$ is said to be an approximation to the identity if, for every $t > 0$, A_t can be represented by a kernel $a_t(x,y)$ in the following sense:

$$A_t(f)(x) = \int_{\mathbb{R}^n} a_t(x,y)f(y)dy \tag{1.1}$$

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for every $f \in L^p(\mathbb{R}^n)$ with $p \geqslant 1$, and $a_t(x,y)$ satisfies

$$|a_t(x,y)| \leqslant h_t(x,y) = ct^{-n/2}s(|x-y|^2/t), \tag{1.2}$$

where s is a positive, bounded, and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\varepsilon}s(r^2) = 0 \tag{1.3}$$

for some $\varepsilon > 0$.

The sharp maximal function $M_A^\sharp$ associated with the approximation to the identity $\{A_t, t > 0\}$ is defined by

$$M_A^\sharp(f)(x) = \sup_{x\in B} \frac{1}{|B|}\int_B |f(y) - A_{t_B}(f)(y)|\, dy, \tag{1.4}$$

where $t_B = r_B^2$ and r_B denotes the radius of B.

DEFINITION 1.2. A linear operator T is called a singular integral operator with non-smooth kernel if T is bounded on $L^2(\mathbb{R}^n)$ and associated with a kernel $K(x,y)$ such that

$$T(f)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \tag{1.5}$$

for every continuous function f with compact support, and for almost all x not in the support of f.

(1) There exists an approximation to the identity $\{B_t, t > 0\}$ such that TB_t has the associated kernel $k_t(x,y)$ and there exist $c_1, c_2 > 0$ so that

$$\int_{|x-y|>c_1t^{1/2}} |K(x,y) - k_t(x,y)| \leqslant c_2 \text{ for all } y \in \mathbb{R}^n. \tag{1.6}$$

(2) There exists an approximation to the identity A_t, $t > 0$ such that A_tT has the associated kernel $K_t(x,y)$ which satisfies

$$|K_t(x,y)| \leqslant c_4t^{-n/2} \text{ if } |x-y| < c_3t^{1/2}, \tag{1.7}$$

and

$$|K(x,y) - K_t(x,y)| \leqslant c_4t^{\delta/2}|x-y|^{-n-\delta} \text{ if } |x-y| \geqslant c_3t^{1/2}, \tag{1.8}$$

for some $\delta > 0$, c_3, $c_4 > 0$.

Let b be a locally integrable function on $\mathbb{R}^n$. The Toeplitz type operator associated to singular integral operator with non-smooth kernel and fractional integral operator I_α is defined by

$$T_b = \sum_{k=1}^m (T^{k,1}M_bI_\alpha T^{k,2} + T^{k,3}I_\alpha M_bT^{k,4}), \tag{1.9}$$

where $T^{k,1}$ are the singular integral operator with non-smooth kernel T or $\pm I$ (the identity operator), $T^{k,2}$ and $T^{k,4}$ are the linear operators, $T^{k,3}=\pm I$, $k=1,\cdots,m$.

Note that the commutators $[b,I_\alpha](f)=bI_\alpha(f)-I_\alpha(bf)$ are the particular operators of the Toeplitz type operator T_b. The Toeplitz type operator T_b is the non-trivial generalization of these commutators.

It is well known that the commutators of fractional integral have been widely studied by many authors. Paluszyński [11] showed that $b\in Lip_\beta$ (homogeneous Lipschitz space) if and only if $[b,I_\alpha]$ is bounded from L^p to L^q, where $0<\beta<1$, $1<p<n/(\alpha+\beta)$ and $1/q=1/p-(\alpha+\beta)/n$. When b belongs to the weighted Lipschitz spaces $Lip_{\beta,\omega}$, Hu and Gu [12] proved that $[b,I_\alpha]$ is bounded from $L^p(\omega)$ to $L^q(\omega^{1-(1-\alpha/n)q})$ for $1/q=1/p-(\alpha+\beta)/n$ with $1<p<n/(\alpha+\beta)$. A similar result obtained when I_α is replaced by the generalized fractional integral operator [13]. In [14], by the sharp maximal function estimates for T_b, the author obtained the boundedness of the operator on Lebesgue space when b belongs to Lipschitz space. Our work is motivated by these papers. In this paper, we establish the sharp maximal function estimates for the Toeplitz type operator associated to singular integral operator with non-smooth kernel, fractional integral operator and weighted Lipschitz spaces. As an application, the boundedness of the operator on weighted Lebesgue space is obtained. The main results are as follows.

THEOREM 1.1. *Suppose T is a singular integral operator with non-smooth kernel. Let $b\in Lip_{\beta,\omega}$ $(0<\beta<1)$, $1/q=1/r_0-\beta/n$, $1/r_0=1/p-\alpha/n$, and $\mu=\omega^{r_0/p}\in A_1$. If $T_1(f)=0$ for any $f\in L^p(\omega)$ $(1<p<n/(\alpha+\beta))$, then there exists a constant $C>0$ such that, for any $1<r<p$ and $x\in\mathbb{R}^n$,*

$$\begin{aligned}
&M_A^\sharp(T_b(f))(x)\\
&\leqslant C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-r_0\alpha\beta/n^2}M_{\beta,\mu,r}(I_\alpha T^{k,2}f)(x)\\
&\quad+C\|b\|_{Lip_{\beta,\omega}}\left(\omega(x)^{1-\alpha/n}M_{\alpha+\beta,\omega,r}(T^{k,4}f)(x)+\omega(x)^{1+\beta/n}M_{\alpha+\beta}(T^{k,4}(f))(x)\right).
\end{aligned}$$

THEOREM 1.2. *Suppose T is a singular integral operator with non-smooth kernel. Let $\omega^{q/p}\in A_1$, $b\in Lip_{\beta,\omega}$ $(0<\beta<1)$ and $1/q=1/p-(\alpha+\beta)/n$. If $T_1(f)=0$ for any $f\in L^p(\omega)$ $(1<p<n/(\alpha+\beta))$, $T^{k,2}$ and $T^{k,4}$ are the bounded operators on $L^p(\omega)$, $k=1,\cdots,m$, then there exists a constant $C>0$ such that,*

$$\|T_b(f)\|_{L^q(\omega^{1-(1-\alpha/n)q})}\leqslant C\|b\|_{Lip_{\beta,\omega}}\|f\|_{L^p(\omega)}.$$

2. Some preliminaries

A weight ω is a nonnegative, locally integrable function on R^n. Let $B=B(x_0,r)$ denote the ball with the center x_0 and radius r. For a given weight function ω and a measurable set E, we also denote the Lebesgue measure of E by $|E|$ and set weighted measure $\omega(E)=\int_E\omega(x)dx$. For any given weight function ω on $\mathbb{R}^n$, $0<p<\infty$, denote by $L^p(\omega)$ the space of all function f satisfying

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx\right)^{1/p} < \infty.$$

A weight ω is said to belong to the Muckenhoupt class A_p for $1 < p < \infty$, if there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \leqslant C \tag{2.1}$$

for every ball $B \subset \mathbb{R}^n$. The class A_1 is defined by replacing the above inequality with

$$\frac{1}{|B|}\int_B \omega(y)dy \leqslant C \cdot \operatorname*{ess\,inf}_{x\in B} w(x) \tag{2.2}$$

for every ball $B \subset \mathbb{R}^n$.

The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p-boundedness of Hardy-Littlewood maximal function in [15]. We also need another weight class $A_{p,q}$ introduced by Muckenhoupt and Wheeden in [16]. Let p' be the dual of p such that $1/p + 1/p' = 1$. A weight function ω belongs to $A_{p,q}$ for $1 < p < q < \infty$, if for every ball B in $\mathbb{R}^n$, there exists a positive constant C which is independent of B such that

$$\left(\frac{1}{|B|}\int_B \omega(y)^{-p'}dy\right)^{1/p'}\left(\frac{1}{|B|}\int_B \omega(y)^q dy\right)^{1/q} \leqslant C. \tag{2.3}$$

From the definition of $\omega \in A_{p,q}$, we can get that

$$\omega \in A_{p,q}, \text{ iff } \omega^q \in A_{1+q/p'}. \tag{2.4}$$

LEMMA 2.1. (see [15]) *Suppose* $\omega \in A_1$. *Then*

(i) there exists a $\varepsilon > 0$ *such that*

$$\left(\frac{1}{|B|}\int_B \omega(x)^{1+\varepsilon}dx\right)^{1/(1+\varepsilon)} \leqslant \frac{C}{|B|}\int_B \omega(x)dx; \tag{2.5}$$

(ii) there exist two constant C_1 *and* C_2, *such that*

$$C_1\omega(B) \leqslant |B| \mathop{inf}_{x\in B}\omega(x) \leqslant C_2\omega(B). \tag{2.6}$$

Let us recall the definition of weighted Lipschitz function space.

DEFINITION 2.1. For $1 \leqslant p < \infty$, $0 < \beta < 1$, and $\omega \in A_\infty$. A locally integrable function b is said to be in the weighted Lipschitz function space if

$$\sup_B \frac{1}{\omega(B)^{\beta/n}}\left[\frac{1}{\omega(B)}\int_B |b(x) - b_B|^p \omega(x)^{1-p}dx\right]^{1/p} \leqslant C < \infty, \tag{2.7}$$

where $b_B = |B|^{-1}\int_B b(y)dy$, and the supremum is taken over all balls $B \subset \mathbb{R}^n$.

The Banach space of such functions modulo constants is denoted by $Lip_{\beta,p}(\omega)$. The smallest bound C satisfying conditions above is then taken to be the norm of b denoted by $\|b\|_{Lip_{\beta,p}(\omega)}$. Put $Lip_{\beta,\omega} = Lip_{\beta,1}(\omega)$. Obviously, for the case $\omega = 1$, the $Lip_{\beta,p}(\omega)$ space is the classical Lip_β space. Let $\omega \in A_1$. García-Cuerva in [17] proved that the spaces $Lip_{\beta,p}(\omega)$ coincide, and the norms $\|b\|_{Lip_{\beta,p}(\omega)}$ are equivalent with respect to different values of p provided that $1 \leqslant p < \infty$. Since we always discuss under the assumption $\omega \in A_1$ in the following, then we denote the norm of $Lip_{\beta,p}(\omega)$ by $\|\cdot\|_{Lip_{\beta,\omega}}$ for $1 \leqslant p < \infty$.

Now we shall introduce the Hardy-Littlewood maximal operator and several variants.

DEFINITION 2.2. The Hardy-Littlewood maximal operator M is defined by

$$M(f)(x) = \sup_{x\in B} \frac{1}{|B|} \int_B |f(y)|dy.$$

For $0 < \alpha < n$, the maximal fractional function $M_\alpha(f)$ is defined by

$$M_\alpha(f)(x) = \sup_{x\in B} \frac{1}{|B|^{1-\alpha/n}} \int_B |f(y)|dy.$$

For $0 < \beta < n$, and $r \geqslant 1$, we define the fractional weighted maximal operator $M_{\beta,r,\omega}$ by

$$M_{\beta,\omega,r}(f)(x) = \sup_{x\in B} \left(\frac{1}{\omega(B)^{1-r\beta/n}} \int_B |f(y)|^r \omega(y)dy \right)^{1/r},$$

where the supremum is taken over all ball B containing x.

DEFINITION 2.3. For $0 < \alpha < n$, the fractional integral operator I_α is defined by

$$I_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

LEMMA 2.2. (see [16]) *Let $0 < \alpha < n$, $1/q = 1/p - \alpha/n$ and $\omega \in A_{p.q}$. Then*

$$\|I_\alpha(f)\|_{L^q(\omega^q)} \leqslant C\|f\|_{L^p(\omega^p)}, \text{ and } \|M_\alpha(f)\|_{L^q(\omega^q)} \leqslant C\|f\|_{L^p(\omega^p)}. \tag{2.8}$$

LEMMA 2.3. *Let I_α be fractional integral operator, and let E be a measurable set in $\mathbb{R}^n$. Then for any $f \in L^1(\mathbb{R}^n)$, there exists a constant C such that*

$$\int_E |I_\alpha f(x)|\, dx \leqslant C\|f\|_{L^1} |E|^{\alpha/n}. \tag{2.9}$$

Proof. Since

$$\begin{aligned} &|\{x\in E: |I_\alpha f(x)|>\lambda\}| \\ &\leqslant |\{x\in R^n: |I_\alpha f(x)|>\lambda\}| \\ &\leqslant C\left(\frac{\|f\|_{L^1}}{\lambda}\right)^{n/(n-\alpha)}, \end{aligned}$$

we have

$$\begin{aligned} \int_E |I_\alpha f(x)|dx &= \int_0^\infty |\{x\in E: |I_\alpha f(x)|>\lambda\}|\,d\lambda \\ &\leqslant \int_0^\infty \min\left\{C\Big(\frac{\|f\|_{L^1}}{\lambda}\Big)^{n/(n-\alpha)},\ |E|\right\}d\lambda \\ &\leqslant \int_0^{C\|f\|_{L^1}|E|^{\alpha/n-1}} |E|d\lambda + \int_{C\|f\|_{L^1}|E|^{\alpha/n-1}}^\infty \left(\frac{\|f\|_{L^1}}{\lambda}\right)^{n/(n-\alpha)} d\lambda \\ &\leqslant C\|f\|_{L^1}|E|^{\alpha/n}. \quad \square \end{aligned}$$

We also need the following conclusions about the sharp maximal function $M_A^\sharp$ associated with the approximation to the identity $\{A_t, t>0\}$.

LEMMA 2.4. [8] *Let $\{A_t, t>0\}$ be an approximation to the identity. Thus, for any $1<p<\infty$, $f\in L^p(\omega)$, and $\omega\in A_p$, we have*

$$\|M(f)\|_{L^p(\omega)} \leqslant C\|M_A^\sharp f\|_{L^p(\omega)}. \tag{2.10}$$

LEMMA 2.5. [18] *Let $\{A_t, t>0\}$ be an approximation to the identity, I_α be fractional integral, and $\tilde{K}_{\alpha,t}(x,y)$ be the kernel of difference operator $I_\alpha - A_t I_\alpha$. Then*

$$|\tilde{K}_{\alpha,t}(x,y)| \leqslant C\frac{t}{|x-y|^{n+2-\alpha}}. \tag{2.11}$$

The following lemma play a key role in the proof of Theorem 1.1.

LEMMA 2.6. *Let $0<\alpha<n$, and let $0<\beta<1$. Suppose $b\in Lip_{\beta,\omega}$, $1/q = 1/r_0-\beta/n$, $1/r_0 = 1/p-\alpha/n$, and $\mu=\omega^{r_0/p}\in A_1$. Then there exist a sufficiently large number s and a constant $C>0$ such that, for every $f\in L^p(\omega)$ with $1<r<p<n/\alpha$, we have*

$$\left(\frac{1}{|B|}\int_B |b(x)-b_B|^{s'}|f(x)|^{s'}dx\right)^{1/s'} \leqslant C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-\alpha\beta r_0/n^2}M_{\beta,\mu,r}(f)(x), \tag{2.12}$$

where $1/s+1/s'=1$.

Proof. Let $r_2 = r/s'$, $r_3 = \varepsilon/(s'-1)$ and $1/r_1 + 1/r_2 + 1/r_3 = 1$, where ε is the constant in (2.5). Choosing a sufficiently large number s such that $1 < s' < r(1+\varepsilon)/(r+\varepsilon)$, then r_1, r_2, $r_3 > 1$. By Hölder's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B |b(x)-b_B|^{s'}|f(x)|^{s'}dx\right)^{1/s'}\\
&= |B|^{-1/s'}\left(\int_B |b(x)-b_B|^{s'}\omega(x)^{1/r_1-s'}|f(x)|^{s'}\omega(x)^{\frac{1}{r_2}}\omega(x)^{s'-1/r_1-1/r_2}dx\right)^{1/s'}\\
&\leqslant C|B|^{-1/s'}\left(\int_B |b(x)-b_B|^{r_1s'}\omega(x)^{1-r_1s'}dx\right)^{1/(r_1s')}\\
&\quad\times\left(\int_B |f(x)|^{r_2s'}\omega(x)dx\right)^{1/(r_2s')}\left(\int_B \omega(x)^{1+r_3(s'-1)}dx\right)^{1/(r_3s')}.
\end{aligned}$$

Since $\mu = \omega^{r_0/p} \in A_1$, then

$$\left(\inf_{x\in B}\omega(x)\right)^{r_0/p-1}\omega(B) \leqslant \mu(B) \leqslant C\left(\inf_{x\in B}\omega(x)\right)^{r_0/p-1}\omega(B). \tag{2.13}$$

By $b \in Lip_{\beta,\omega}$, and (2.5), we have

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B |b(x)-b_B|^{s'}|f(x)|^{s'}dx\right)^{1/s'}\\
&\leqslant C\|b\|_{Lip_{\beta,\omega}}|B|^{-1/s'}\omega(B)^{\beta/n+1/(r_1s')}\\
&\quad\times\left(\int_B |f(x)|^r\omega(x)dx\right)^{1/r}\left(\int_B \omega(x)^{1+\varepsilon}dx\right)^{1/(r_3s')}\\
&\leqslant C\|b\|_{Lip_{\beta,\omega}}\frac{\omega(B)^{1+\beta/n}}{|B|}\left(\frac{1}{\omega(B)}\int_B |f(x)|^r\omega(x)dx\right)^{1/r}\\
&\leqslant C\|b\|_{Lip_{\beta,\omega}}\frac{\omega(B)^{1+\beta/n}}{|B|}\left(\frac{1}{\mu(B)}\int_B |f(x)|^r\mu(x)dx\right)^{1/r}.
\end{aligned}$$

But

$$\begin{aligned}
\omega(B)^{\beta/n} &= \left(\inf_{x\in B}\omega(x)\right)^{-\alpha\beta r_0/n^2}\left(\int_B \omega(x)\left(\inf_{x\in B}\omega(x)\right)^{\alpha r_0/n}dx\right)^{\beta/n}\\
&\leqslant C\left(\inf_{x\in B}\omega(x)\right)^{-\alpha\beta r_0/n^2}\mu(B)^{\beta/n}.
\end{aligned} \tag{2.14}$$

Since $1/q = 1/r_0 - \beta/n$, we have $1 - \alpha\beta r_0/n^2 > 0$. Then

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B |b(y)-b_B|^{s'}|f(y)|^{s'}dy\right)^{1/s'} \\
&\leqslant C\|b\|_{Lip_{\beta,\omega}}\left(\inf_{x\in B}\omega(x)\right)^{1-\alpha\beta r_0/n^2} M_{\beta,\mu,r}(f)(x) \\
&\leqslant C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-\alpha\beta r_0/n^2}M_{\beta,\mu,r}(f)(x). \quad \square
\end{aligned}$$

3. Proof of Theorems

Proof of Theorem 1.1. Fix a ball $B = B(x_0, r_B)$ and $B \ni x$. It suffices to prove for $f \in C_0^\infty(\mathbb{R}^n)$, the following inequality holds:

$$\begin{aligned}
&\frac{1}{|B|}\int_B |T_b(f)(y) - A_{t_B}(T_b(f)(y))|dy \\
&\leqslant C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-r_0\alpha\beta/n^2}M_{\beta,\mu,r}(I_\alpha T^{k,2}f)(x) \\
&\quad + C\|b\|_{Lip_{\beta,\omega}}\left(\omega(x)^{1-\alpha/n}M_{\alpha+\beta,\omega,r}(T^{k,4}f)(x) + \omega(x)^{1+\beta/n}M_{\alpha+\beta}(T^{k,4}(f))(x)\right),
\end{aligned}$$

where $t_B = r_B^2$. Without loss generality, we may assume $T^{k,1}$ are T $(k = 1,\cdots,m)$. We write, by $T_1(f) = 0$,

$$\begin{aligned}
T_b(f)(y) &= \sum_{k=1}^m T^{k,1}M_bI_\alpha T^{k,2}(f)(y) + \sum_{k=1}^m T^{k,3}I_\alpha M_b T^{k,2}(f)(y) \\
&= \mathscr{U}_b(y) + \mathscr{V}_b(y) = \mathscr{U}_{b-b_{2B}}(y) + \mathscr{V}_{b-b_{2B}}(y),
\end{aligned}$$

where,

$$\begin{aligned}
\mathscr{U}_{b-b_{2B}}(y) &= \sum_{k=1}^m T^{k,1}M_{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f)(y) + \sum_{k=1}^m T^{k,1}M_{(b-b_{2B})\chi_{(2B)^c}}I_\alpha T^{k,2}(f)(y) \\
&= \mathrm{U}_1(y) + \mathrm{U}_2(y).
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{V}_{b-b_{2B}}(y) &= \sum_{k=1}^m T^{k,3}I_\alpha M_{(b-b_{2B})\chi_{2B}}T^{k,4}(f)(y) + \sum_{k=1}^m T^{k,3}I_\alpha M_{(b-b_{2B})\chi_{(2B)^c}}T^{k,4}(f)(y) \\
&= \mathrm{V}_1(y) + \mathrm{V}_2(y).
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{|B|}\int_B |T_b(f)(y) - A_{t_B}(T_b(f))(y)|dy \\
&\leqslant \frac{1}{|B|}\int_B |\mathrm{U}_1(y)|dy + \frac{1}{|B|}\int_B |\mathrm{V}_1(y)|dy \\
&\quad + \frac{1}{|B|}\int_B |A_{t_B}(\mathrm{U}_1)(y)|dy + \frac{1}{|B|}\int_B |A_{t_B}(\mathrm{V}_1)(y)|dy \\
&\quad + \frac{1}{|B|}\int_B |\mathrm{U}_2(y) - A_{t_B}(\mathrm{U}_2)(y)|dy + \frac{1}{|B|}\int_B |\mathrm{V}_2(y) - A_{t_B}(\mathrm{V}_2)(y)|dy \\
&= M_1 + M_2 + M_3 + M_4 + M_5 + M_6.
\end{aligned}$$

We are going to estimate each term, respectively. Choosing a sufficiently large number $s,$ by Hölder's inequality, the boundedness of $T^{k,1}$ and Lemma 2.6, we have

$$\begin{aligned}
&\frac{1}{|B|}\int_B |T^{k,1}M_{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f)(y)|dy \\
&\leqslant \left(\frac{1}{|B|}\int_B |T^{k,1}M_{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f)(y)|^{s'}dy\right)^{1/s'} \\
&\leqslant C\left(\frac{1}{|B|}\int_{R^n} |M_{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f)(y)|^{s'}dy\right)^{1/s'} \\
&\leqslant C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-r_0\alpha\beta/n^2}M_{\beta,\mu,r}(I_\alpha T^{k,2}f)(x).
\end{aligned}$$

Then

$$\begin{aligned}
M_1 &\leqslant \sum_{k=1}^m \frac{1}{|B|}\int_B |T^{k,1}M_{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f)(y)|dy \\
&\leqslant C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-r_0\alpha\beta/n^2}M_{\beta,\mu,r}(I_\alpha T^{k,2}f)(x).
\end{aligned}$$

By Lemma 2.3 and Hölder's inequality, we deduce that

$$\begin{aligned}
M_2 &\leqslant \sum_{k=1}^m \frac{1}{|B|}\int_B |I_\alpha M_{(b-b_{2B})\chi_{2B}}T^{k,4}(f)(y)|dy \\
&\leqslant \sum_{k=1}^m \frac{C}{|B|^{1-\alpha/n}}\int_{R^n} |M_{(b-b_{2B})\chi_{2B}}T^{k,4}(f)(y)|dy \\
&\leqslant \sum_{k=1}^m \frac{C}{|B|^{1-\alpha/n}}\left(\int_{2B}|b(y)-b_{2B}|^{r'}\omega(y)^{1-r'}dy\right)^{1/r'}\left(\int_{2B}|T^{k,4}(f)(y)|^r\omega(y)dy\right)^{1/r} \\
&\leqslant C\|b\|_{Lip_{\beta,\omega}}M_{\alpha+\beta,\omega,r}(T^{k,4}f)(x)\left(\frac{\omega(B)}{|B|}\right)^{1-\alpha/n} \\
&\leqslant C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-\alpha/n}M_{\alpha+\beta,\omega,r}(T^{k,4}f)(x).
\end{aligned}$$

For M_3, by (1.2), we have $h_{t_B}(y,z) \leqslant Ct_B^{-n/2}s(2^{2(j-1)})$ for $y \in B$, $z \in 2^{j+1}B \setminus 2^jB$. Then, by (1.3) and the estimates for M_1, we get

$$\begin{aligned}
&\frac{1}{|B|}\int_B |A_{t_B}(T^{k,1}M_{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f))(y)|dy \\
\leqslant\ & \frac{C}{|B|}\int_B\int_{\mathbb{R}^n} h_{t_B}(y,z)|T^{k,1}M_{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f)(z)|dzdy \\
\leqslant\ & \frac{C}{|B|}\int_B\int_{2B} r_B^{-n}|T^{k,1}M_{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f)(z)|dzdy \\
&+C\sum_{j=1}^{\infty} r_B^{-n}s(2^{2(j-1)})\frac{1}{|B|}\int_B\int_{2^{j+1}B\setminus 2^jB}|T^{k,1}M_{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f)(z)|dzdy \\
\leqslant\ & \frac{C}{|B|}\int_{2B}|T^{k,1}M_{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f)(z)|dz \\
&+C\sum_{j=1}^{\infty} 2^{(j-1)(n+\varepsilon)}s(2^{2(j-1)})2^{-j\varepsilon}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|T^{k,1}M_{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f)(z)|dz \\
\leqslant\ & C\left(\frac{1}{|B|}\int_{R^n}|M_{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f)(z)|^{s'}dz\right)^{1/s'} \\
&+C\sum_{j=1}^{\infty}2^{-j\varepsilon}\left(\frac{1}{|2^{j+1}B|}\int_{R^n}|M_{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f)(z)|^{s'}dz\right)^{1/s'} \\
\leqslant\ & C(1+\sum_{j=1}^{\infty}2^{-j\varepsilon})\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-r_0\alpha\beta/n^2}M_{\beta,\mu,r}(I_\alpha T^{k,2}f)(x) \\
\leqslant\ & C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-r_0\alpha\beta/n^2}M_{\beta,\mu,r}(I_\alpha T^{k,2}f)(x).
\end{aligned}$$

Thus

$$\begin{aligned}
M_3 &\leqslant \sum_{k=1}^{m}\frac{1}{|B|}\int_B|A_{t_B}(T^{k,1}M_{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f))(y)|dy \\
&\leqslant C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-r_0\alpha\beta/n^2}M_{\beta,\mu,r}(I_\alpha T^{k,2}f)(x).
\end{aligned}$$

Since $T^{k,3} = \pm I$, by the estimates in M_2, we have

$$\begin{aligned}
M_4 \leqslant & \sum_{k=1}^{m}\frac{1}{|B|}\int_B|A_{t_B}(T^{k,3}I_\alpha M_{(b-b_{2B})\chi_{2B}}T^{k,4}(f))(y)|dy \\
\leqslant\ & \sum_{k=1}^{m}\frac{C}{|B|}\int_B\int_{\mathbb{R}^n}h_{t_B}(y,z)|T^{k,3}I_\alpha M_{(b-b_{2B})\chi_{2B}}T^{k,4}(f)(z)|dzdy \\
\leqslant\ & \sum_{k=1}^{m}\frac{C}{|B|}\int_{2B}|I_\alpha M_{(b-b_{2B})\chi_{2B}}T^{k,4}(f)(z)|dz \\
&+C\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{(j-1)(n+\varepsilon)}s(2^{2(j-1)})2^{-j\varepsilon}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|I_\alpha M_{(b-b_{2B})\chi_{2B}}T^{k,4}(f)(z)|dz
\end{aligned}$$

$$\begin{aligned}
&\leqslant \sum_{k=1}^{m} \frac{C}{|B|^{1-\alpha/n}} \int_{R^n} |M_{(b-b_{2B})\chi_{2B}} T^{k,4}(f)(z)|dz \\
&\quad + C\sum_{k=1}^{m}\sum_{j=1}^{\infty} 2^{-j\varepsilon} \frac{1}{|2^{j+1}B|^{1-\alpha/n}} \int_{R^n} |M_{(b-b_{2B})\chi_{2B}} T^{k,4}(f)(z)|dz \\
&\leqslant C\|b\|_{Lip_{\beta,\omega}} \omega(x)^{1-\alpha/n} M_{\alpha+\beta,\omega,r}(T^{k,4}f)(x).
\end{aligned}$$

For M_5, by (1.8), we get,

$$\begin{aligned}
&|T^{k,1}M_{(b-b_{2B})\chi_{(2B)^c}} I_\alpha T^{k,2}(f)(y) - A_{t_B}(T^{k,1}M_{(b-b_{2B})\chi_{(2B)^c}} I_\alpha T^{k,2}(f))(y)| \\
&\leqslant C\int_{(2B)^c} |b(z)-b_{2B}||K(y-z)-K_{t_B}(y-z)||I_\alpha T^{k,2}(f)(z)|dz \\
&\leqslant C\sum_{j=1}^{\infty} \int_{2^{j+1}B\setminus 2^jB} |b(z)-b_{2B}| \frac{r_B^\delta}{|z-x_0|^{n+\delta}} |I_\alpha T^{k,2}(f)(z)|dz \\
&\leqslant C\sum_{j=1}^{\infty} 2^{-j\delta} \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b(z)-b_{2B}||I_\alpha T^{k,2}(f)(z)|dz \\
&\leqslant C\sum_{j=1}^{\infty} 2^{-j\delta} |b_{2^{j+1}B}-b_{2B}| \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |I_\alpha T^{k,2}(f)(z)|dz \\
&\quad + C\sum_{j=1}^{\infty} 2^{-j\delta} \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b(z)-b_{2^{j+1}B}||I_\alpha T^{k,2}(f)(z)|dz \\
&= M_{51} + M_{52}.
\end{aligned}$$

Note that

$$\begin{aligned}
\omega(2^{k+1}B)^{\beta/n} &\leqslant C \inf_{x\in 2^{k+1}B} \omega(x)^{-\alpha\beta r_0/n^2} \mu(2^{k+1}B)^{\beta/n} \\
&\leqslant C \inf_{x\in 2^{k+1}B} \omega(x)^{-\alpha\beta r_0/n^2} \mu(2^{j+1}B)^{\beta/n},
\end{aligned}$$

then

$$\begin{aligned}
|b_{2^{j+1}B} - b_{2B}| &\leqslant \sum_{k=1}^{j} \frac{1}{|2^kB|} \int_{2^{k+1}B} |b(z)-b_{2^{k+1}B}|dz \\
&\leqslant C\|b\|_{Lip_{\beta,\omega}} \sum_{k=1}^{j} \frac{\omega(2^{k+1}B)^{1+\beta/n}}{|2^kB|} \\
&\leqslant C\|b\|_{Lip_{\beta,\omega}} \sum_{k=1}^{j} \inf_{x\in 2^{k+1}B} \omega(x)\omega(2^{k+1}B)^{\beta/n} \\
&\leqslant C\|b\|_{Lip_{\beta,\omega}} \sum_{k=1}^{j} \inf_{x\in 2^{k+1}B} (\omega(x))^{1-\alpha\beta r_0/n^2} \mu(2^jB)^{\beta/n} \\
&\leqslant Cj\|b\|_{Lip_{\beta,\omega}} \omega(x)^{1-\alpha\beta r_0/n^2} \mu(2^jB)^{\beta/n}.
\end{aligned}$$

Thus

$$\begin{aligned}
M_{51} \leqslant & C\sum_{j=1}^{\infty} 2^{-j\delta}|b_{2^{j+1}B}-b_{2B}|\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|I_\alpha T^{k,2}(f)(z)|dz \\
\leqslant & C\|b\|_{Lip_{\beta,\omega}}\sum_{j=1}^{\infty} j2^{-j\delta}\omega(x)^{1-\alpha\beta r_0/n^2}\mu(2^jB)^{\beta/n}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|I_\alpha T^{k,2}(f)(z)|dz \\
\leqslant & C\|b\|_{Lip_{\beta,\omega}}\sum_{j=1}^{\infty} j2^{-j\delta}\omega(x)^{1-\alpha\beta r_0/n^2}\left(\frac{1}{\mu(2^jB)^{1-\beta r/n}}\int_{2^{j+1}B}|I_\alpha T^{k,2}(f)(z)|^r\mu(z)dz\right)^{1/r} \\
\leqslant & C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-\alpha\beta r_0/n^2}M_{\beta,\mu,r}(I_\alpha T^{k,2}(f))(x).
\end{aligned}$$

Similar to the proof of Lemma 2.6, we have

$$\begin{aligned}
& \frac{\omega(2^{j+1}B)^{1+\beta/n}}{|2^{j+1}B|}\left(\frac{1}{\omega(2^{j+1}B)}\int_{2^{j+1}B}|I_\alpha T^{k,2}(f)(z)|^r\omega(z)dz\right)^{1/r} \\
\leqslant & \, C\omega(x)^{1-r_0\alpha\beta/n^2}M_{\beta,\mu,r}(I_\alpha T^{k,2}(f))(x).
\end{aligned}$$

Then, by Hölder's inequality,

$$\begin{aligned}
M_{52} \leqslant & \, C\sum_{j=1}^{\infty} 2^{-j\delta}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}||I_\alpha T^{k,2}(f)(z)|dz \\
\leqslant & \, C\sum_{j=1}^{\infty} 2^{-j\delta}\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}|^{r'}\omega(z)^{1-r'}dz\right)^{1/r'} \\
& \times\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|I_\alpha T^{k,2}(f)(z)|^r\omega(z)dz\right)^{1/r} \\
\leqslant & \, C\|b\|_{Lip_{\beta,\omega}}\sum_{j=1}^{\infty} 2^{-j\delta}\frac{\omega(2^{j+1}B)^{1+\beta/n}}{|2^{j+1}B|}\left(\frac{1}{\omega(2^{j+1}B)}\int_{2^{j+1}B}|I_\alpha T^{k,2}(f)(z)|^r\omega(z)dz\right)^{1/r} \\
\leqslant & \, C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-r_0\alpha\beta/n^2}M_{\beta,\mu,r}(I_\alpha T^{k,2}(f))(x)\sum_{j=1}^{\infty}2^{-j\delta} \\
\leqslant & \, C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-r_0\alpha\beta/n^2}M_{\beta,\mu,r}(I_\alpha T^{k,2}(f))(x).
\end{aligned}$$

Hence,

$$M_5 \leqslant C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-r_0\alpha\beta/n^2}\left(M_{\beta,\mu,r}(I_\alpha T^{k,2}(f))(x)+M_{\beta,\mu,1}(I_\alpha T^{k,2}(f))(x)\right).$$

Since $T^{k,3}=\pm I$, by Lemma 2.5, we get

$$\begin{aligned}
& |T^{k,3}M_{(b-b_{2B})\chi_{(2B)^c}}I_\alpha T^{k,4}(f)(y)-A_{t_B}(T^{k,3}M_{(b-b_{2B})\chi_{(2B)^c}}I_\alpha T^{k,4}(f))(y)| \\
\leqslant & \, C\int_{(2B)^c}|b(y)-b_{2B}||\tilde{K}_{t_B}(y-z)||T^{k,4}(f)(z)|dz
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^{j}B}|b(z)-b_{2B}|\frac{r_B^2}{|z-x_0|^{n+2-\alpha}}|T^{k,4}(f)(z)|dz\\
&\leqslant C\sum_{j=1}^{\infty}2^{-2j}\frac{1}{|2^{j+1}B|^{1-\frac{\alpha}{n}}}\int_{2^{j+1}B}|b(z)-b_{2B}||T^{k,4}(f)(z)|dz\\
&\leqslant C\sum_{j=1}^{\infty}2^{-2j}|b_{2^{j+1}B}-b_{2B}|\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|T^{k,4}(f)(z)|dz\\
&\quad+C\sum_{j=1}^{\infty}2^{-2j}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}||T^{k,4}(f)(z)|dz\\
&=M_{61}+M_{62}.
\end{aligned}
$$

Since

$$
\begin{aligned}
|b_{2^{j+1}B}-b_{2B}| &\leqslant C\|b\|_{Lip_{\beta,\omega}}\sum_{k=1}^{j}\inf_{x\in 2^{k+1}B}\omega(x)\omega(2^{k+1}B)^{\beta/n}\\
&\leqslant Cj\|b\|_{Lip_{\beta,\omega}}\omega(x)\left(\inf_{x\in 2^{j+1}B}\omega(x)\right)^{\beta/n}|2^{j+1}B|^{\beta/n}\\
&\leqslant Cj\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1+\beta/n}|2^{j+1}B|^{\beta/n},
\end{aligned}
$$

we get

$$
\begin{aligned}
M_{61} &\leqslant C\sum_{j=1}^{\infty}2^{-2j}|b_{2^{j+1}B}-b_{2B}|\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|T^{k,4}(f)(z)|dz\\
&\leqslant C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1+\beta/n}\sum_{j=1}^{\infty}j2^{-2j}\frac{1}{|2^{j+1}B|^{1-(\alpha+\beta)/n}}\int_{2^{j+1}B}|T^{k,4}(f)(z)|dz\\
&\leqslant C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1+\beta/n}M_{\alpha+\beta}(T^{k,4}(f))(x)\sum_{j=1}^{\infty}j2^{-2j}\\
&\leqslant C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1+\beta/n}M_{\alpha+\beta}(T^{k,4}(f))(x).
\end{aligned}
$$

By Hölder's inequality,

$$
\begin{aligned}
M_{62} &\leqslant C\sum_{j=1}^{\infty}2^{-2j}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}||T^{k,4}(f)(z)|dz\\
&\leqslant C\sum_{j=1}^{\infty}2^{-2j}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\left(\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}|^{r'}\omega(z)^{1-r'}dz\right)^{1/r'}\\
&\qquad\times\left(\int_{2^{j+1}B}|T^{k,2}(f)(z)|^{r}\omega(z)dz\right)^{1/r}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\|b\|_{Lip_{\beta,\omega}} \sum_{j=1}^{\infty} 2^{-2j} \left(\frac{\omega(2^{j+1}B)}{|2^{j+1}B|} \right)^{1-\alpha/n} \\
&\quad \times \left(\frac{1}{\omega(2^{j+1}B)^{1-(\alpha+\beta)r/n}} \int_{2^{j+1}B} |T^{k,2}(f)(z)|^r \omega(z)dz \right)^{1/r} \\
&\leqslant C\|b\|_{Lip_{\beta,\omega}} \omega(x)^{1-\alpha/n} M_{\alpha+\beta,\omega,r}(T^{k,4}(f))(x).
\end{aligned}
$$

Then

$$
M_6 \leqslant C\|b\|_{Lip_{\beta,\omega}} \omega(x)^{1-\alpha/n} \Big(M_{\alpha+\beta,\omega,r}(T^{k,4}(f))(x) + M_{\alpha+\beta,\omega,1}(T^{k,4}(f))(x) \Big).
$$

Combining the above estimates for $M_1 - M_6$, the proof of Theorem 1.1 is finished. $\square$

Proof of Theorem 1.2. Let $1/q = 1/r_0 - \beta/n$, and $1/r_0 = 1/p - \alpha/n$. Then

$$
(1 - \alpha\beta r_0/n^2)q + 1 - (1 - \alpha/n)q = r_0/p,
$$

and

$$
(1 + \beta/n)q + 1 - (1 - \alpha/n)q = q/p.
$$

Since $\omega^{q/p} \in A_1$, we have $\mu = \omega^{r_0/p} \in A_1$, and $\omega \in A_1$. By (2.4) we have $\omega^{1/p} \in A_{r_0,p}$. Thus, by Lemma 2.4, Theorem 1.1 and Lemma 2.2, we have

$$
\begin{aligned}
&\|T_b(f)\|_{L^q(\omega^{1-(1-\alpha/n)q})} \\
&\leqslant C\|M_A^{\sharp} T_b(f)\|_{L^q(\omega^{1-(1-\alpha/n)q})} \\
&\leqslant C\|b\|_{Lip_{\beta,\omega}} \Big(\Big\| \omega(\cdot)^{1-\alpha\beta r_0/n^2} M_{\beta,\mu,r}(I_\alpha T^{k,4} f) \Big\|_{L^q(\omega^{1-(1-\alpha/n)q})} \\
&\quad + \Big\| \omega(\cdot)^{1-\alpha/n} M_{\alpha+\beta,\omega,r}(T^{k,4} f) \Big\|_{L^q(\omega^{1-(1-\alpha/n)q})} \\
&\quad + \Big\| \omega(\cdot)^{1+\beta/n} M_{\alpha+\beta}(T^{k,4}(f)) \Big\|_{L^q(\omega^{1-(1-\alpha/n)q})} \Big) \\
&\leqslant C\|b\|_{Lip_{\beta,\omega}} \Big(\Big\| M_{\beta,\mu,r}(I_\alpha T^{k,4} f) \Big\|_{L^q(\mu)} \\
&\quad + \|M_{\alpha+\beta,\omega,r}(T^{k,4} f)\|_{L^q(\omega)} + \|M_{\alpha+\beta}(T^{k,4}(f))\|_{L^p(\omega^{q/p})} \Big) \\
&\leqslant C\|b\|_{Lip_{\beta,\omega}} \Big(\|I_\alpha T^{k,4} f\|_{L^{r_0}(\mu)} + \|T^{k,4}(f)\|_{L^p(\omega)} \Big) \\
&\leqslant C\|b\|_{Lip_{\beta,\omega}} \|f\|_{L^p(\omega)}. \quad \square
\end{aligned}
$$

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