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ON CERTAIN ANALYTIC FUNCTIONS

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Abstract. We apply Nunokawa's lemma from the paper: On Properties of Non-Carathéodory Functions, Proc. Japan Acad. 68, Ser. A (1992) 152–153, to prove some new results.

1. Introduction

For integer $n \geqslant 0$, denote by Σ_n the class of meromorphic functions, defined in $\dot{\mathbb{U}} = \{z : 0 < |z| < 1\}$, which are of the form

$$F(z) = \frac{1}{z} + a_n z^n + a_{n+1} z^{n+1} + \cdots.$$

A function $F \in \Sigma_0$ is said to be starlike if it is univalent and the complement of $F(\dot{\mathbb{U}})$ is starlike with respect to the origin. Denote by Σ_0^* the class of such functions. If $F \in \Sigma_0$, then it is well-known that $F \in \Sigma_0^*$ if and only if

$$\mathfrak{Re}\left\{-\frac{zF'(z)}{F(z)}\right\} > 0$$

for $z \in \dot{\mathbb{U}}$. For $\alpha < 1$, let

$$\Sigma_{n,\alpha}^* = \left\{F \in \Sigma_n : \mathfrak{Re}\left\{-\frac{zF'(z)}{F(z)}\right\} > \alpha,\ z \in \dot{\mathbb{U}}\right\},$$

the class of meromorphic-starlike functions of order α. For $0 < \alpha \leqslant 1$, let

$$\Sigma_n^*(\alpha) = \left\{F \in \Sigma_n : \left|\arg\left\{-\frac{zF'(z)}{F(z)}\right\}\right| < \frac{\alpha\pi}{2},\ z \in \dot{\mathbb{U}}\right\} \tag{1.1}$$

the class of meromorphic-strongly starlike functions of order α.

Let p be positive integer and let $\mathscr{A}(p)$ be the class of functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} c_n z^n,$$

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which are analytic in the unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. Furthermore, denote by $\mathscr{A}$ the class of analytic functions in $\mathbb{D}$ and usually normalized, i.e. $\mathscr{A}=\{f\in\mathscr{H}: f(0)=0, f'(0)=1\}$. We say that the $f\in\mathscr{H}$ is subordinate to $g\in\mathscr{H}$ in the unit disc $\mathbb{D}$, written $f\prec g$ if and only if there exists an analytic function $w\in\mathscr{H}$ such that $|w(z)|\leqslant|z|$ and $f(z)=g[w(z)]$ for $z\in\mathbb{D}$. Therefore $f\prec g$ in $\mathbb{D}$ implies $f(\mathbb{D})\subset g(\mathbb{D})$. In particular if g is univalent in $\mathbb{D}$ then the Subordination Principle says that $f\prec g$ if and only if $f(0)=g(0)$ and $f(|z|<r)\subset g(|z|<r)$, for all $r\in(0,1]$.

The subclass of $\mathscr{A}(p)$ consisting of p-valently starlike functions is denoted by $\mathscr{S}^*(p)$. An analytic description of $\mathscr{S}^*(p)$ is given by

$$\mathscr{S}^*(p)=\left\{f\in\mathscr{A}(p):\ \left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{\pi}{2},\ z\in\mathbb{D}\right\}.$$

The subclass of $\mathscr{A}(p)$ consisting of p-valently and strongly starlike functions of order α, $0<\alpha\leqslant 1$ is denoted by $\mathscr{S}^*_\alpha(p)$. An analytic description of $\mathscr{S}^*_\alpha(p)$ is given by

$$\mathscr{S}^*_\alpha(p)=\left\{f\in\mathscr{A}(p):\ \left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{\alpha\pi}{2},\ z\in\mathbb{D}\right\}.$$

The subclass of $\mathscr{A}(p)$ consisting of p-valently convex functions and p-valently strongly convex functions of order α, $0<\alpha\leqslant 1$ are denoted by $\mathscr{C}^*(p)$ and $\mathscr{C}^*_\alpha(p)$ respectively. The analytic descriptions of $\mathscr{C}^*(p)$ and $\mathscr{C}^*_\alpha(p)$ are given by

$$\mathscr{C}^*(p)=\left\{f\in\mathscr{A}(p):\ \left|\arg\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right|<\frac{\pi}{2},\ z\in\mathbb{D}\right\}$$

and

$$\mathscr{C}^*_\alpha(p)=\left\{f\in\mathscr{A}(p):\ \left|\arg\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right|<\frac{\alpha\pi}{2},\ z\in\mathbb{D}\right\}.$$

2. Main result

To prove the main results, we also need the following generalization of Nunokawa's lemma, [3], [4], see also [2].

LEMMA 2.1. [5] *Let $p(z)$ be of the form*

$$p(z)=1+\sum_{n=m\geqslant 1}^{\infty}a_nz^n,\qquad a_m\neq 0,\quad (|z|<1), \tag{2.1}$$

with $p(z)\neq 0$ in $|z|<1$. If there exists a point z_0, $|z_0|<1$, such that

$$|\arg\{p(z)\}|<\pi\alpha/2\quad in\quad |z|<|z_0|$$

and

$$|\arg\{p(z_0)\}|=\pi\alpha/2$$

for some $\alpha > 0$, *then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where

$$k \geqslant m\left(a^2+1\right)/(2a) \quad when \;\; \arg\{p(z_0)\} = \pi\alpha/2 \tag{2.2}$$

and

$$k \leqslant -m\left(a^2+1\right)/(2a) \quad when \quad \arg\{p(z_0)\} = -\pi\alpha/2, \tag{2.3}$$

where

$$\{p(z_0)\}^{1/\alpha} = \pm ia, \; a > 0.$$

3. Main result

THEOREM 3.1. *Let* $p(z)$ *be analytic in* $\mathbb{D}$ *with* $p(0)-1 = p'(0) = 0$. *Assume that* $\alpha \in [0,1/2]$. *If for* $z \in \mathbb{D}$

$$\left|\arg\left\{p(z) - zp'(z)\right\}\right| < \arctan(2\alpha) - \frac{\alpha\pi}{2}, \tag{3.1}$$

then

$$|\arg\{p(z)\}| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{D}). \tag{3.2}$$

Proof. If there exists a point z_0, $|z_0| < 1$, such that

$$|\arg\{p(z)\}| < \pi\alpha/2 \quad (|z| \leqslant |z_0|)$$

and

$$|\arg\{p(z_0)\}| = \pi\alpha/2,$$

then from Nunokawa's lemma 2.1, with $m = 2$, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where k is a real number

$$k \geqslant (a^2+1)/a, \text{ when } \arg\{p(z_0)\} = \pi\alpha/2$$

and

$$k \leqslant -(a^2+1)/a, \text{ when } \arg\{p(z_0)\} = -\pi\alpha/2$$

and where $p(z_0)^{1/\alpha} = \pm ia$, $a > 0$. For the case $\arg\{p(z_0)\} = \pi\alpha/2$, we have

$$\begin{aligned} p(z_0) - z_0 p'(z_0) &= p(z_0)\left(1 - \frac{z_0 p'(z_0)}{p(z_0)}\right) \\ &= (ia)^{\alpha}(1 - i\alpha k), \end{aligned} \tag{3.3}$$

where $k \geqslant (a^2+1)/a$ and $0 < a$. Because, $k \geqslant (a^2+1)/a \geqslant 2$, we have

$$-\frac{\pi}{2} < \arg(1 - i\alpha k) \leqslant -\arctan(2\alpha). \tag{3.4}$$

It is easy to see that for $\alpha \in [0,1/2]$ we have

$$\arctan(2\alpha) - \frac{\alpha\pi}{2} \geqslant 0.$$

Therefore, using (3.3) and (3.4), we obtain

$$\begin{aligned} \arg\{p(z_0) - z_0 p'(z_0)\} &= \arg\{p(z_0)\} + \arg\left\{1 - \frac{z_0 p'(z_0)}{p(z_0)}\right\} \\ &= \arg\{(ia)^\alpha\} + \arg\{1 - i\alpha k\} \\ &\leqslant -\left\{\arctan(2\alpha) - \frac{\alpha\pi}{2}\right\}. \end{aligned} \tag{3.5}$$

This is a contradiction with (3.1). For the case $\arg\{p(z_0)\} = -\pi\alpha/2$, we have

$$\begin{aligned} p(z_0) - z_0 p'(z_0) &= p(z_0)\left(1 - \frac{z_0 p'(z_0)}{p(z_0)}\right) \\ &= (-ia)^\alpha (1 - i\alpha k), \end{aligned} \tag{3.6}$$

where $k \leqslant -(a^2+1)/a \leqslant -2$. We also have

$$\arctan(2\alpha) \leqslant \arg(1 - i\alpha k) < \frac{\pi}{2}. \tag{3.7}$$

Therefore, using (3.6) and (3.7) and applying the same method as above, we obtain

$$\begin{aligned} \arg\{p(z_0) - z_0 p'(z_0)\} &= \arg\{p(z_0)\} + \arg\left\{1 - \frac{z_0 p'(z_0)}{p(z_0)}\right\} \\ &= \arg\{(-ia)^\alpha\} + \arg\{1 - i\alpha k\} \\ &\geqslant \arctan(2\alpha) - \frac{\alpha\pi}{2}. \end{aligned} \tag{3.8}$$

This is also a contradiction with (3.1), and it completes the proof. □

Let us put $p(z) = e^{-i\beta} q(e^{i\beta} z)$ in Theorem 3.1.

COROLLARY 3.2. *Let* $p(z) = e^{-i\beta} q(e^{i\beta} z)$, $\beta \in \mathbb{R}$, *be analytic in* $\mathbb{D}$ *with* $p(0) - 1 = p'(0) = 0$. *Assume that* $\alpha \in [0,1/2]$. *If for* $z \in \mathbb{D}$

$$\left|\arg\left\{e^{-i\beta} q(e^{i\beta} z) - z q'(e^{i\beta} z)\right\}\right| < \arctan(2\alpha) - \frac{\alpha\pi}{2}, \tag{3.9}$$

then

$$\left|\arg\left\{e^{-i\beta} q(e^{i\beta} z)\right\}\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{D}). \tag{3.10}$$

COROLLARY 3.3. *Let* $p(z) = e^{-i\beta}q(e^{i\beta}z)$, $\beta \in \mathbb{R}$, *be analytic in* $\mathbb{D}$ *with* $p(0) - 1 = p'(0) = 0$. *Assume that* $\alpha \in [0,1/2]$. *If for* $z \in \mathbb{D}$

$$\left|\arg\left\{q(e^{i\beta}z) - e^{i\beta}zq'(e^{i\beta}z)\right\} - \beta\right| < \arctan(2\alpha) - \frac{\alpha\pi}{2}, \tag{3.11}$$

then

$$|\arg\left\{q(e^{i\beta}z)\right\} - \beta| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{D}). \tag{3.12}$$

COROLLARY 3.4. *Let* $q(z)$ *be analytic in* $\mathbb{D}$ *with* $q(0) = e^{i\beta}$, $q'(0) = 0$, $\beta \in \mathbb{R}$. *Assume that* $\alpha \in [0,1/2]$. *If for* $z \in \mathbb{D}$

$$\left|\arg\left\{q(z) - zq'(z)\right\} - \beta\right| < \arctan(2\alpha) - \frac{\alpha\pi}{2}, \tag{3.13}$$

then

$$|\arg\{q(z)\} - \beta| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{D}). \tag{3.14}$$

THEOREM 3.5. *Let*

$$F(z) = \frac{1}{z} + a_1z + a_2z^2 + \cdots, \quad z \in \dot{\mathbb{U}}.$$

Assume that $\alpha \in [0,1/2]$. *If for* $z \in \mathbb{D}$

$$\left|\arg\left\{-z^2F'(z)\right\}\right| < \arctan(2\alpha) - \frac{\alpha\pi}{2}, \tag{3.15}$$

then

$$|\arg\{zF(z)\}| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{D}). \tag{3.16}$$

Proof. Let

$$p(z) = zF(z) = 1 + a_1z^2 + a_2z^3 + \cdots, \quad p(0) = 1.$$

Then

$$p(z) - zp'(z) = -z^2F'(z).$$

Applying Theorem 3.1 we obtain the result. □

COROLLARY 3.6. *Let*

$$F(z) = \frac{1}{z} + a_1z + a_2z^2 + \cdots, \quad z \in \dot{\mathbb{U}}.$$

Assume that $\alpha \in [0,1/2]$. *If for* $z \in \mathbb{D}$

$$\left|\arg\left\{-z^2F'(z)\right\}\right| < \arctan(2\alpha) - \frac{\alpha\pi}{2}, \tag{3.17}$$

then $F(z)$ *is meromorphic-strongly starlike function of order* $\arctan(2\alpha)$.

Proof. For showing that $F(z)$ is meromorphic-strongly starlike function we need to show (1.1). By Theorem 3.5 we have $|\arg\{zF(z)\}| < \alpha\pi/2$, since this and since (3.17), we obtain

$$\begin{aligned}\left|\arg\left\{\frac{-zF'(z)}{F(z)}\right\}\right| &= \left|\arg\left\{\frac{-z^2F'(z)}{zF(z)}\right\}\right| \\ &\leqslant \left|\arg\left\{-z^2F'(z)\right\}\right| + |\arg\{zF(z)\}| \\ &< \arctan(2\alpha) - \frac{\alpha\pi}{2} + \frac{\alpha\pi}{2} \\ &= \arctan(2\alpha).\end{aligned}$$

Therefore, $F(z)$ is meromorphic-strongly starlike function of order $\arctan(2\alpha)$. □

For $\alpha \in [0,1/2]$ we have

$$0 \leqslant \arctan(2\alpha) - \frac{\alpha\pi}{2} \leqslant \arctan(2\alpha_0) - \frac{\alpha_0\pi}{2},$$

where

$$\alpha_0 = \sqrt{\frac{4-\pi}{4\pi}} = 0.26\ldots.$$

This gives for $\alpha \in [0,1/2]$ the inequalities

$$0 \leqslant \arctan(2\alpha) - \frac{\alpha\pi}{2} < 0.07\ldots.$$

THEOREM 3.7. *Let*

$$F(z) = \frac{1}{z} + a_1 z + a_2 z^2 + \cdots, \quad z \in \dot{\mathbb{U}}. \tag{3.18}$$

Assume that $\alpha \in [0,1/2]$. *If for* $z \in \mathbb{D}$

$$\left|\arg\left\{\frac{z^2(-F''(z) - (F'(z))^2)}{F^2(z)}\right\}\right| < \arctan(2\alpha) - \frac{\alpha\pi}{2}, \tag{3.19}$$

then

$$|\arg\left\{\frac{-zF'(z)}{F(z)}\right\}| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{D}). \tag{3.20}$$

Proof. Let $p(z) = -zF'(z)/F(z)$. By (3.18) we have that

$$p(z) = 1 + p_2 z^2 + p_3 z^3 + \cdots.$$

Moreover

$$p(z) - zp'(z) = \frac{z^2(-F''(z) - (F'(z))^2)}{F^2(z)}.$$

Applying Theorem 3.1 we obtain the result. □

THEOREM 3.8. *Let $p(z)$ be analytic in $\mathbb{D}$ with $p(0)-1=p'(0)=0$. Assume that*

$$v(\alpha) = -\frac{2\alpha}{1-\alpha}\left(\frac{1-\alpha}{1+\alpha}\right)^{(1+\alpha)/2}\cos\frac{\pi\alpha}{2} \tag{3.21}$$

and

$$u(\alpha) = 1-\frac{2\alpha}{1-\alpha}\left(\frac{1-\alpha}{1+\alpha}\right)^{(1+\alpha)/2}\sin\frac{\pi\alpha}{2}. \tag{3.22}$$

Assume that α_0 is the smallest positive root of the equation

$$u(\alpha)=0, \tag{3.23}$$

$(0.72<\alpha_0<0.75)$.

If for $z\in\mathbb{D}$

$$\left|\arg\left\{p(z)-\frac{zp'(z)}{p(z)}\right\}\right| < \begin{cases} \tan^{-1}\frac{-v(\alpha)}{u(\alpha)}-\frac{\alpha\pi}{2} \text{ when } 0<\alpha<\alpha_0, \\ \pi/2-\frac{\alpha\pi}{2} \text{ when } \alpha=\alpha_0, \\ \pi+\tan^{-1}\frac{-v(\alpha)}{u(\alpha)}-\frac{\alpha\pi}{2} \text{ when } \alpha_0<\alpha<1, \end{cases} \tag{3.24}$$

then

$$|\arg\{p(z)\}| < \frac{\alpha\pi}{2} \quad (z\in\mathbb{D}). \tag{3.25}$$

Proof. If there exists a point z_0, $|z_0|<1$, such that

$$|\arg\{p(z)\}| < \pi\alpha/2 \quad (|z|\leqslant|z_0|)$$

and

$$|\arg\{p(z_0)\}| = \pi\alpha/2,$$

then from Nunokawa's lemma, with $m=2$, we have

$$\frac{z_0p'(z_0)}{p(z_0)} = ik\alpha,$$

where k is a real number

$$k\geqslant(a^2+1)/a, \text{ when } \arg\{p(z_0)\}=\pi\alpha/2$$

and

$$k\leqslant-(a^2+1)/a, \text{ when } \arg\{p(z_0)\}=-\pi\alpha/2$$

and where $p(z_0)^{1/\alpha}=\pm ia$, $a>0$. For the case $\arg\{p(z_0)\}=\pi\alpha/2$, we have

$$\begin{aligned} p(z_0)-\frac{z_0p'(z_0)}{p(z_0)} &= p(z_0)\left(1-\frac{z_0p'(z_0)}{p^2(z_0)}\right) \\ &= (ia)^\alpha\left(1-\frac{i\alpha k}{(ia)^\alpha}\right) \\ &= (ia)^\alpha\left(1-i^{1-\alpha}\frac{\alpha k}{a^\alpha}\right), \end{aligned} \tag{3.26}$$

where $k \geqslant (a^2+1)/a$ and $0 < a$. Then we have

$$\frac{\alpha k}{a^{\alpha}} \geqslant \alpha\left(a^{1-\alpha}+a^{-1-\alpha}\right).$$

Let us put

$$g(a) = a^{1-\alpha}+a^{-1-\alpha}, \quad a > 0.$$

Then, by easy calculation, we have

$$g'(a) = (1-\alpha)a^{-\alpha}-(1+\alpha)a^{-2-\alpha},$$

and $g(a)$ takes its minimum value at $a_0 = \sqrt{(1+\alpha)/(1-\alpha)}$ since

$$g(a) = a^{1-\alpha}+a^{-1-\alpha} \geqslant g(a_0) = \frac{2}{1-\alpha}\left(\frac{1-\alpha}{1+\alpha}\right)^{(1+\alpha)/2},$$

and

$$\frac{\alpha k}{a^{\alpha}} \geqslant \frac{2\alpha}{1-\alpha}\left(\frac{1-\alpha}{1+\alpha}\right)^{(1+\alpha)/2}. \tag{3.27}$$

Because of the facts

$$0 \leqslant \arg\{p(z_0)\} = \pi\alpha/2 \leqslant \pi/2$$

and

$$-\pi \leqslant \arg\left\{1-\frac{z_0p'(z_0)}{p^2(z_0)}\right\} = \arg\left\{1-i^{1-\alpha}\frac{\alpha k}{a^{\alpha}}\right\} < 0,$$

we have

$$\arg\left\{p(z_0)-\frac{z_0p'(z_0)}{p(z_0)}\right\} = \arg\{p(z_0)\}+\arg\left\{1-\frac{z_0p'(z_0)}{p^2(z_0)}\right\} \tag{3.28}$$

$$= \frac{\alpha\pi}{2}+\arg\left\{1-i^{1-\alpha}\frac{\alpha k}{a^{\alpha}}\right\}. \tag{3.29}$$

Observe that

$$1-i^{1-\alpha}\frac{\alpha k}{a^{\alpha}}$$

lies on a half-line with the end point $u(\alpha)+iv(\alpha)$ because of the inequality (3.27). Note that $v(\alpha) < 0$ while $u(\alpha) > 0$ for $0 < \alpha < \alpha_0$ and $u(\alpha) < 0$ for $\alpha_0 < \alpha < 1$, so in the sequel we will have 3 of cases.

$$-\pi+\frac{(1-\alpha)\pi}{2} \leqslant \arg\left\{1-i^{1-\alpha}\frac{\alpha k}{a^{\alpha}}\right\} \leqslant \begin{cases} -\pi-\tan^{-1}\frac{-v(\alpha)}{u(\alpha)} \text{ when } u(\alpha) < 0, \\ -\pi/2 \text{ when } u(\alpha) = 0, \\ -\tan^{-1}\frac{-v(\alpha)}{u(\alpha)} \text{ when } u(\alpha) > 0. \end{cases}$$

Therefore, using this in (3.28) we obtain

$$\begin{aligned}\arg\left\{p(z_0)-\frac{z_0p'(z_0)}{p(z_0)}\right\} &= \arg\{p(z_0)\}+\arg\left\{1-\frac{z_0p'(z_0)}{p^2(z_0)}\right\}\\ &= \arg\{(ia)^{\alpha}\}+\arg\left\{1-\frac{i\alpha k}{(ia)^{\alpha}}\right\}\\ &\leqslant \begin{cases}\frac{\alpha\pi}{2}-\pi-\tan^{-1}\frac{-v(\alpha)}{u(\alpha)} \text{ when } u(\alpha)<0,\\ \frac{\alpha\pi}{2}-\pi/2 \text{ when } u(\alpha)=0,\\ \frac{\alpha\pi}{2}-\tan^{-1}\frac{-v(\alpha)}{u(\alpha)} \text{ when } u(\alpha)>0.\end{cases}\end{aligned} \tag{3.30}$$

It is easy to see that $(\alpha\pi)/2-\pi-\tan^{-1}\frac{-v(\alpha)}{u(\alpha)}<0$, $u(\alpha)<0$, and $(\alpha\pi)/2-\pi/2<0$. Now we shall show that also the third bound is negative. Namely, if $u(\alpha)>0$, then

$$\frac{\alpha\pi}{2}-\tan^{-1}\frac{-v(\alpha)}{u(\alpha)}<0 \;\Leftrightarrow\; \frac{2\alpha}{1-\alpha}\left(\frac{1-\alpha}{1+\alpha}\right)^{(1+\alpha)/2}-\sin\frac{\alpha\pi}{2}>0.$$

Therefore, it suffices to show that

$$\frac{2\alpha}{1-\alpha}\left(\frac{1-\alpha}{1+\alpha}\right)^{(1+\alpha)/2}-\frac{\alpha\pi}{2}>0 \quad \alpha\in[0,\alpha_0)$$

or

$$\frac{2}{1-\alpha}\left(\frac{1-\alpha}{1+\alpha}\right)^{(1+\alpha)/2}-\frac{\pi}{2}>0 \quad \alpha\in[0,\alpha_0).$$

Let $G(\alpha)$ be defined by

$$G(\alpha)=\frac{2}{1-\alpha}\left(\frac{1-\alpha}{1+\alpha}\right)^{(1+\alpha)/2}-\frac{\pi}{2},\quad \alpha\in[0,\alpha_0).$$

Then we have

$$G'(\alpha)=\frac{2}{(1-\alpha)^2}\left(\frac{1-\alpha}{1+\alpha}\right)^{(1+\alpha)/2}\left\{1+\log\left(\frac{1-\alpha}{1+\alpha}\right)^{(1-\alpha)/2}\right\},\quad \alpha\in[0,\alpha_0).$$

For $\alpha\in[0,\alpha_0)$ we have

$$\left(\frac{1-\alpha}{1+\alpha}\right)^{(1-\alpha)/2}>\frac{1}{e}$$

hence $G'(\alpha)>0$, $\alpha\in[0,\alpha_0)$. Because $G(0)=2-\pi/2>0$ we finally obtain $G(\alpha)>0$, $\alpha\in[0,\alpha_0)$. Thus (3.30) is a contradiction with (3.24). For the case $\arg\{p(z_0)\}=-\pi\alpha/2$, applying the same method as the above, we can have the contradiction. □

It should be noted that the known results connected with the Briot-Boquet differential subordination

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec h(z)$$

are usually proved under the assumptions that h, is a convex function and $\mathfrak{Re}\{\beta h(z)+\gamma\}$ >0. It is easy to see that the assumptions of Theorem 3.8 are of another type.

COROLLARY 3.9. *Let*

$$F(z)=\frac{1}{z}+a_1z+a_2z^2+\cdots,\quad z\in\dot{\mathbb{U}}.$$

Under the assumptions of Theorem 3.8, if for $z\in\mathbb{D}$

$$\left|\arg\left\{-1-\frac{zF''(z)}{F(z)}\right\}\right|<\begin{cases}\tan^{-1}\frac{-v(\alpha)}{u(\alpha)}-\frac{\alpha\pi}{2} & \text{when } 0<\alpha<\alpha_0,\\ \pi/2-\frac{\alpha\pi}{2} & \text{when } \alpha=\alpha_0,\\ \pi+\tan^{-1}\frac{-v(\alpha)}{u(\alpha)}-\frac{\alpha\pi}{2} & \text{when } \alpha_0<\alpha<1,\end{cases}\tag{3.31}$$

then $F(z)$ is meromorphic-strongly starlike function of order α.

Proof. Let $p(z)=-zF'(z)/F(z)$. By (3.18) we have that

$$p(z)=1+p_2z^2+p_3z^3+\cdots.$$

Moreover,

$$p(z)-\frac{zp'(z)}{p(z)}=-1-\frac{zF''(z)}{F(z)}.$$

Therefore, it is sufficient to apply Theorem 3.8. □

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