BOUNDENNESS FROM BELOW OF COMPOSITION OPERATORS ON $\alpha$–LOGARITHMIC BLOCH SPACES

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Abstract. In this paper, we characterize the boundedness from below of composition operator on $\alpha$-logarithmic Bloch spaces.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $\partial \mathbb{D}$ be its boundary. Let $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ be the set of analytic self-maps of $\mathbb{D}$. For $a \in \mathbb{D}$, let $\sigma_a$ be the automorphism of $\mathbb{D}$ exchanging 0 for $a$, namely $\sigma_a(z) = \frac{a-z}{1-\overline{a}z}$, $z \in \mathbb{D}$. It is well known that $\sigma_a^{-1} = \sigma_a$,

$$\frac{|\sigma_a'(z)|}{1 - |\sigma_a(z)|^2} = \frac{1}{1 - |z|^2}, \quad \frac{(1 - |z|^2)(1 - |a|^2)}{|1 - \overline{a}z|} = 1 - |\sigma_a(z)|^2. \quad (1)$$

Let $\psi$ be a M"obius mapping such that $a = \psi(0)$. For $z \in \mathbb{D}$, by (1) we get

$$\frac{1 - |a|}{1 + |a|} \leq \frac{1 - |\psi(z)|^2}{1 - |z|^2} = |\psi'(z)| \leq \frac{1 + |a|}{1 - |a|}. \quad (2)$$

Moreover, let $\psi, \sigma \in S(\mathbb{D})$ such that $\sigma(0) = 0$ and $\psi = \sigma_a \circ \sigma$. Then by the Schwarz lemma and (2), we obtain

$$\frac{1 - |z|^2}{1 - |\psi(z)|^2} = \frac{1 - |\sigma(z)|^2}{1 - |\sigma_a(z)|^2} \leq \frac{1 + |a|}{1 - |a|}, \quad z \in \mathbb{D}. \quad (3)$$

For $z, w \in \mathbb{D}$, the pseudo distance on $\mathbb{D}$ is defined by $\rho(z, w) = |\frac{z - w}{1 - \overline{w}z}|$. Let $E \subset \mathbb{D}$. For every $w \in \mathbb{D}$, if there exists a $z \in E$ such that $\rho(z, w) \leq r$, then $E$ is called a pseudo $r$-net. For $w \in \mathbb{C}$ and $t > 0$, we denote by $D(w, t)$ the disk with radius $t$ and centered at $w$. For $w \in \mathbb{D}$ and $r \in (0, 1)$, we denote by $\Delta(w, r)$ the pseudo disk, where $\Delta(w, r) = \{z \in \mathbb{D} : \rho(z, w) < r\}$. Let $\overline{D}(w, t)$ and $\overline{\Delta}(w, r)$ denote the closures of $D(w, t)$ and $\Delta(w, r)$, respectively.

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An $f \in H(\mathbb{D})$ is said to belong to the $\alpha$-logarithmic Bloch space, denoted by $L^{\mathcal{B}^{\alpha}} = \mathcal{L}^{\mathcal{B}^{\alpha}}(\mathbb{D})$, if
\[
\| f \| \mathcal{L}^{\mathcal{B}^{\alpha}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2} |f'(z)| < \infty.
\]
The space $\mathcal{L}^{\mathcal{B}^{\alpha}}$ becomes a Banach space with the norm $\| f \| \mathcal{L} = |f(0)| + \| f \| \mathcal{L}^{\mathcal{B}^{\alpha}}$. If $\alpha = 1$, the space $\mathcal{L}^{\mathcal{B}^{\alpha}}$ is the logarithmic Bloch space, which will be denoted by $\mathcal{L}^{\mathcal{B}}$. From [1] or [11], we see that $\mathcal{L}^{\mathcal{B}} \cap H^{\infty}$ is the space of multipliers of the Bloch space $\mathcal{B}$. Here $H^{\infty}$ is the space consisting of all bounded analytic functions and the Bloch space is defined as follows.

$$\mathcal{B} = \{ f \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} |f'(z)(1 - |z|^2) < \infty \}.$$ 

The composition operator $C_\varphi$ defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$. The main subject in the study of composition operators is to describe operator theoretic properties of $C_\varphi$ in terms of function theoretic properties of $\varphi$. See [5] and the references therein for the study of the composition operator.

See [8, 9, 10] for the study of composition operators on the logarithmic Bloch space. In [8], Ye studied the boundedness and compactness of composition operator on $\alpha$-logarithmic Bloch spaces. For example, he proved that $C_\varphi$ is bounded on $\mathcal{L}^{\mathcal{B}^{\alpha}}$ if and only if $\sup_{z \in \mathbb{D}} \tau_\varphi(z) < \infty$, where
\[
\tau_\varphi(z) = \frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2} |\varphi'(z)|}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}}.
\]

Recall that the operator $C_\varphi : \mathcal{L}^{\mathcal{B}^{\alpha}} \to \mathcal{L}^{\mathcal{B}^{\alpha}}$ is said to be bounded, if there exists a $C > 0$, such that $\| C_\varphi f \| \mathcal{L}^{\mathcal{B}^{\alpha}} \leq C \| f \| \mathcal{L}^{\mathcal{B}^{\alpha}}$ for all $f \in \mathcal{L}^{\mathcal{B}^{\alpha}}$. A bounded composition operator $C_\varphi$ on $\mathcal{L}^{\mathcal{B}^{\alpha}}$ is said to be bounded below, if there exists a $\delta > 0$ such that
\[
\| C_\varphi f \| \mathcal{L}^{\mathcal{B}^{\alpha}} \geq \delta \| f \| \mathcal{L}^{\mathcal{B}^{\alpha}}
\]
for all $f \in \mathcal{L}^{\mathcal{B}^{\alpha}}$.

The boundedness from below of composition operator $C_\varphi$ on $\mathcal{B}$ was first studied by Gathage, Yan and Zheng in [6]. See [2, 3, 6, 7] for some characterizations of the boundedness from below of composition operator on $\mathcal{B}$. The boundedness from below of multiplication operator on $\mathcal{B}$ was studied in [4].

In this paper, we investigate the condition for the boundedness from below of composition operator on $\mathcal{L}^{\mathcal{B}^{\alpha}}$. We extend the results in [3] to the case of $\mathcal{L}^{\mathcal{B}^{\alpha}}$.

Throughout the paper, we denote by $C$ a positive constant which may differ from one occurrence to the next.
2. Main results and proofs

In this section, we give our main results and their proofs. Before stating these results, we need some auxiliary results, which are incorporated in the lemmas which follow.

**Lemma 1.** Let $\alpha > 0$ and $0 \leq x, y < 1$. Let $f(x) = (1 - x)^{\alpha} \ln \frac{2}{1 - x}$. Then

$$\frac{1}{\alpha e \ln 2} + \left(\frac{1 - y}{1 - x}\right)^{\alpha} \leq \frac{f(x)}{f(y)} \leq \frac{1}{\alpha e \ln 2} + \left(\frac{1 - x}{1 - y}\right)^{\alpha}.$$

**Proof.** A short calculation shows that

$$\left(\frac{1 - x}{1 - y}\right)^{\alpha} \ln \frac{2}{1 - y} \leq \frac{1}{\alpha e}.$$

Hence we get

$$\frac{f(x)}{f(y)} = \frac{(1 - x)^{\alpha} \ln \frac{2}{1 - x}}{(1 - y)^{\alpha} \ln \frac{2}{1 - y}} = \left(\frac{1 - x}{1 - y}\right)^{\alpha} \frac{\ln \frac{2}{1 - x} - \ln \frac{2}{1 - y}}{\ln \frac{2}{1 - y}} + \left(\frac{1 - x}{1 - y}\right)^{\alpha} \leq \frac{1}{\alpha e \ln 2} + \left(\frac{1 - x}{1 - y}\right)^{\alpha}.$$

The other inequality follows in a similar way. \qed

**Lemma 2.** [3, Lemma 4.1] Let $h \subset S(D)$ such that $h(0) = 0$. If $|h'(0)| \geq \epsilon > 0$, then there exist $\delta_1, \delta_2 > 0$, depending only on $\epsilon$, such that

(a) $|h'(z)| \geq \frac{\epsilon}{2}$ for $z \in D(0, \delta_1)$;

(b) $\overline{D}(0, \delta_2) \subset h(D(0, \delta_1))$.

To state our main results, we need a definition and some notations. For $\alpha > 0$, a subset $F$ of $\mathbb{D}$ is called a sampling set for $\mathcal{L}B^{\alpha}$, if there exists a $k > 0$ such that

$$\|f\|_{\mathcal{L}B^{\alpha}} \leq k \sup_{z \in F} (1 - |z|^2)^{\alpha} \ln \frac{2}{1 - |z|^2} |f'(z)|$$

hold for $f \in \mathcal{L}B^{\alpha}$. For $\epsilon > 0$, let

$$\Omega_{\epsilon}^{\alpha} = \{z \in \mathbb{D} : \tau_{\phi}^{\alpha}(z) \geq \epsilon\}, \quad G_{\epsilon}^{\alpha} = \varphi(\Omega_{\epsilon}^{\alpha}).$$

**Theorem 1.** Let $\alpha > 0$ and $\varphi \in S(\mathbb{D})$ such that $C_{\varphi}$ is bounded on $\mathcal{L}B^{\alpha}$. Then $C_{\varphi}$ is bounded below on $\mathcal{L}B^{\alpha}$ if and only if there exists an $\epsilon > 0$ such that $G_{\epsilon}^{\alpha}$ is a sampling set for $\mathcal{L}B^{\alpha}$.
Proof. Suppose that $G^\alpha_\varepsilon$ with an $\varepsilon > 0$ is a sampling set for $\mathcal{L}B^\alpha$. Then for any $f \in \mathcal{L}B^\alpha$, there is a $z_f \in \mathbb{D}$ such that

$$\varphi(z_f) \in G^\alpha_\varepsilon, \quad \tau^\alpha_\varepsilon(z_f) \geq \varepsilon$$

and

$$\|f\|_{\mathcal{L}B^\alpha} \leq k \sup_{w \in G^\alpha_\varepsilon} (1 - |w|^2)^\alpha \frac{2}{1 - |w|^2} |f'(w)|$$

$$\leq 2k(1 - |\varphi(z_f)|^2)^\alpha \frac{2}{1 - |\varphi(z_f)|^2} |f'(\varphi(z_f))|,$$

where $k > 0$ is independent of $f$. So,

$$\|C_\varphi(f)\|_{\mathcal{L}B^\alpha} \geq (1 - |z_f|^2)^\alpha \frac{2}{1 - |z_f|^2} |(C_\varphi(f))'(z_f)|$$

$$= \tau^\alpha_\varphi(z_f)(1 - |\varphi(z_f)|^2)^\alpha \frac{2}{1 - |\varphi(z_f)|^2} |f'(\varphi(z_f))|$$

$$\geq \frac{\varepsilon \|f\|_{\mathcal{L}B^\alpha}}{2k}.$$

Thus, $C_\varphi$ is bounded below on $\mathcal{L}B^\alpha$.

Conversely, we assume that $C_\varphi$ is bounded below on $\mathcal{L}B^\alpha$, i.e., $\|C_\varphi(f)\|_{\mathcal{L}B^\alpha} \geq \delta \|f\|_{\mathcal{L}B^\alpha}$ for any $f \in \mathcal{L}B^\alpha$ with $\delta > 0$ independent of $f$. Then, for any $f \in \mathcal{L}B^\alpha$ with $\|f\|_{\mathcal{L}B^\alpha} > 0$, there is a $z_f \in \mathbb{D}$ such that

$$\tau^\alpha_\varphi(z_f)(1 - |\varphi(z_f)|^2)^\alpha \frac{2}{1 - |\varphi(z_f)|^2} |f'(\varphi(z_f))|$$

$$= \frac{(1 - |z_f|^2)^\alpha \frac{2}{1 - |z_f|^2} |\varphi'(z_f)|}{(1 - |\varphi(z_f)|^2)^\alpha \frac{2}{1 - |\varphi(z_f)|^2}} \frac{1}{1 - |\varphi(z_f)|^2} (1 - |\varphi(z_f)|^2)^\alpha \frac{2}{1 - |\varphi(z_f)|^2} |f'(\varphi(z_f))|$$

$$= (1 - |z_f|^2)^\alpha \frac{2}{1 - |z_f|^2} |(C_\varphi(f))'(z_f)|$$

$$\geq \frac{\delta}{2} \|f\|_{\mathcal{L}B^\alpha}.$$

In addition,

$$(1 - |\varphi(z_f)|^2)^\alpha \frac{2}{1 - |\varphi(z_f)|^2} |f'(\varphi(z_f))| \leq \|f\|_{\mathcal{L}B^\alpha}.$$

Thus,

$$\tau^\alpha_\varphi(z_f) \geq \frac{\delta}{2} \text{ and } (1 - |\varphi(z_f)|^2)^\alpha \frac{2}{1 - |\varphi(z_f)|^2} |f'(\varphi(z_f))| \geq \frac{\delta}{2M} \|f\|_{\mathcal{L}B^\alpha}.$$

Here $M : = \sup_{z \in \mathbb{D}} \tau^\alpha_\varphi(z) < \infty$. Taking $\varepsilon = \frac{\delta}{2M}$, we see that $G^\alpha_\varepsilon$ contains all $\varphi(z_f)$ and is a sampling set for $\mathcal{L}B^\alpha$. □
THEOREM 2. Let $\alpha > 1$. A sampling set for $\mathcal{L} \mathcal{B}^\alpha$ is a pseudo $r$-net. Conversely, if $E$ is a pseudo $r$-net, then for any $\delta > 0$, the set $E_\delta = \bigcup_{z \in E} A(z, \delta)$ is a sampling set for $\mathcal{L} \mathcal{B}^\alpha$.

Proof. Let $w \in \mathbb{D}$. Take
\[
f_w(z) = 2 \int_0^z \frac{1 - |w|^2}{\ln \frac{4}{1-\bar{w}u}} \frac{1}{(1-\bar{w}u)^{\alpha+1}} du, \quad z \in \mathbb{D}.
\]
It is easy to check that $f_w \in \mathcal{L} \mathcal{B}^\alpha$ and $\|f_w\|_{\mathcal{L} \mathcal{B}^\alpha} \geq 1$. Suppose that $F$ is a sampling set for $\mathcal{L} \mathcal{B}^\alpha$, then there exists a $z \in F$ such that
\[
\|f_w\|_{\mathcal{L} \mathcal{B}^\alpha} \leq k (1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2} |f_w'(z)|.
\]

Similarly to the proof of Lemma 2.2 in [9], we get
\[
1 \leq k (1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2} |f_w'(z)| = 2 k (1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2} \left(1 - |w|^2\right) \leq 2 k \ln \frac{2}{1 - |z|^2} \left(1 - |z|^2\right)^{\alpha-1} \frac{1 - |w|^2}{1 - |z|^2}.\]

Thus,
\[
\rho(z, w) = |\sigma_w(z)| \leq r = \sqrt{1 - 1/C_\alpha},
\]
where
\[
C_\alpha = 2 k \max \left\{1, \frac{2^{\alpha-1}e^{1/(\alpha-1)}}{(\alpha-1)e \ln 4}\right\} (1 - |\sigma_w(z)|^2).
\]

Therefore, $F$ is a pseudo $r$-net.

Now, suppose that $E$ is a pseudo $r$-net. We next to prove that $E_\delta$ is a sampling set for $\mathcal{L} \mathcal{B}^\alpha$ for any $\delta > 0$. Suppose on the contrary that there are a $\delta$ and a sequence $f_n \in \mathcal{L} \mathcal{B}^\alpha$ such that
\[
\|f_n\|_{\mathcal{L} \mathcal{B}^\alpha} = 1 \quad \text{for } n = 1, 2, 3, \ldots
\]
and
\[
\sup_{z \in E_\delta} (1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2} |f_n'(z)| = \varepsilon_n \to 0 \quad \text{as } n \to \infty.
\]
For $n = 1, 2, 3, \ldots$, let $z_n \in \mathbb{D}$ satisfy
\[
(1 - |z_n|^2)^{\alpha} \ln \frac{2}{1 - |z_n|^2} |f'_n(z_n)| \geq \frac{1}{2}. \tag{7}
\]
Since $E$ is a pseudo $r$-net, we have a sequence $z'_n \in E$ such that $\rho(z_n, z'_n) \leq r$ for $n = 1, 2, 3, \ldots$. Let $t_n = \sigma_{z_n}(z_n)$ and $g_n = f_n \circ \sigma_{z_n}$ for $n = 1, 2, 3, \ldots$.

Let $n \geq 1$ be fixed. We have
\[
|t_n| = |\sigma_{z_n}(z_n)| = \rho(z_n, z'_n) \leq r. \tag{8}
\]
For $w \in \mathbb{D}$, we obtain
\[
(1 - |w|^2)^{\alpha} \ln \frac{2}{1 - |w|^2} |g'_n(w)| \tag{9}
\]
\[
= (1 - |w|^2)^{\alpha-1} (1 - |w|^2) \ln \frac{2}{1 - |w|^2} |g'_n(w)|
= (1 - |w|^2)^{\alpha-1} (1 - |\sigma_{z_n}(w)|^2) \ln \frac{2}{1 - |w|^2} |f'_n(\sigma_{z_n}(w))|
= \frac{(1 - |w|^2)^{\alpha-1} \ln \frac{2}{1 - |w|^2}}{(1 - |\sigma_{z_n}(w)|^2)^{\alpha-1} \ln \frac{2}{1 - |\sigma_{z_n}(w)|^2}} \times (1 - |\sigma_{z_n}(w)|^2)^{\alpha} \ln \frac{2}{1 - |\sigma_{z_n}(w)|^2} |f'_n(\sigma_{z_n}(w))|.
\]
By (3), (5) and Lemma 1, we have
\[
(1 - |w|^2)^{\alpha} \ln \frac{2}{1 - |w|^2} |g'_n(w)| \leq \frac{1}{(\alpha - 1)e \ln 2} + \left(\frac{1 - |w|^2}{1 - |\sigma_{z_n}(w)|^2}\right)^{\alpha-1}
\leq \frac{1}{(\alpha - 1)e \ln 2} + \frac{(1 + |z'_n|)^{\alpha-1}}{(1 + |z'_n|^{\alpha-1})}
\leq \frac{1}{(\alpha - 1)e \ln 2} + \frac{4^{\alpha-1}}{(1 + |z'_n|^2)^{\alpha-1}}. \tag{10}
\]
Set $w = t_n$ in (9). By (1), (7) and (8), we get
\[
(1 - |t_n|^2)^{\alpha} \ln \frac{2}{1 - |t_n|^2} |g'_n(t_n)| \tag{11}
\]
\[
= \frac{(1 - |t_n|^2)^{\alpha-1} \ln \frac{2}{1 - |t_n|^2}}{(1 - |\sigma_{z_n}(t_n)|^2)^{\alpha-1} \ln \frac{2}{1 - |\sigma_{z_n}(t_n)|^2}} \times (1 - |\sigma_{z_n}(t_n)|^2)^{\alpha} \ln \frac{2}{1 - |\sigma_{z_n}(t_n)|^2} |f'_n(\sigma_{z_n}(t_n))|
\geq \frac{1}{2 \left(\frac{1}{(\alpha - 1)e \ln 2} + \frac{(1 + |z'_n|^2)^{\alpha-1}}{(1 + |z'_n|^2)^{\alpha-1}}\right)} \geq \frac{1}{2 \left(\frac{1}{(\alpha - 1)e \ln 2} + \frac{4^{\alpha-1}}{(1 + |z'_n|^2)^{\alpha-1}}\right)}.
\]
If $|w| \leq \delta$, then

$$\rho(\sigma_{\epsilon_n}(w), z_{n}') = \rho(\sigma_{\epsilon_n}(w), \varphi_{\epsilon_n}(0)) = |w| \leq \delta.$$ 

Hence $\sigma_{\epsilon_n}(w) \in E_\delta$. Thus, by (6),

$$(1 - |\sigma_{\epsilon_n}(w)|^2)^{\alpha} \ln \frac{2}{1 - |\sigma_{\epsilon_n}(w)|^2} |f_n'(\sigma_{\epsilon_n}(w))| \leq \varepsilon_n,$$

which combining with (10) and (3) imply

$$(1 - |w|^2)^{\alpha} \ln \frac{2}{1 - |w|^2} |g_n'(w)| \leq \varepsilon_n,$$  \hspace{1cm} (12)

For $n = 1, 2, \ldots$, let

$$h_n(w) = \left(\frac{1}{(\alpha - 1)e \ln 2} + \frac{4^{\alpha - 1}}{(1 - |z_{n}'|^2)^{\alpha - 1}((\alpha - 1)e \ln 2)}\right)^{-1} g_n'(w), \quad w \in \mathbb{D}.$$ 

By (10), $h_n$ is bounded locally uniformly in $\mathbb{D}$. Using Montel’s theorem, we may assume that $h_n$ converges to a $h \in H(\mathbb{D})$ locally uniformly in $\mathbb{D}$, and $t_n \to t_0$ with $|t_0| \leq r$ by (10). Letting $n \to \infty$ in (11) and (12), we have

$$h(t_0) \geq \frac{1}{2((\alpha - 1)e \ln 2 + \frac{1}{(1 - r)^2(\alpha - 1)}) (1 - |t_0|^2)^{\alpha - 1}} \ln \frac{2}{1 - |t_0|^2},$$

and $h(w) = 0$. Thus, we arrive at a contradiction. Therefore $E_\delta$ is a sampling set for any $\delta$. \hfill \Box

**Lemma 3.** Let $\alpha > 1$. For $\varepsilon > 0$, there exist $\delta$, $\varepsilon' > 0$, which only depend on $\varphi(0)$, $\varepsilon$ and $\alpha$ only, such that

$$(G_{\alpha}^\varepsilon)_\delta = \bigcup_{w' \in G_{\alpha}^\varepsilon} \overline{\Delta}(w', \delta) \subset G_{\varepsilon'}^{\alpha}.$$
Proof. Let \( w' = \varphi(z') \in G_{e}^{\alpha} \), \( \tau_{\varphi}^{\alpha}(z') \geq \varepsilon \) and \( h = \sigma_{w'} \circ \varphi \circ \sigma_{z'} \). By Lemma 1 and (3), we get
\[
|h'(0)| = \frac{(1 - |z'|^2)|\varphi'(z')|}{1 - |w'|^2} = \frac{(1 - |w'|)^{\alpha-1} \ln \frac{2}{1 - |w'|^2} \tau_{\varphi}^{\alpha}(z')}{(1 - |z'|)^{\alpha-1} \ln \frac{2}{1 - |z'|^2}} \geq \frac{1}{(\alpha-1)e\ln 2} + \frac{(1 - |z'|)^{\alpha-1} \tau_{\varphi}^{\alpha}(z')}{\ln \frac{2}{1 - |z'|^2}} \geq \left( \frac{1}{(\alpha-1)e\ln 2} + \frac{(1 - |\varphi(0)|)^{\alpha-1}}{1 - |\varphi(0)|} \right) \varepsilon = \varepsilon_{1}.
\]
By Lemma 2, there exists \( t_{1}, t_{2} > 0 \) satisfying (a) and (b) of Lemma 2 with \( \varepsilon \) replaced by \( \varepsilon_{1} \).

For \( w \in \bar{\Delta}(w', t_{2}) \), let \( \lambda = \sigma_{w'}(w) \in \bar{D}(0, t_{2}) \). By Lemma 2, there exists a \( \xi \in D(0, t_{1}) \) such that \( h(\xi) = \lambda \) and \( h'(\xi) \geq \varepsilon_{1}/2 \). Let \( z = \sigma_{z'}(\xi) \). Then \( \varphi(z) = w \). By Lemma 1, (1) and (3), we get
\[
\tau_{\varphi}^{\alpha}(z) = \frac{(1 - |z|^{2})^{\alpha} \ln \frac{2}{1 - |z|^2} |\varphi'(z)|}{(1 - |w|^{2})^{\alpha} \ln \frac{2}{1 - |w|^2}} \geq \frac{1 - |z|^{2} - |\xi|^2 - |\lambda|^2}{1 - |z|^{2} - |\xi|^2 - |\lambda|^2} \frac{1 - |\varphi'(\lambda)|}{1 - |\varphi(\lambda)|} \frac{1 - |\lambda|^{2}}{1 - |\lambda|^2} \frac{1}{(\alpha-1)e\ln 2} + \frac{(1 - |\varphi(0)|)^{\alpha-1}}{1 - |\varphi(0)|} \geq \frac{\varepsilon_{1}}{2} \frac{1}{(\alpha-1)e\ln 2} + \frac{(1 - |w'|^2)^{\alpha-1}}{1 - |w'|^2},
\]
which implies that \( \bar{\Delta}(w', t_{2}) \subset G_{e}^{\alpha} \), for \( w' \in G_{e}^{\alpha} \). The proof of the lemma is finished. \( \Box \)

**Theorem 3.** Let \( \alpha > 1 \) and \( \varphi \in S(\mathbb{D}) \) such that \( C_{\varphi} \) is bounded on \( \mathcal{L} \mathcal{B}^{\alpha} \). Then, \( C_{\varphi} \) is bounded below on \( \mathcal{L} \mathcal{B}^{\alpha} \) if and only if there exist an \( \varepsilon > 0 \) and an \( r \) with \( 0 < r < 1 \) such that \( G_{e}^{\alpha} \) is a pseudo \( r \)-net.

**Proof.** First we assume that there exist an \( \varepsilon > 0 \) and an \( r \in (0, 1) \) such that \( G_{e}^{\alpha} \) is a pseudo \( r \)-net. By Lemma 3, there exist \( \delta, e' > 0 \) such that \( (G_{e}^{\alpha})_{\delta} \subset G_{e}^{\alpha} \). From Theorem 2, we see that \( (G_{e}^{\alpha})_{\delta} \) is a sampling set for \( \mathcal{L} \mathcal{B}^{\alpha} \). Since \( (G_{e}^{\alpha})_{\delta} \subset G_{e}^{\alpha} \), we see that \( G_{e}^{\alpha} \) is a sampling set for \( \mathcal{L} \mathcal{B}^{\alpha} \).

Conversely, assume that \( C_{\varphi} \) is bounded below. Using Theorem 1, there exists an \( \varepsilon > 0 \) such that \( G_{e}^{\alpha} \) is a sampling set for \( \mathcal{L} \mathcal{B}^{\alpha} \). Therefore \( G_{e}^{\alpha} \) is a pseudo \( r \)-net with \( r \in (0, 1) \) by Theorem 2, completing the proof. \( \Box \)
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