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SCHUR CONVEXITY OF STOLARSKY'S EXTENDED MEAN VALUES

K. MURALI AND K. M. NAGARAJA

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Abstract. In the recent years, the Schur convexity and Schur geometrically convexity of Stolarsky's mean values have grabed the focus of many mathematicians and researchers. In this article, the Schur convexity of Stolarsky's extended type mean values are discussed.

1. Introduction

The importance and applications of means and their inequalities to science and technology is explored by prominent researchers and scholars, see [1]. In literature ([8], [9], [15]) various results on means and inequalities including contra harmonic mean have been studied. In [20], reseachers have investigated the different properties of the Stolarsky (extended) two parameter mean values, which are defined for positive values of a, b and $a \neq b$ as follows;

$$E_{p,q}(a,b) = \begin{cases} \left[\frac{p}{q}\frac{(a^q-b^q)}{(a^p-b^p)}\right]^{\frac{1}{q-p}} & pq(q-p) \neq 0; \\ e^{\left(\frac{-1}{q}+\frac{(a^q\ln a - b^q \ln b)}{(a^q-b^q)}\right)} & p = q \neq 0; \\ \left(\frac{(a^q-b^q)}{q(\ln a - \ln b)}\right)^{\frac{1}{q}} & p=0,\ q\neq 0; \\ \sqrt{ab} & p=q=0; \\ a & a=b>0. \end{cases} \tag{1.1}$$

Here let us recall some of the popular means which are essential for this paper. The Weighted arithmetic mean is

$$A_{r,s}(a,b) = ra+sb = A(a,b;r,s).$$

Where $a,b>0$ and r, s are the weights such that $r+s=1$.

The Stolarsky means $E_{p,q}(a,b) \in C^{\infty}$ on the domain $(p,q,a,b) : p,q \in R;\ a,b > 0$. Clearly, Stolarsky means $E_{p,q}(a,b)$ are symmetric with respect to a,b and p,q.

Many of the classical two variable means can be deduced from $E_{p,q}(a,b)$.

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For example, we have

$$E_{1,2}(a,b) = \frac{a+b}{2},$$

$$E_{0,0}(a,b) = \sqrt{ab},$$

$$E_{-1,-2}(a,b) = \frac{2ab}{a+b},$$

$$E_{0,1}(a,b) = \frac{a-b}{\ln a - \ln b},$$

$$E_{1,1}(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}},$$

and

$$E_{r,2r}(a,b) = \left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}}.$$

The above are Arithmetic, Geometric, Harmonic, Logarithmic, Identric and r^th power means respectively.

The primitive properties of Stolarsky means, comparison theorems, log- convexities, and inequalities are discussed in papers ([6], [22], [23]).

In the recent past, the Schur convexity and Schur geometrically convexity of various means have grabed the focus of a number of researchers ([2]–[4], [7], [11]–[14], [16], [17]). Qi [18] initially proved that the Stolarsky means $E_{p,q}(a,b)$ are Schur convex on $(-\infty,0] \times (-\infty,0]$ and Schur concave on $[0,\infty) \times [0,\infty)$ with respect to (p,q) for fixed $a,b>0$ with $a \neq b$. Yang [25] improved Qi's result and proved that Stolarsky means $E_{p,q}(a,b)$ are Schur convex with respect to (p,q) for fixed $a,b>0$ with $a \neq b$ if and only if $p+q<0$ and Schur concave iff $p+q>0$.

Qi et al. [17] tried to obtain the Schur convexity of $E_{p,q}(a,b)$ with respect to (a,b) for fixed (p,q). Shi et al. [19] worked on the same and obtained a sufficient condition for the Schur convexity of $E_{p,q}(a,b)$ with respect to (a,b). Chu and Zhang [3] improved Shi's results and gave a necessary and sufficient condition. This complectly solved the Schur convexity of $E_{p,q}(a,b)$ with respect to (a,b).

For the Schur geometrical convexity, Zhang and Chu [2] proved that Stolarsky means $E_{p,q}(a,b)$ are Schur geometrically convex with respect to $(a,b) \in (0,\infty)\times(0,\infty)$ if $p+q \geqslant 0$ and Schur geometrically concave if $p+q \leqslant 0$. Li et al. [7] also studied the Schur geometrical convexity of generalized exponent mean $I_p(a,b)$.

The purpose of this paper is to investigate Schur convexity of Stolarsky's extended type mean values $N_{p,q}(a,b;r,s)$.

In [1], the weighted contra harmonic mean is defined on the basis of proportions by

$$C_{r,s}(a,b) = \frac{ra^2+sb^2}{ra+sb} = C(a,b;r,s);$$

where $a,b>0$ and r, s are weights such that $r+s=1$.

This work has led to introduce the Stolarsky's extended type mean values in weighted forms in *two* and n variables.

For any $a,b>0$, $p,q\in R$ and r,s are weights such that $r+s=1$. Consider the new mean in the form;

$$N_{p,q}(a,b;r,s)=\left[\frac{p^2}{q^2}\frac{C(a^q,b^q;r,s)-A(a^q,b^q;r,s)}{C(a^p,b^p;r,s)-A(a^p,b^p;r,s)}\right]^{\frac{1}{q-p}}.$$

Which is equivalently,

$$N_{p,q}(a,b;r,s)=\left[\frac{p^2}{q^2}\left(\frac{ra^p+sb^p}{ra^q+sb^q}\right)\left(\frac{ra^{2q}+sb^{2q}-(ra^q+sb^q)^2}{ra^{2p}+sb^{2p}-(ra^p+sb^p)^2}\right)\right]^{\frac{1}{q-p}}.$$

Which is equivalently,

$$N_{p,q}(a,b;r,s)=\left[\frac{p^2}{q^2}\left(\frac{ra^p+sb^p}{ra^q+sb^q}\right)\left(\frac{a^q-b^q}{a^p-b^p}\right)^2\right]^{\frac{1}{q-p}}.$$

In [21], author has introduced and investigated the different properties and log-convexity results of the class W of weighted two parameter means which are defined as below;

$$W_{p,q}(a,b;r,s)=\begin{cases}\left[\frac{p^2}{q^2}\left(\frac{ra^q+sb^q-a^{rq}b^{sq}}{ra^p+sb^p-a^{rp}b^{sp}}\right)\right]^{\frac{1}{q-p}} & pq(q-p)(a-b)\neq 0;\\ \left[\frac{2}{\ln^2(a/b)}\left(\frac{ra^q+sb^q-a^{rq}b^{sq}}{rsq^2}\right)\right]^{\frac{1}{q}} & q(a-b)\neq 0,\ p=0;\\ exp\left(\frac{-2}{q}+\frac{ra^q\ln a+sb^q\ln b-(r\ln a+s\ln b)a^{rq}b^{sq}}{ra^q+sb^q-a^{rq}b^{sq}}\right) & p=q,\ q\neq 0;\\ a^{(r+1)/3}b^{(s+1)/3} & a\neq b,\ p=q=0;\\ a & a=b>0.\end{cases} \tag{1.2}$$

The $N_{p,q}(a,b;r,s)$ can be arranged in the following form;

$$N_{p,q}(a,b;r,s)=\left[\frac{ra^p+sb^p}{ra^q+sb^q}\right]^{\frac{1}{q-p}}\left[\left(\frac{p}{q}\frac{(a^q-b^q)}{(a^p-b^p)}\right)^{\frac{1}{q-p}}\right]^2.$$

The different properties and identities related to $N_{p,q}(a,b;r,s)$ are also studied by K. M. Nagaraja and et. al.

The laborious calculations give the following different cases of the mean value

$N_{p,q}(a,b;r,s)$.

$$N_{p,q}(a,b;r,s)=\begin{cases}\left[\frac{p^2}{q^2}\left(\frac{ra^p+sb^p}{ra^q+sb^q}\right)\left(\frac{a^q-b^q}{a^p-b^p}\right)^2\right]^{\frac{1}{q-p}} & pq(q-p)(a-b)\neq 0;\\ \left[\frac{1}{\ln^2(a/b)}\left(\frac{1}{ra^q+sb^q}\right)\left(\frac{a^q-b^q}{q}\right)^2\right]^{\frac{1}{q}} & q(a-b)\neq 0,\ p=0;\\ exp\left(\frac{-2}{q}-\frac{ra^q\ln a+sb^q\ln b}{ra^q+sb^q}+2\frac{a^q\ln a-b^q\ln b}{a^q-b^q}\right) & p=q,\ q\neq 0;\\ a^{1-r}b^{1-s} & a\neq b,\ p=q=0;\\ a & a=b>0.\end{cases} \tag{1.3}$$

2. Definition and properties

Schur convexity was introduced by Schur in 1923 [10], and it has many important applications in analytic inequalities [5], linear regression, graphs and matrices, combinatorial optimization, information theoretic topics, Gamma functions, stochastic orderings, reliability , and other related fields. For convenience of readers, we recall some of the definitions.

DEFINITION 2.1. [10] Let $x=(x_1,x_2,...,x_n)$ and $y=(y_1,y_2,...,y_n)\in R^n$

1. x is majorized by y (in symbol $x\prec y$) If $\sum_{i=1}^k x_{[i]}\leqslant\sum_{i=1}^k y_{[i]}$, and $\sum_{i=1}^n x_{[i]}=\sum_{i=1}^n y_{[i]}$, where $x_{[1]}\geqslant,...,\geqslant x_{[n]}$ and $y_{[1]}\geqslant,...,\geqslant y_{[n]}$ are rearrangements of x and y in descending order.

2. $x\geqslant y$ means $x_i\geqslant y_i$ for all $i=1,2,....,n$. Let $\Omega\in R^n(n\geqslant 2)$. The function $\varphi:\Omega\to R$ is said to be increasing if $x\geqslant y$ implies $\varphi(x)\geqslant\varphi(y)$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

3. $\Omega\subseteq R^n$ is called a convex set if $(\alpha x_1+\beta y_1,....,\alpha x_n+\beta y_n)\in\Omega$ for every x and $y\in\Omega$ where $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$.

4. Let $\Omega\subseteq R^n$. The function $\varphi:\Omega\to R$ be said to be a Schur convex function on Ω if $x\prec y$ on Ω implies $\varphi(x)\leqslant\varphi(y)$. φ is said to be a Schur concave function on Ω if and only if $-\varphi$ is Schur convex.

DEFINITION 2.2. [24] Let $x=(x_1,x_2,...,x_n)$ and $y=(y_1,y_2,...,y_n)\in R_+^n\Omega\subseteq R^n$ is called geometrically convex set if $(x_1^\alpha y_1^\beta,....,x_1^\alpha y_1^\beta)\in\Omega$ for all x and $y\in\Omega$ where $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$.

Let $\Omega\subseteq R_+^n$. The function $\varphi:\Omega\to R_+$ is said to be Schur geometrically convex function on Ω if $(\ln x_1,...,\ln x_n)\prec(\ln y_1,...,\ln y_n)$ on Ω implies $\varphi(x)\leqslant\varphi(y)$. φ is said to be a Schur geometrically concave function on Ω if and only if $-\varphi$ is Schur geometrically convex.

DEFINITION 2.3. $\Omega \subseteq R^n$ is called symmetric set if $x \in \Omega$ implies $Px \in \Omega$ for every $n \times n$ permutation matrix P.

The function $\varphi : \Omega \to R$ is called symmetric if for every permutation matrix P, $\varphi(Px) = \varphi(x)$ for all $x \in \Omega$.

LEMMA 2.1. [25] *Let $\Omega \subseteq R^n$ be symmetric convex set with non empty interior set Ω^0 and let $\varphi : \Omega \to R$ be continuous on Ω and differentiable in Ω^0. Then φ is Schur convex (Schur concave) on Ω if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geqslant 0 (\leqslant 0). \tag{2.1}$$

holds for any $x = (x_1, x_2, ..., x_n) \in \Omega^0$.

3. Main results

Here, we shall prove some of the lemmas needed for establishing the main theorem.

LEMMA 3.1. *Stolarsky's extended type mean $N_{p,q}(a,b;r,s)$ are Schur convex or Schur concave with respect to $(a,b) \in (0,\infty) \times (0,\infty)$ if and only if $g(t) \geqslant 0$ or $g(t) \leqslant 0$ for all $t > 0$, where*

$$g(t) = g_{p,q}(t)$$
$$= \begin{cases} \left[\dfrac{(p-q)\operatorname{Sinh} At\{\frac{1}{2}\operatorname{Cosh}(p+q)t + \frac{3}{2}\operatorname{Cosh}(p-q)t\} - (p\operatorname{Sinh} Ct + q\operatorname{Sinh} Bt)\{\frac{3}{2}\operatorname{Cosh}(p+q)t + \frac{1}{2}\operatorname{Cosh}(p-q)t\}}{pq(p-q)}\right] & pq(p-q) \neq 0; \\ \left[\dfrac{3qt\operatorname{Cosh} t + qt\operatorname{Cosh}(1-2q)t + \operatorname{Sinh}(1-2q)t - \operatorname{Sinh}(1+2q)t}{q^2}\right] & p = 0,\ q \neq 0; \\ \left[\dfrac{3pt\operatorname{Cosh} t + pt\operatorname{Cosh}(1-2p)t + \operatorname{Sinh}(1-2p)t - \operatorname{Sinh}(1+2p)t}{p^2}\right] & p \neq 0,\ q = 0; \\ \left[\dfrac{\operatorname{Sinh} t(\frac{1}{2} + \frac{3}{2}\operatorname{Cosh} 2qt) - qt\operatorname{Cosh} t(1 + 3\operatorname{Cosh} 2qt) - \operatorname{Sinh}(1-2q)t(\frac{3}{2} + \frac{1}{2}\operatorname{Cosh} 2qt)}{q^2}\right] & p = q \neq 0; \\ \left[-2t^2\operatorname{Sinh} t\right] & p = q = 0; \end{cases} \tag{3.1}$$

and

$$A = p + q - 1, \quad B = p - q + 1, \quad C = p - q - 1. \tag{3.2}$$

Proof. We have the Stolarsky's extended type means $N = N_{p,q}(a,b;r,s)$ for $pq(q-p) \neq 0$ as;

$$N = N_{p,q}(a,b;r,s) = \left[\frac{p^2}{q^2}\left(\frac{ra^p + sb^p}{ra^q + sb^q}\right)\left(\frac{a^q - b^q}{a^p - b^p}\right)^2\right]^{\frac{1}{q-p}} \tag{3.3}$$

Let $r = s$.

Taking log on both sides and differentiating partially with respect to a give

$$\frac{\partial N}{\partial a}=\frac{N}{p-q}\left[\frac{qa^{q-1}}{a^q+b^q}-\frac{pa^{p-1}}{a^p+b^p}+2\frac{pa^{p-1}}{a^p-b^p}-2\frac{qa^{q-1}}{a^q-b^q}\right]. \tag{3.4}$$

Similarly,

$$\frac{\partial N}{\partial b}=\frac{N}{p-q}\left[\frac{qb^{q-1}}{a^q+b^q}-\frac{pb^{p-1}}{a^p+b^p}-2\frac{pb^{p-1}}{a^p-b^p}+2\frac{qb^{q-1}}{a^q-b^q}\right] \tag{3.5}$$

then,

$$(a-b)\left(\frac{\partial N}{\partial a}-\frac{\partial N}{\partial b}\right)=\frac{(a-b)N}{p-q}[\Delta] \tag{3.6}$$

where,

$$\Delta=\left[\frac{q(a^{q-1}-b^{q-1})}{a^q+b^q}-\frac{p(a^{p-1}-b^{p-1})}{a^p+b^p}+2\frac{p(a^{p-1}+b^{p-1})}{a^p-b^p}-2\frac{q(a^{q-1}+b^{q-1})}{a^q-b^q}\right].$$

Substituting $\ln\sqrt{a/b}=t$ and using $\operatorname{Sinh}x=\frac{1}{2}(e^x-e^{-x})$; $\operatorname{Cosh}x=\frac{1}{2}(e^x+e^{-x})$, we have

$$\Delta=\frac{1}{\sqrt{ab}}\left[q\frac{\operatorname{Sinh}(q-1)t}{\operatorname{Cosh}qt}-p\frac{\operatorname{Sinh}(p-1)t}{\operatorname{Cosh}pt}+2p\frac{\operatorname{Cosh}(p-1)t}{\operatorname{Sinh}pt}-2q\frac{\operatorname{Cosh}(q-1)t}{\operatorname{Sinh}qt}\right].$$

For $pq(p-q)\neq 0$, using the product-sum formula of hyperbolic functions, we get

$$(a-b)\left(\frac{\partial N}{\partial a}-\frac{\partial N}{\partial b}\right)=\frac{(pq)N(a-b)}{\sqrt{ab}2\operatorname{Sinh}pt\operatorname{Sinh}qt\operatorname{Cosh}pt\operatorname{Cosh}qt}[g_{p,q}(t)], \tag{3.7}$$

where,

$$g_{p,q}(t)=\left[\frac{(p-q)\operatorname{Sinh}At\{\frac{1}{2}\operatorname{Cosh}(p+q)t+\frac{3}{2}\operatorname{Cosh}(p-q)t\}}{pq(p-q)}\right]-\left[\frac{(p\operatorname{Sinh}Ct+q\operatorname{Sinh}Bt)\{\frac{3}{2}\operatorname{Cosh}(p+q)t+\frac{1}{2}\operatorname{Cosh}(p-q)t\}}{pq(p-q)}\right]. \tag{3.8}$$

In case of $p\neq q=0$. Since $N_{p,q}\in C^\infty$, we have

$$\frac{\partial N_{p,0}}{\partial a}=\lim_{q\to 0}\frac{\partial N_{p,q}}{\partial a},\qquad \frac{\partial N_{p,0}}{\partial b}=\lim_{q\to 0}\frac{\partial N_{p,q}}{\partial b},$$

$$\frac{\partial N_{p,p}}{\partial a}=\lim_{q\to p}\frac{\partial N_{p,q}}{\partial a},\qquad \frac{\partial N_{p,p}}{\partial b}=\lim_{q\to p}\frac{\partial N_{p,q}}{\partial b},$$

$$\frac{\partial N_{0,0}}{\partial a}=\lim_{p\to 0}\frac{\partial N_{p,p}}{\partial a},\qquad \frac{\partial N_{0,0}}{\partial b}=\lim_{p\to 0}\frac{\partial N_{p,p}}{\partial b}.$$

Thus for $p \neq q = 0$, we have

$$
\begin{aligned}
(a-b)\left(\frac{\partial N_{p,0}}{\partial a}-\frac{\partial N_{p,0}}{\partial b}\right) &= \lim_{q\to 0}\left[(a-b)\left(\frac{\partial N_{p,q}}{\partial a}-\frac{\partial N_{p,q}}{\partial b}\right)\right] \\
&= \lim_{q\to 0}\left(\frac{(pq)N_{p,q}(a-b)}{\sqrt{ab}2\mathrm{Sinh}\,pt\mathrm{Sinh}\,qt\mathrm{Cosh}\,pt\mathrm{Cosh}\,qt}\left[g_{p,q}(t)\right]\right) \\
&= \frac{(p)N_{p,0}(a-b)\mathrm{Csch}\,2pt}{\sqrt{abt}}\left[g_{p,0}(t)\right].
\end{aligned}
$$

likewise for $q \neq p = 0$

$$
\begin{aligned}
(a-b)\left(\frac{\partial N_{0,q}}{\partial a}-\frac{\partial N_{0,q}}{\partial b}\right) &= \lim_{p\to 0}\left[(a-b)\left(\frac{\partial N_{p,q}}{\partial a}-\frac{\partial N_{p,q}}{\partial b}\right)\right] \\
&= \lim_{p\to 0}\left(\frac{(pq)N_{p,q}(a-b)}{\sqrt{ab}2\mathrm{Sinh}\,pt\mathrm{Sinh}\,qt\mathrm{Cosh}\,pt\mathrm{Cosh}\,qt}\left[g_{p,q}(t)\right]\right) \\
&= \frac{(q)N_{0,q}(a-b)\mathrm{Csch}\,2qt}{\sqrt{abt}}\left[g_{0,q}(t)\right].
\end{aligned}
$$

for $q = p \neq 0$

$$
\begin{aligned}
(a-b)\left(\frac{\partial N_{p,p}}{\partial a}-\frac{\partial N_{p,p}}{\partial b}\right) &= \lim_{q\to p}\left(\frac{(pq)N_{p,q}(a-b)}{\sqrt{ab}2\mathrm{Sinh}^2 pt\mathrm{Cosh}^2 pt}\left[g_{p,q}(t)\right]\right) \\
&= \frac{2p^2N_{p,p}(a-b)\mathrm{Csch}^2 2pt}{\sqrt{ab}}\left[g_{p,p}(t)\right].
\end{aligned}
$$

for $q = p = 0$

$$
\begin{aligned}
(a-b)\left(\frac{\partial N_{0,0}}{\partial a}-\frac{\partial N_{0,0}}{\partial b}\right) &= \lim_{p\to 0}\left(\frac{(p^2)N_{p,p}(a-b)}{2\sqrt{ab}\mathrm{Sinh}^2 pt\mathrm{Cosh}^2 pt}\left[g_{p,p}(t)\right]\right) \\
&= \frac{N_{0,0}(a-b)}{\sqrt{ab}2t^2}\left[g_{0,0}(t)\right].
\end{aligned}
$$

From all the above cases, we conclude that

$$
(a-b)\left(\frac{\partial N}{\partial a}-\frac{\partial N}{\partial b}\right) = \begin{cases}
\frac{2pqN(a-b)\mathrm{Csch}\,2pt\mathrm{Csch}\,2qt}{\sqrt{ab}}\left[g_{p,q}(t)\right] & pq(p-q)\neq 0; \\
\frac{qN_{0,q}(a-b)\mathrm{Csch}\,2qt}{\sqrt{abt}}\left[g_{0,q}(t)\right] & p=0,\ q\neq 0; \\
\frac{pN_{p,0}(a-b)\mathrm{Csch}\,2pt}{\sqrt{abt}}\left[g_{p,0}(t)\right] & p\neq 0,\ q=0; \\
\frac{2p^2N_{p,p}(a-b)\mathrm{Csch}^2 2pt}{\sqrt{ab}}\left[g_{p,p}(t)\right] & p=q\neq 0; \\
\frac{N_{0,0}(a-b)}{\sqrt{ab}2t^2}\left[g_{0,0}(t)\right] & p=q=0.
\end{cases} \tag{3.9}
$$

Since $(a-b)\left(\frac{\partial N}{\partial a}-\frac{\partial N}{\partial b}\right)$ is symmetric with respect to a and b, without loss of generality we assume $a > b$, then $t = \ln\sqrt{a/b} > 0$. It is easy to verify that $\frac{2N(a-b)}{\sqrt{ab}} > 0$,

$\frac{p}{\text{Sinh}\,2pt}$, $\frac{q}{\text{Sinh}\,2qt} > 0$ if $pq \neq 0$ for $t > 0$. Thus by Lemma 2.1 $N_{p,q}(a,b)$ are Schur convex (Schur concave) with respect to $(a,b) \in (0,\infty)\times(0,\infty)$ iff $(a-b)\left(\frac{\partial N}{\partial a} - \frac{\partial N}{\partial b}\right)$ $(\geqslant)(\leqslant)0$, if and only if $g(t) = g_{p,q}(t)(\geqslant)(\leqslant)0$ for all $t > 0$.

Hence the proof of Lemma 3.1. □

LEMMA 3.2. *The function* $g(t) = g_{p,q}(t)$ *and* $g^{\prime}(t) = \frac{\partial g_{p,q}(t)}{\partial t}$ *both are symmetric with respect to* p *and* q, *and continuous with respect to* p *and* q *on* $R\times R$.

Proof. It is can be easily verified that $g_{p,q}(t)$ and $\frac{\partial g_{p,q}(t)}{\partial t}$ are symmetric with respect to p and q. That is $\frac{\partial g_{p,q}(t)}{\partial t} = \frac{\partial g_{q,p}(t)}{\partial t}$.

From lemma 3.1, we observe that $g(t) = g_{p,q}(t)$ is continuous with respect to (p, q) on $R\times R$. Finally, we shall prove that $g^{\prime}(t) = \frac{\partial g_{p,q}(t)}{\partial t}$ is also continuous with respect to (p, q) on $R\times R$.

Easy calculations result the following cases.

Case (i) $pq(p-q) \neq 0$

$$\begin{aligned} g^{\prime}(t) = & \frac{A\text{Cosh}At[1.5\text{Cosh}(p-q)t + 0.5\text{Cosh}(p+q)t]}{pq} \\ & - \frac{(pC\text{Cosh}Ct + Bq\text{Cosh}Bt)[0.5\text{Cosh}(p-q)t + 1.5\text{Cosh}(p+q)t]}{p(p-q)q} \\ & + \frac{\text{Sinh}At[1.5(p-q)\text{Sinh}(p-q)t + 0.5(p+q)\text{Sinh}(p+q)t]}{pq} \\ & - \frac{(p\text{Sinh}Ct + q\text{Sinh}Bt)[0.5(p-q)\text{Sinh}(p-q)t + 1.5(p+q)\text{Sinh}(p+q)t]}{p(p-q)q}. \end{aligned}$$

Case (ii) $q = 0,\ p \neq 0$

$$\begin{aligned} g^{\prime}(t) = & \frac{\partial g_{p,q}(t)}{\partial t} \\ = & \frac{3p\text{Cosh}t + (1-2p)\text{Cosh}(1-2p)t + p\text{Cosh}(1-2p)t - (1+2p)\text{Cosh}(1+2p)t}{p^2} \\ & + \frac{3pt\text{Sinh}t + (1-2p)pt\text{Sinh}(1-2p)t}{p^2}. \end{aligned}$$

Case (iii) $p = 0,\ q \neq 0$

$$\begin{aligned} g^{\prime}(t) = & \frac{\partial g_{p,q}(t)}{\partial t} \\ = & \frac{3q\text{Cosh}t + (1-2q)\text{Cosh}(1-2q)t + q\text{Cosh}(1-2q)t - (1+2q)\text{Cosh}(1+2q)t}{q^2} \\ & + \frac{3qt\text{Sinh}t + (1-2q)qt\text{Sinh}(1-2q)t}{q^2}. \end{aligned}$$

Case (iv) $p = q \neq 0$

$$
\begin{aligned}
g'(t) &= \frac{\partial g_{p,q}(t)}{\partial t} \\
&= \frac{0.5\mathrm{Cosh}\,t - q\mathrm{Cosh}\,t + 1.5\mathrm{Cosh}\,t\mathrm{Cosh}\,2qt - 3q\mathrm{Cosh}\,t\mathrm{Cosh}\,2qt}{q^2} \\
&\quad + \frac{-1.5(1-2q)\mathrm{Cosh}\,(1-2q)t - 0.5(1-2q)\mathrm{Cosh}\,2qt\mathrm{Cosh}\,(1-2q)t - qt\mathrm{Sinh}\,t}{q^2} \\
&\quad + \frac{-3qt\mathrm{Cosh}\,2qt\mathrm{Sinh}\,t - 6q^2t\mathrm{Cosh}\,t\mathrm{Sinh}\,2qt + 3q\mathrm{Sinh}\,t\mathrm{Sinh}\,2qt}{q^2} \\
&\quad - \frac{q\mathrm{Sinh}\,2qt\mathrm{Sinh}\,(1-2q)t}{q^2}.
\end{aligned}
$$

Case (v) $p = q = 0$

$$g'(t) = \frac{\partial g_{p,q}(t)}{\partial t} = -2t^2\mathrm{Cosh}\,t - 4t\mathrm{Sinh}\,t.$$

It is clear that $\frac{\partial g_{p,q}(t)}{\partial t}$ is continuous with respect to (p, q) on $R \times R$, with $pq(p-q) \neq 0$. Further, we have

$$\mathrm{Lim}_{q\to 0}\frac{\partial g_{p,q}(t)}{\partial t} = \frac{\partial g_{p,0}(t)}{\partial t}; \qquad \mathrm{Lim}_{p\to 0}\frac{\partial g_{p,q}(t)}{\partial t} = \frac{\partial g_{0,q}(t)}{\partial t};$$

$$\mathrm{Lim}_{q\to p}\frac{\partial g_{p,q}(t)}{\partial t} = \frac{\partial g_{p,p}(t)}{\partial t}; \qquad \mathrm{Lim}_{p\to 0}\frac{\partial g_{p,p}(t)}{\partial t} = \frac{\partial g_{0,0}(t)}{\partial t}.$$

Hence, $g'(t)$ is continuous for all $(p, q) \in R \times R$. □

Hence the proof of Lemma 3.2.

LEMMA 3.3. $\mathrm{Lim}_{t\to 0, t>0}\frac{g_{p,q}(t)}{t^3} = \frac{-2}{3}(p+q+3)$.

Proof. We see that 1^{st} and 2^{nd} derivatives of $g_{p,q}(t)$ are zero at $t = 0$. Now, we shall find the limiting value of $\frac{g_{p,q}(t)}{t^3}$, by considering the following cases.

Case (i) $pq(p-q) \neq 0$. Using L-Hospital's rule (three times), we have;

$$\mathrm{Lim}_{t\to 0,t>0}\frac{g_{p,q}(t)}{t^3} = \mathrm{Lim}_{t\to 0,t>0}\frac{g'_{p,q}(t)}{3t^2} = \ldots = \mathrm{Lim}_{t\to 0,t>0}\frac{g'''_{p,q}(t)}{6} = \frac{-2}{3}(p+q+3).$$

Case (ii) $p = 0$, $q \neq 0$

$$\mathrm{Lim}_{t\to 0,t>0}\frac{g_{0,q}(t)}{t^3} = \mathrm{Lim}_{t\to 0,t>0}\frac{g'_{0,q}(t)}{3t^2} = \ldots = \mathrm{Lim}_{t\to 0,t>0}\frac{g'''_{0,q}(t)}{6} = \frac{-2}{3}(q+3).$$

Case (iii) $q=0,\ p\neq 0$

$$\mathrm{Lim}_{t\to 0,t>0}\frac{g_{p,0}(t)}{t^3}=\mathrm{Lim}_{t\to 0,t>0}\frac{g_{p,0}^{'}(t)}{3t^2}=\ldots=\mathrm{Lim}_{t\to 0,t>0}\frac{g_{p,0}^{'''}(t)}{6}=\frac{-2}{3}(p+3).$$

Case (iv) $p=q\neq 0$

$$\mathrm{Lim}_{t\to 0,t>0}\frac{g_{p,p}(t)}{t^3}=\mathrm{Lim}_{t\to 0,t>0}\frac{g_{p,p}^{'}(t)}{3t^2}=\ldots=\mathrm{Lim}_{t\to 0,t>0}\frac{g_{p,p}^{'''}(t)}{6}=\frac{-2}{3}(2p+3).$$

Case (v) $p=q=0$

$$\mathrm{Lim}_{t\to 0,t>0}\frac{g_{0,0}(t)}{t^3}=\mathrm{Lim}_{t\to 0,t>0}\frac{g_{0,0}^{'}(t)}{3t^2}=\ldots=\mathrm{Lim}_{t\to 0,t>0}\frac{g_{0,0}^{'''}(t)}{6}=-2. \quad \square$$

By summarizing the above cases, we conclude the proof of Lemma 3.3.
The proof of our main theorem stated below follows from the preceding lemmas:

THEOREM 3.1. *For fixed* $(p,q)\in R\times R$ *and* $r=s$*,*

1. *Stolarsky's extended type means* $N_{p,q}(a,b;r,s)$ *are Schur convex with respect to* (a,b) *if* $p+q+3\leqslant 0$.

2. *Stolarsky's extended type means* $N_{p,q}(a,b;r,s)$ *are Schur concave if* $p+q+3\geqslant 0$.

4. Conclusion

In the case when $r\neq s$, the convexity and concavity of Stolarsky's extended type means $N_{p,q}(a,b;r,s)$ is an open problem.

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K. Murali
Research & Development Centre, Bharathiar University
Coimbatore – 641 046, Tamil Nadu
and
Department of Mathematics, Amrita School of Engineering
Amrita Vishwa Vidyapeetham (University)
Bengaluru-35, Karanataka, India
e-mail: murali_k73@yahoo.co.in

K. M. Nagaraja
Department of Mathematics, JSS Academy of Technical Education
Uttarahalli-Kengeri Main Road, Bangalore-60, Karanataka, India
e-mail: nagkmn@gmail.com