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HEISENBERG UNCERTAINTY INEQUALITY FOR GABOR TRANSFORM

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Abstract. We discuss the Heisenberg uncertainty inequality for groups of the form $K \ltimes \mathbb{R}^n$, K is a separable unimodular locally compact group of type I. This inequality is also proved for Gabor transform for several classes of groups of the form $K \ltimes \mathbb{R}^n$.

1. Introduction

The uncertainty principle states that a non-zero function and its Fourier transform cannot both be sharply localized. The most precise way of formulating this principle quantitatively is the inequality known as *Heisenberg uncertainty inequality*. Let f be any function in $L^2(\mathbb{R})$. The Fourier transform of f is defined as

$$\widehat{f}(\omega) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i\omega x}\, dx.$$

The following theorem gives the Heisenberg uncertainty inequality for the Fourier transform on $\mathbb{R}$:

THEOREM 1.1. *For any $f \in L^2(\mathbb{R})$, we have*

$$\frac{\|f\|_2^2}{4\pi} \leqslant \left(\int_{\mathbb{R}} x^2\, |f(x)|^2\, dx\right)^{1/2} \left(\int_{\mathbb{R}} \omega^2\, |\widehat{f}(\omega)|^2\, d\omega\right)^{1/2}, \tag{1.1}$$

where $\|\cdot\|_2$ denotes the L^2-norm.

For proof of the theorem, refer to [7]. Various uncertainty inequalities were presented in [3] and [4] on stratified Lie groups and groups of polynomial volume growth respectively.

The representation of f as a function of x is usually called its *time-representation*, while the representation of $\hat{f}$ as a function of ω is called its *frequency-representation*. The Fourier transform has been the most commonly used tool for analyzing frequency properties of a given signal, but the problem with this tool is that after transformation, the information about time is lost and it is hard to tell where a certain frequency occurs.

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To counter this problem, we can use *joint time-frequency representation*, i.e., Gabor transform.

Let $\psi \in L^2(\mathbb{R})$ be a fixed non-zero function usually called a *window function*. The Gabor transform of a function $f \in L^2(\mathbb{R})$ with respect to the window function ψ is defined by

$$G_\psi f : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{C}$$

such that

$$G_\psi f(t,\omega) = \int_{\mathbb{R}} f(x)\, \overline{\psi(x-t)}\, e^{-2\pi i \omega x}\, dx,$$

for all $(t,\omega) \in \mathbb{R} \times \widehat{\mathbb{R}}$. The following uncertainty inequality of Heisenberg-type has been proved by Wilczok [15].

THEOREM 1.2. *Let ψ be a window function. Then, for arbitrary $f \in L^2(\mathbb{R})$, the following inequality holds*

$$\frac{\|\psi\|_2\, \|f\|_2^2}{4\pi} \leqslant \left(\int_{\mathbb{R}} x^2\, |f(x)|^2\, dx \right)^{1/2} \left(\int_{\mathbb{R}^2} \omega^2\, |G_\psi f(t,\omega)|^2\, dt\, d\omega \right)^{1/2}. \tag{1.2}$$

The continuous Gabor transform for second countable, non-abelian, unimodular and type I groups has been defined by Farashahi and Kamyabi-Gol in [5].

In section 2, we shall state the Heisenberg uncertainty inequality for Fourier transform on the groups of the form $K \ltimes \mathbb{R}^n$, where K is a separable unimodular locally compact group of type I and prove it for the semi-direct product $K \ltimes \mathbb{R}^n$ (where K is a compact subgroup of the group of automorphisms of $\mathbb{R}^n$). In section 3, we shall discuss continuous Gabor transform and prove Heisenberg uncertainty inequality for Gabor transform on $K \ltimes \mathbb{R}^n$ (where K is a separable unimodular locally compact group of type I) that satisfy the Heisenberg uncertainty inequality for Fourier transform. The explicit forms of Heisenberg uncertainty inequality for Gabor transform are obtained for $K \ltimes \mathbb{R}^n$, K is a compact subgroup of $\mathrm{Aut}(\mathbb{R}^n)$; $\mathbb{R}^n \times K$, K is separable unimodular locally compact group of type I; Heisenberg group $\mathbb{H}_n$; Thread-like nilpotent Lie groups; 2-NPC nilpotent Lie groups and several classes of connected, simply connected nilpotent Lie groups.

2. Extensions of $\mathbb{R}^n$

Let $G = K \ltimes \mathbb{R}^n$, where K is a separable unimodular locally compact group of type I. For $\gamma \in \widehat{\mathbb{R}^n}$, let G_γ, K_γ denote the stabilizer subgroup of γ in G and K respectively and let

$$\check{G}_\gamma = \{\nu \in \widehat{G_\gamma} : \nu|_{\mathbb{R}^n} \text{ is a finite multiple of } \gamma\}.$$

Then for $\nu \in \check{G}_\gamma$, the representation $\pi_\nu = \mathrm{ind}_{G_\gamma}^G \nu$ is irreducible and

$$\widehat{G} = \cup_{\widehat{\mathbb{R}^n}/G} \{\pi_\nu : \nu \in \check{G}_\gamma\}.$$

Since $\mathbb{R}^n$ is abelian, any $\nu \in \check{G}_\gamma$ is of the form $\nu = \sigma \otimes \gamma$, $\nu(kx) = \sigma(k)\gamma(x)$, $k \in K_\gamma$, $x \in \mathbb{R}^n$ and $\sigma \in \widehat{K_\gamma}$.

We consider the induced representations

$$\pi_{\gamma,\sigma} = \mathrm{ind}_{G_\gamma}^G(\gamma \otimes \sigma).$$

The Plancherel formula for G (for details, see [9]) takes the following form:

PROPOSITION 2.1. (Plancherel formula) *For all $f \in L^2(G)$, we have*

$$\int_G |f(g)|^2\, dg = \int_{\widehat{\mathbb{R}^n}/G} \int_{\widehat{K_\gamma}} \|\pi_{\gamma,\sigma}(f)\|_2^2\, d\mu_\gamma(\sigma)\, d\overline{\mu}_{\mathbb{R}^n}(\overline{\gamma}).$$

We now state the Heisenberg uncertainty inequality for Fourier transform on G which has been proved, in particular cases of $\mathbb{R}^n$ (see [7]); Heisenberg group (see [14], [12] and [16]); $\mathbb{R}^n \times K$ (where K is a separable unimodular locally compact group of type I), Euclidean motion group $M(n) = SO(n) \ltimes \mathbb{R}^n$ and several general classes of nilpotent Lie groups which include thread-like nilpotent Lie groups, 2-NPC nilpotent Lie groups and several low-dimensional nilpotent Lie groups (see [2]).

THEOREM 2.2. *For any $f \in L^2(G)$ and $a,b \geqslant 1$, we have*

$$\|f\|_2^{\left(\frac{1}{a}+\frac{1}{b}\right)} \leqslant C\left(\int_{K\times\mathbb{R}^n} \|x\|^{2a}\, |f(k,x)|^2\, dx\, dk\right)^{\frac{1}{2a}}$$
$$\times\left(\int_{\widehat{\mathbb{R}^n}/G} \int_{\widehat{K_\gamma}} \|\gamma\|^{2b} \|\pi_{\gamma,\sigma}(f)\|_{HS}^2\, d\mu_\gamma(\sigma)\, d\overline{\mu}_{\mathbb{R}^n}(\overline{\gamma})\right)^{\frac{1}{2b}}, \tag{H}$$

where C is a constant.

We do not know whether the inequality (H) is true for $K \ltimes \mathbb{R}^n$, however we now prove the Heisenberg uncertainty inequality for Fourier transform when K is a compact subgroup of $\mathrm{Aut}(\mathbb{R}^n)$.

Let G be the semi-direct product $K \ltimes \mathbb{R}^n$, where K is a compact subgroup of $\mathrm{Aut}(\mathbb{R}^n)$. The Haar measure on G is $dg = d\nu(k)\, dx$, where $d\nu(k)$ denotes the normalized Haar measure of K and dx denotes the Lebesgue measure on $\mathbb{R}^n$. We shall now give more explicit description of the unitary dual space of the group G in this case which can be determined by Mackey's theory. For more details, refer to [10].

Let ℓ be a non-zero real linear form on $\mathbb{R}^n$ and let χ_ℓ be the unit character of $\mathbb{R}^n$ defined by $\chi_\ell(x) = e^{i\langle l,x\rangle}$. The natural action $g \cdot \ell$ of G on the dual vector space of $\mathbb{R}^n$ is given by

$$\langle g\cdot\ell, x\rangle = \langle \ell, g^{-1}xg\rangle,$$

for $g \in G$ and $x \in \mathbb{R}^n$. Therefore, if g acts on $\widehat{\mathbb{R}^n}$ by

$$g \cdot \chi_\ell(x) := \chi_\ell(g^{-1}xg),$$

we get $g \cdot \chi_\ell = \chi_{g\cdot\ell}$. Define

$$K_\ell = \{k \in K : k \cdot \chi_\ell = \chi_\ell\}.$$

Then, the subgroup $K_\ell \ltimes \mathbb{R}^n$ is the stabilizer of χ_ℓ in G. We take the normalized Haar measure $d\nu_\ell$ on K_ℓ and a normalized K-invariant measure $d\dot{\nu}_\ell$ on K/K_ℓ so that

$$\int_K \xi(k)\, d\nu(k) = \int_{K/K_\ell} \int_{K_\ell} \xi(kk')\, d\nu_\ell(k')\, d\dot{\nu}_\ell(kK_\ell).$$

Regarding the action of K on $\widehat{\mathbb{R}^n}$ which is isomorphic to $\mathbb{R}^n$, we set by $d\bar{\ell}$ the image of the Lebesgue measure on $\mathbb{R}^n/K$ by the canonical projection $\mathbb{R}^n \ni \ell \mapsto \bar{\ell} := K.\ell \in \mathbb{R}^n/K$ such that

$$\int_{\mathbb{R}^n} \varphi(\ell)\, d\ell = \int_{\mathbb{R}^n/K} \int_K \varphi(k.\ell)\, d\nu(k)\, d\bar{\ell}.$$

Let σ be an irreducible unitary representation of K_ℓ and $\mathscr{H}_{\ell,\sigma}$ be the completion of the vector space of all continuous mapping $\xi : K \to \mathscr{H}_\sigma$ which satisfies $\xi(ks) = \sigma(s)^*(\xi(k))$ for $k \in K$ and $s \in K_\ell$ with respect to the norm

$$\|\xi\|_2 = \left(\int_K \|\xi(k)\|^2_{\mathscr{H}_\sigma}\, d\nu(k) \right)^{1/2}.$$

The induced representation

$$\pi_{\ell,\sigma} := \mathrm{ind}^G_{K_\ell \ltimes \mathbb{R}^n}(\sigma \otimes \chi_\ell),$$

realized on the Hilbert space $\mathscr{H}_{\ell,\sigma}$ by

$$\pi_{\ell,\sigma}(k,x)\xi(s) = e^{i\langle \ell, s^{-1}xs\rangle}\xi(k^{-1}s) = e^{i\langle s.\ell, x\rangle}\xi(k^{-1}s),$$

for $\xi \in \mathscr{H}_{\ell,\sigma}$, $(k,x) \in G$ and $s \in K$, is an irreducible representation of G. Furthermore, every infinite dimensional irreducible unitary representation of G is equivalent to some representation $\pi_{\ell,\sigma}$.

The Plancherel formula [6, Theorem 7.44] can be stated in this particular case as follows:

PROPOSITION 2.3. (Plancherel formula) *Let* $f \in L^1(G) \cap L^2(G)$, *then*

$$\int_{K\times\mathbb{R}^n} |f(k,x)|^2\, dx\, dk = \int_{\mathbb{R}^n/K} \sum_{\sigma \in \widehat{K_\ell}} \|\pi_{\ell,\sigma}(f)\|^2_{HS}\, d\bar{\ell}. \tag{2.1}$$

The proof of the following Heisenberg uncertainty inequality for Fourier transform on G is similar in nature to that for the Euclidean motion group which has been proved in [2], so we only outline the proof.

THEOREM 2.4. *For any $f \in L^2(G)$ and $a,b \geqslant 1$, we have*

$$\|f\|_2^{\left(\frac{1}{a}+\frac{1}{b}\right)} \leqslant C\left(\int_{K\times\mathbb{R}^n} \|x\|^{2a}\,|f(k,x)|^2\,dx\,dk\right)^{\frac{1}{2a}} \times \left(\int_{\mathbb{R}^n/K} \sum_{\sigma\in\widehat{K_\ell}} \|\ell\|^{2b}\|\pi_{\ell,\sigma}(f)\|_{HS}^2\,d\overline{\ell}\right)^{\frac{1}{2b}}, \tag{2.2}$$

where C is a constant.

Proof. As in [2], it suffices to prove the inequality (2.2) for functions in $\mathscr{S}(G)$, the space of C^∞-functions which are rapidly decreasing on G.

Let $f \in \mathscr{S}(G)$. Proceeding as in [2, Theorem 3.2]), we obtain

$$\frac{\|f\|_2^{1+\frac{1}{a}}}{2} \leqslant \left(\int_{K\times\mathbb{R}^n} \|x\|^{2a}\,|f(k,x)|^2\,dx\,dk\right)^{\frac{1}{2a}} \left(\int_{\mathbb{R}^n/K} \sum_{\sigma\in\widehat{K_\ell}} \left\|\pi_{\ell,\sigma}\left(\frac{\partial f}{\partial x_1}\right)\right\|_{\mathrm{HS}}^2 d\overline{\ell}\right)^{1/2}. \tag{2.3}$$

For each non-zero linear form ℓ on $\mathbb{R}^n$ and each irreducible unitary representation σ of K_ℓ, consider the representation $\pi_{\ell,\sigma}$ realized on the Hilbert space $\mathscr{H}_{\ell,\sigma}$ as

$$\pi_{\ell,\sigma}(k,x)\xi(s) = e^{i\langle \ell, s^{-1}xs\rangle}\xi(k^{-1}s) = e^{i\langle s.\ell,x\rangle}\xi(k^{-1}s),$$

for $\xi \in \mathscr{H}_{\ell,\sigma}$, $(k,x)\in G$ and $s\in K$. For $h\in\mathbb{R}$ and $x=(x_1,x_2,\ldots,x_n)$, $e_1=(1,0,0,\ldots,0)$ $\in\mathbb{R}^n$, we can write

$$\pi_{\ell,\sigma}(k,x_1-h,x_2,\ldots,x_n)^*\xi(s) = e^{ih\langle \ell,s^{-1}e_1s\rangle}\,\pi_{\ell,\sigma}(k,x_1,x_2,\ldots,x_n)^*\xi(s).$$

Since $f\in\mathscr{S}(G)$, we observe that

$$\pi_{\ell,\sigma}\left(\frac{\partial f}{\partial x_1}\right)\xi(s) = i\langle \ell, s^{-1}e_1s\rangle\,\pi_{\ell,\sigma}(f)\xi(s).$$

Since $s\mapsto s^{-1}e_1s$ is a continuous map from K to $\mathbb{R}^n$, so $\{s^{-1}e_1s : s\in K\}$ is bounded. For any orthonormal basis $\{\xi_j\}$ of $\mathscr{H}_{\ell,\sigma}$, we have

$$\begin{aligned}\left\|\pi_{\ell,\sigma}\left(\frac{\partial f}{\partial x_1}\right)\right\|_{\mathrm{HS}}^2 &= \sum_j \int_K |i\langle \ell,s^{-1}e_1s\rangle\,\pi_{\ell,\sigma}(f)\xi_j(s)|^2\,ds\\ &\leqslant const.\,\|\ell\|^2\sum_j\int_K |\pi_{\ell,\sigma}(f)\xi_j(s)|^2\,ds = const.\,\|\ell\|^2\|\pi_{\ell,\sigma}(f)\|_{\mathrm{HS}}^2.\end{aligned}$$

So, (2.3) can be written as

$$\|f\|_2^{1+\frac{1}{a}} \leqslant C\left(\int_{K\times\mathbb{R}^n} \|x\|^{2a}\,|f(k,x)|^2\,dx\,dk\right)^{\frac{1}{2a}} \left(\int_{\mathbb{R}^n/K} \sum_{\sigma\in\widehat{K_\ell}} \|\ell\|^2\left\|\pi_{\ell,\sigma}(f)\right\|_{\mathrm{HS}}^2 d\overline{\ell}\right)^{1/2}. \tag{2.4}$$

Using Hölder's inequality in the second integral on R.H.S. of the inequality (2.4), we obtain the required inequality (2.2). □

3. Continuous Gabor transform

Let $\mathscr{H}$ be a separable Hilbert space and let $\mathscr{B}(\mathscr{H})$ denotes the set of all bounded linear operators on $\mathscr{H}$. An operator $T \in \mathscr{B}(\mathscr{H})$ is called Hilbert-Schmidt operator if and only if

$$\sum_k \|Te_k\|^2 < \infty,$$

for some, and hence for any, orthonormal basis $\{e_k\}$ of $\mathscr{H}$. We denote the set of all Hilbert-Schmidt operators on $\mathscr{H}$ by $\mathrm{HS}(\mathscr{H})$. For each $T \in \mathrm{HS}(\mathscr{H})$, the Hilbert-Schmidt norm $\|T\|_{\mathrm{HS}}$ of T is defined as

$$\|T\|_{\mathrm{HS}}^2 := \sum_k \|Te_k\|^2.$$

Also, $\mathrm{HS}(\mathscr{H})$ forms a Hilbert space with the inner product given by

$$\langle T, S\rangle_{\mathrm{HS}(\mathscr{H})} = \mathrm{tr}\,(S^*T).$$

For more details, refer to [6].

Let G be a second countable, non-abelian, unimodular and type I group. Let dx be the Haar measure on G. Let $d\pi$ be the Plancherel meaure on $\widehat{G}$. For each $(x,\pi) \in G \times \widehat{G}$, let

$$\mathscr{H}_{(x,\pi)} = \pi(x)\mathrm{HS}(\mathscr{H}_\pi),$$

where $\pi(x)\mathrm{HS}(\mathscr{H}_\pi) = \{\pi(x)T : T \in \mathrm{HS}(\mathscr{H}_\pi)\}$. Then, $\mathscr{H}_{(x,\pi)}$ is a Hilbert space with the inner product given by

$$\langle \pi(x)T, \pi(x)S\rangle_{\mathscr{H}_{(x,\pi)}} = \mathrm{tr}\,(S^*T) = \langle T, S\rangle_{\mathrm{HS}(\mathscr{H}_\pi)}.$$

One can easily verify that $\mathscr{H}_{(x,\pi)} = \mathrm{HS}(\mathscr{H}_\pi)$ for all $(x,\pi) \in G \times \widehat{G}$. The family $\{\mathscr{H}_{(x,\pi)}\}_{(x,\pi)\in G\times\widehat{G}}$ of Hilbert spaces indexed by $G \times \widehat{G}$ is a field of Hilbert spaces over $G \times \widehat{G}$. Let $\mathscr{H}^2(G \times \widehat{G})$ denote the direct integral of $\{\mathscr{H}_{(x,\pi)}\}_{(x,\pi)\in G\times\widehat{G}}$ with respect to the product measure $dx\,d\pi$, i.e., the space of all measurable vector fields F on $G \times \widehat{G}$ such that

$$\|F\|^2_{\mathscr{H}^2(G\times\widehat{G})} = \int_{G\times\widehat{G}} \|F(x,\pi)\|^2_{(x,\pi)}\, dx\, d\pi < \infty.$$

$\mathscr{H}^2(G \times \widehat{G})$ is a Hilbert space with the inner product given by

$$\langle F, K\rangle_{\mathscr{H}^2(G\times\widehat{G})} = \int_{G\times\widehat{G}} \mathrm{tr}\,[F(x,\pi)K(x,\pi)^*]\, dx\, d\pi.$$

Let $f \in C_c(G)$, the set of all continuous complex-valued functions on G with compact supports and ψ be a fixed non-zero function in $L^2(G)$ which is sometimes called a window function. For $(x,\pi) \in G \times \widehat{G}$, the continuous *Gabor Transform* of f with respect to the window function ψ can be defined as a measurable field of operators on $G \times \widehat{G}$ by

$$G_\psi f(x,\pi) := \int_G f(y)\, \overline{\psi(x^{-1}y)}\, \pi(y)^*\, dy. \tag{3.1}$$

The operator-valued integral (3.1) is considered in the weak-sense, i.e., for each $(x,\pi) \in G \times \widehat{G}$ and $\xi, \eta \in \mathscr{H}_\pi$, we have

$$\langle G_\psi f(x,\pi)\xi, \eta\rangle = \int_G f(y)\, \overline{\psi(x^{-1}y)}\, \langle \pi(y)^*\xi, \eta\rangle\, dy.$$

For each $x \in G$, define $f_x^\psi : G \to \mathbb{C}$ by

$$f_x^\psi(y) := f(y)\, \overline{\psi(x^{-1}y)}.$$

Since, $f \in C_c(G)$ and $\psi \in L^2(G)$, we have $f_x^\psi \in L^1(G) \cap L^2(G)$, for all $x \in G$. The Fourier transform is given by

$$\widehat{f_x^\psi}(\pi) = \int_G f_x^\psi(y)\, \pi(y)^*\, dy = \int_G f(y)\, \overline{\psi(x^{-1}y)}\, \pi(y)^*\, dy = G_\psi f(x,\pi).$$

Also, using Plancherel theorem [6, Theorem 7.44], we see that $\widehat{f_x^\psi}(\pi)$ is a Hilbert-Schmidt operator for almost all $\pi \in \widehat{G}$. Therefore, $G_\psi f(x,\pi)$ is a Hilbert-Schmidt operator for all $x \in G$ and for almost all $\pi \in \widehat{G}$. As in [5], for $f \in C_c(G)$ and a window function $\psi \in L^2(G)$, we have

$$\|G_\psi f\|_{\mathscr{H}^2(G\times\widehat{G})} = \|\psi\|_2\, \|f\|_2.$$

The above equality shows that the continuous Gabor transform $G_\psi : C_c(G) \to \mathscr{H}^2(G \times \widehat{G})$ defined by $f \mapsto G_\psi f$ is a multiple of an isometry. So, we can extend G_ψ uniquely to a bounded linear operator from $L^2(G)$ into a closed subspace H of $\mathscr{H}^2(G \times \widehat{G})$ which we still denote by G_ψ and this extension satisfies

$$\|G_\psi f\|_{\mathscr{H}^2(G\times\widehat{G})} = \|\psi\|_2\, \|f\|_2, \tag{3.2}$$

for each $f \in L^2(G)$. We now prove an important lemma.

LEMMA 3.1. *Let $f \in L^2(G)$ and $\psi \in L^2(G)$ be a window function. Then*

$$G_\psi f(x,\pi) = \widehat{f_x^\psi}(\pi).$$

Proof. Let $f \in L^2(G)$. Since $C_c(G)$ is dense in $L^2(G)$, there exists a sequence $\{\phi_n\}$ in $C_c(G)$ such that $f = \lim_{n\to\infty} \phi_n$ in the L^2-norm. It follows that

$$G_\psi : L^2(G) \to H \subseteq \mathscr{H}^2(G \times \widehat{G})$$

satisfies $G_\psi f = \lim_{n\to\infty} G_\psi \phi_n$ in the $\mathscr{H}^2(G\times\widehat{G})$-norm and

$$G_\psi \phi_n(x,\pi) = \widehat{(\phi_n)_x^\psi}(\pi).$$

$$\text{Now,}\quad \|G_\psi f - G_\psi \phi_n\|^2_{\mathscr{H}^2(G\times\widehat{G})} = \int_G\int_{\widehat{G}} \|G_\psi f(x,\pi) - G_\psi \phi_n(x,\pi)\|^2_{\mathrm{HS}}\, dx\, d\pi$$
$$= \int_G\int_{\widehat{G}} \|G_\psi f(x,\pi) - \widehat{(\phi_n)_x^\psi}(\pi)\|^2_{\mathrm{HS}}\, dx\, d\pi$$

$$\text{and}\quad \|\psi\|_2^2\, \|f-\phi_n\|_2^2 = \int_G |\psi(x)|^2\, dx \int_G |(f-\phi_n)(y)|^2\, dy$$
$$= \int_G\int_G |(f-\phi_n)(y)|^2\, |\overline{\psi(x^{-1}y)}|^2\, dx\, dy$$
$$= \int_G\int_G |f(y)\,\overline{\psi(x^{-1}y)} - \phi_n(y)\,\overline{\psi(x^{-1}y)}|^2\, dx\, dy$$
$$= \int_G\int_G |(f_x^\psi - (\phi_n)_x^\psi)(y)|^2\, dx\, dy$$
$$= \int_G\int_{\widehat{G}} \|\widehat{f_x^\psi}(\pi) - \widehat{(\phi_n)_x^\psi}(\pi)\|^2_{\mathrm{HS}}\, dx\, d\pi.$$

Hence, $G_\psi f(x,\pi) = \widehat{f_x^\psi}(\pi)$ for all $f\in L^2(G)$. $\square$

We now establish Heisenberg uncertainty inequality for Gabor transform on $G = K\ltimes\mathbb{R}^n$, where K is a separable unimodular locally compact group of type I. The continuous *Gabor Transform* of f with respect to the window function ψ can be defined as follows:

$$G_\psi f(u,t,\gamma,\sigma) := \int_G f_{u,t}^\psi(k,x)\, \pi_{\gamma,\sigma}(k,x)^*\, dx\, dk, \tag{3.3}$$

where $f_{u,t}^\psi(k,x) = f(k,x)\,\overline{\psi((u,t)^{-1}(k,x))}$, $(u,t)\in G$, $\gamma\in\widehat{\mathbb{R}^n}$ and $\sigma\in\widehat{K_\gamma}$. Also, the equality in Lemma 3.1 takes the following form:

$$G_\psi f(u,t,\gamma,\sigma) = \pi_{\gamma,\sigma}(f_{u,t}^\psi). \tag{3.4}$$

THEOREM 3.2. *Let $G = K\ltimes\mathbb{R}^n$ satisfies the inequality* (H) *and ψ be a window function. For $a,b\geqslant 1$, we have*

$$\|\psi\|_2^{\frac{1}{b}}\, \|f\|_2^{\left(\frac{1}{a}+\frac{1}{b}\right)} \leqslant C\left(\int_{K\times\mathbb{R}^n} \|x\|^{2a}\, |f(k,x)|^2\, dx\, dk\right)^{\frac{1}{2a}}$$
$$\times\left(\int_{K\times\mathbb{R}^n}\int_{\widehat{\mathbb{R}^n}/G}\int_{\widehat{K_\gamma}} \|\gamma\|^{2b}\, \|G_\psi f(u,t,\gamma,\sigma)\|^2_{HS}\, d\mu_\gamma(\sigma)\, d\overline{\mu}_{\mathbb{R}^n}(\overline{\gamma})\, du\, dt\right)^{\frac{1}{2b}}. \tag{3.5}$$

Proof. Assume that both integrals on the right-hand side of (3.5) are finite. Since $f_{u,t}^{\psi} \in L^2(G)$ for all $(u,t) \in G$, so by using inequality (H) for $a = b = 1$, we have

$$\|f_{u,t}^{\psi}\|_2^2 \leqslant C\left(\int_{K\times\mathbb{R}^n} \|x\|^2\, |f_{u,t}^{\psi}(k,x)|^2\, dx\, dk\right)^{1/2} \times \left(\int_{\widehat{\mathbb{R}^n}/G}\int_{\widehat{K_\gamma}} \|\gamma\|^2\, \|\pi_{\gamma,\sigma}(f_{u,t}^{\psi})\|_{\mathrm{HS}}^2\, d\mu_\gamma(\sigma)\, d\overline{\mu}_{\mathbb{R}^n}(\overline{\gamma})\right)^{1/2}. \tag{3.6}$$

Also, by Proposition 2.3 and (3.4), we have

$$\int_{\widehat{\mathbb{R}^n}/G}\int_{\widehat{K_\gamma}} \|G_\psi f(u,t,\gamma,\sigma)\|_{\mathrm{HS}}^2\, d\mu_\gamma(\sigma)\, d\overline{\mu}_{\mathbb{R}^n}(\overline{\gamma}) = \int_{\widehat{\mathbb{R}^n}/G}\int_{\widehat{K_\gamma}} \|\pi_{\gamma,\sigma}(f_{u,t}^{\psi})\|_{\mathrm{HS}}^2\, d\mu_\gamma(\sigma)\, d\overline{\mu}_{\mathbb{R}^n}(\overline{\gamma}) = \|f_{u,t}^{\psi}\|_2^2. \tag{3.7}$$

On combining (3.6) and (3.7), we obtain

$$\int_{\widehat{\mathbb{R}^n}/G}\int_{\widehat{K_\gamma}} \|G_\psi f(u,t,\gamma,\sigma)\|_{\mathrm{HS}}^2\, d\mu_\gamma(\sigma)\, d\overline{\mu}_{\mathbb{R}^n}(\overline{\gamma}) \leqslant C\left(\int_{K\times\mathbb{R}^n} \|x\|^2\, |f_{u,t}^{\psi}(k,x)|^2\, dx\, dk\right)^{1/2} \times \left(\int_{\widehat{\mathbb{R}^n}/G}\int_{\widehat{K_\gamma}} \|\gamma\|^2\, \|\pi_{\gamma,\sigma}(f_{u,t}^{\psi})\|_{\mathrm{HS}}^2\, d\mu_\gamma(\sigma)\, d\overline{\mu}_{\mathbb{R}^n}(\overline{\gamma})\right)^{1/2},$$

which holds for almost all $(u,t) \in G$. Integrating both sides with respect to $du\, dt$ and then applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\int_{K\times\mathbb{R}^n}\int_{\widehat{\mathbb{R}^n}/G}\int_{\widehat{K_\gamma}} \left\|G_\psi f(u,t,\gamma,\sigma)\right\|_{\mathrm{HS}}^2\, d\mu_\gamma(\sigma)\, d\overline{\mu}_{\mathbb{R}^n}(\overline{\gamma})\, du\, dt \\
&\leqslant C\left(\int_{K\times\mathbb{R}^n}\int_{K\times\mathbb{R}^n} \|x\|^2\, |f_{u,t}^{\psi}(k,x)|^2\, dx\, dk\, du\, dt\right)^{1/2} \\
&\quad \times \left(\int_{K\times\mathbb{R}^n}\int_{\widehat{\mathbb{R}^n}/G}\int_{\widehat{K_\gamma}} \|\gamma\|^2\, \|\pi_{\gamma,\sigma}(f_{u,t}^{\psi})\|_{\mathrm{HS}}^2\, d\mu_\gamma(\sigma)\, d\overline{\mu}_{\mathbb{R}^n}(\overline{\gamma})\, du\, dt\right)^{1/2} \\
&= C\left(\int_{K\times\mathbb{R}^n}\int_{K\times\mathbb{R}^n} \|x\|^2\, |f(k,x)\, \overline{\psi((u,t)^{-1}(k,x))}|^2\, dx\, dk\, du\, dt\right)^{1/2} \\
&\quad \times \left(\int_{K\times\mathbb{R}^n}\int_{\widehat{\mathbb{R}^n}/G}\int_{\widehat{K_\gamma}} \|\gamma\|^2\, \|\pi_{\gamma,\sigma}(f_{u,t}^{\psi})\|_{\mathrm{HS}}^2\, d\mu_\gamma(\sigma)\, d\overline{\mu}_{\mathbb{R}^n}(\overline{\gamma})\, du\, dt\right)^{1/2} \\
&= C\|\psi\|_2 \left(\int_{K\times\mathbb{R}^n} \|x\|^2\, |f(k,x)|^2\, dx\, dk\right)^{1/2} \\
&\quad \times \left(\int_{K\times\mathbb{R}^n}\int_{\widehat{\mathbb{R}^n}/G}\int_{\widehat{K_\gamma}} \|\gamma\|^2\, \|\pi_{\gamma,\sigma}(f_{u,t}^{\psi})\|_{\mathrm{HS}}^2\, d\mu_\gamma(\sigma)\, d\overline{\mu}_{\mathbb{R}^n}(\overline{\gamma})\, du\, dt\right)^{1/2}.
\end{aligned}$$

Using (3.2) and (3.4), we get

$$\|\psi\|_2 \, \|f\|_2^2 \leqslant C\left(\int_{K\times\mathbb{R}^n} \|x\|^2 \, |f(k,x)|^2 \, dx\, dk\right)^{1/2} \times \left(\int_{K\times\mathbb{R}^n}\int_{\widehat{\mathbb{R}^n}/G}\int_{\widehat{K_\gamma}} \|\gamma\|^2 \, \|G_\psi f(u,t,\gamma,\sigma)\|_{\mathrm{HS}}^2 \, d\mu_\gamma(\sigma)\, d\overline{\mu}_{\mathbb{R}^n}(\overline{\gamma})\, du\, dt\right)^{1/2}. \tag{3.8}$$

Applying Hölder's inequality, we have

$$\left(\int_{K\times\mathbb{R}^n} \|x\|^{2a} \, |f(k,x)|^2 \, dx\, dk\right)^{\frac{1}{a}} \left(\int_{K\times\mathbb{R}^n} |f(k,x)|^2 \, dx\, dk\right)^{1-\frac{1}{a}} \geqslant \int_{K\times\mathbb{R}^n} \|x\|^2 \, |f(k,x)|^2 \, dx\, dk \tag{3.9}$$

and

$$\left(\int_{K\times\mathbb{R}^n}\int_{\widehat{\mathbb{R}^n}/G}\int_{\widehat{K_\gamma}} \|\gamma\|^{2b} \, \|G_\psi f(u,t,\gamma,\sigma)\|_{\mathrm{HS}}^2 \, d\mu_\gamma(\sigma)\, d\overline{\mu}_{\mathbb{R}^n}(\overline{\gamma})\, du\, dt\right)^{\frac{1}{b}} \times \left(\int_{K\times\mathbb{R}^n}\int_{\widehat{\mathbb{R}^n}/G}\int_{\widehat{K_\gamma}} \|G_\psi f(u,t,\gamma,\sigma)\|_{\mathrm{HS}}^2 \, d\mu_\gamma(\sigma)\, d\overline{\mu}_{\mathbb{R}^n}(\overline{\gamma})\, du\, dt\right)^{1-\frac{1}{b}} \geqslant \int_{K\times\mathbb{R}^n}\int_{\widehat{\mathbb{R}^n}/G}\int_{\widehat{K_\gamma}} \|\gamma\|^2 \, \|G_\psi f(u,t,\gamma,\sigma)\|_{\mathrm{HS}}^2 \, d\mu_\gamma(\sigma)\, d\overline{\mu}_{\mathbb{R}^n}(\overline{\gamma})\, du\, dt. \tag{3.10}$$

Combining (3.8), (3.9) and (3.10), we have

$$\|\psi\|_2 \, \|f\|_2^2 \leqslant C\left(\int_{K\times\mathbb{R}^n} \|x\|^{2a} \, |f(k,x)|^2 \, dx\, dk\right)^{\frac{1}{2a}} (\|f\|_2^2)^{\frac{1}{2}-\frac{1}{2a}} \times \left(\int_{K\times\mathbb{R}^n}\int_{\widehat{\mathbb{R}^n}/G}\int_{\widehat{K_\gamma}} \|\gamma\|^{2b} \, \|G_\psi f(u,t,\gamma,\sigma)\|_{\mathrm{HS}}^2 \, d\mu_\gamma(\sigma)\, d\overline{\mu}_{\mathbb{R}^n}(\overline{\gamma})\, du\, dt\right)^{\frac{1}{2b}} \times (\|\psi\|_2^2 \, \|f\|_2^2)^{\frac{1}{2}-\frac{1}{2b}}.$$

Thus, we have the required inequality (3.5). □

EXAMPLE 3.3. We give the explicit expression of the Heisenberg uncertainty inequality for Gabor transform in the following cases:

1. Euclidean group $\mathbb{R}^n$.

$$\|\psi\|_2^{\frac{1}{b}} \, \|f\|_2^{\left(\frac{1}{a}+\frac{1}{b}\right)} \leqslant C\left(\int_{\mathbb{R}^n} \|x\|^{2a} \, |f(x)|^2 \, dx\right)^{\frac{1}{2a}} \times \left(\int_{\mathbb{R}^n}\int_{\widehat{\mathbb{R}^n}} \|\omega\|^{2b} \, \|G_\psi f(t,\omega)\|_{\mathrm{HS}}^2 \, dt\, d\omega\right)^{\frac{1}{2b}}.$$

2. $\mathbb{R}^n \times K$, where K is a separable unimodular locally compact group of type I.

$$\|\psi\|_2^{\frac{1}{b}} \, \|f\|_2^{\left(\frac{1}{a}+\frac{1}{b}\right)} \leqslant C\left(\int_{\mathbb{R}^n \times K} \|x\|^{2a} \, |f(x,k)|^2 \, dx \, dk\right)^{\frac{1}{2a}} \times \left(\int_{\mathbb{R}^n \times K} \int_{\mathbb{R}^n \times \widehat{K}} \|z\|^{2b} \, \|G_\psi f(t,u,z,\gamma)\|_{\mathrm{HS}}^2 \, dz \, d\gamma \, dt \, du\right)^{\frac{1}{2b}}.$$

3. Heisenberg Group $\mathbb{H}_n$ (see [14]).

$$\|\psi\|_2^{\frac{1}{b}} \, \|f\|_2^{\left(\frac{1}{a}+\frac{1}{b}\right)} \leqslant C\left(\int_{\mathbb{H}_n} |t|^{2a} \, |f(z,t)|^2 \, dz \, dt\right)^{\frac{1}{2a}} \times \left(\int_{\mathbb{H}_n} \int_{\mathbb{R}^*} |\lambda|^{2b} \, \|G_\psi f(z',t',\lambda)\|_{\mathrm{HS}}^2 \, |\lambda|^n \, d\lambda \, dz' \, dt'\right)^{\frac{1}{2b}}.$$

4. $K \ltimes \mathbb{R}^n$, where K is a compact subgroup of the group of automorphisms of $\mathbb{R}^n$.

$$\|\psi\|_2^{\frac{1}{b}} \, \|f\|_2^{\left(\frac{1}{a}+\frac{1}{b}\right)} \leqslant C\left(\int_{K \times \mathbb{R}^n} \|x\|^{2a} \, |f(k,x)|^2 \, dx \, dk\right)^{\frac{1}{2a}} \times \left(\int_{K \times \mathbb{R}^n} \int_{\widehat{\mathbb{R}^n}/G} \sum_{\sigma \in \widehat{K_\ell}} \|\ell\|^{2b} \, \|G_\psi f(u,t,\ell,\sigma)\|_{\mathrm{HS}}^2 \, d\overline{\ell} \, du \, dt\right)^{\frac{1}{2b}}.$$

5. A class of connected, simply connected nilpotent Lie groups G for which the Hilbert-Schmidt norm of the group Fourier transform $\pi_\xi(f)$ of f attains a particular form (see [2]).

$$\|\psi\|_2^{\frac{1}{b}} \, \|f\|_2^{\left(\frac{1}{a}+\frac{1}{b}\right)} \leqslant C\left(\int_G \|x\|^{2a} \, |f(x)|^2 \, dx\right)^{\frac{1}{2a}} \times \left(\int_G \int_{\mathscr{W}} \|\xi\|^{2b} \|G_\psi f(y,\xi)\|_{\mathrm{HS}}^2 \frac{1}{|h(\xi)|^b |\operatorname{Pf}(\xi)|^{b-1}} d\xi \, dy\right)^{\frac{1}{2b}}.$$

6. For thread-like nilpotent Lie groups (see [8]).

$$\|\psi\|_2^{\frac{1}{b}} \, \|f\|_2^{\left(\frac{1}{a}+\frac{1}{b}\right)} \leqslant C\left(\int_G \|x\|^{2a} \, |f(x)|^2 \, dx\right)^{\frac{1}{2a}} \times \left(\int_G \int_{\mathscr{W}} \|\xi\|^{2b} \, \|G_\psi f(y,\xi)\|_{\mathrm{HS}}^2 \, |\xi_1| \, d\xi\right)^{\frac{1}{2b}}.$$

7. For 2-NPC nilpotent Lie groups (see [1]), let $\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$ be a Jordan-Hölder sequence in $\mathfrak{g}$ such that $\mathfrak{g}_m = \mathfrak{z}(g)$ and $\mathfrak{h} = \mathfrak{g}_{n-2}$. We have the following two cases:

(a) $\dim[\mathfrak{g},\mathfrak{g}_{m+1}]=2$.

$$\|\psi\|_2^{\frac{1}{b}}\,\|f\|_2^{\left(\frac{1}{a}+\frac{1}{b}\right)}\leqslant C\left(\int_G\|x\|^{2a}\,|f(x)|^2\,dx\right)^{\frac{1}{2a}}$$
$$\times\left(\int_G\int_{\mathscr{W}}\|\xi\|^{2b}\,\|G_\psi f(y,\xi)\|_{\mathrm{HS}}^2\,\frac{1}{|h(\xi)|^b|\,\mathrm{Pf}(\xi)|^{b-1}}\,d\xi\right)^{\frac{1}{2b}}.$$

(b) $\dim[\mathfrak{g},\mathfrak{g}_{m+1}]=1$.

$$\|\psi\|_2^{\frac{1}{b}}\,\|f\|_2^{\left(\frac{1}{a}+\frac{1}{b}\right)}\leqslant C\left(\int_G\|x\|^{2a}\,|f(x)|^2\,dx\right)^{\frac{1}{2a}}$$
$$\times\left(\int_G\int_{\mathscr{W}}\|\xi\|^{2b}\,\|G_\psi f(y,\xi)\|_{\mathrm{HS}}^2\,|\mathrm{Pf}(\xi)|\,d\xi\right)^{\frac{1}{2b}}.$$

8. For connected simply connected nilpotent Lie groups $G=\exp\mathfrak{g}$ such that $\mathfrak{g}(\xi)\subset[\mathfrak{g},\mathfrak{g}]$ for all $\xi\in\mathscr{U}$ (see [13]).

$$\|\psi\|_2^{\frac{1}{b}}\,\|f\|_2^{\left(\frac{1}{a}+\frac{1}{b}\right)}\leqslant C\left(\int_G\|x\|^{2a}\,|f(x)|^2\,dx\right)^{\frac{1}{2a}}$$
$$\times\left(\int_G\int_{\mathscr{W}}\|\xi\|^{2b}\,\|G_\psi f(y,\xi)\|_{\mathrm{HS}}^2\,\frac{|\,\mathrm{Pf}(\xi)|^{b+1}}{|\xi([X_{j_1},X_n])|^b}\,d\xi\right)^{\frac{1}{2b}}.$$

9. For low-dimensional nilpotent Lie groups of dimension less than or equal to 6 (for details, see [11]) except for $G_{6,8}$, $G_{6,12}$, $G_{6,14}$, $G_{6,15}$, $G_{6,17}$, one can write an explicit Heisenberg uncertainty inequality for Gabor transform.

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