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A NEW OSTROWSKI TYPE INEQUALITY ON TIME SCALES

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Abstract. In this paper, by introducing a technique of parameter functions, we establish a new Ostrowski type inequality on time scales and unify corresponding continuous and discrete versions. Furthermore, some particular integral inequalities on time scales are given as special cases.

1. Introduction

In 1938, Ostrowski established the following interesting integral inequality, which is a relationship between the value of a function f on some point in (a,b) and the integration on $[a,b]$:

Suppose that $f:[a,b]\to R$ is continuous on $[a,b]$ and differentiable in (a,b), and that its derivative $f':(a,b)\to R$ is bounded in (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)| < \infty$. Then for any $t\in[a,b]$, the following inequality holds:

$$\left|(b-a)f(t)-\int_a^b f(s)ds\right| \leqslant \left[\frac{(b-a)^2}{4}+\left(t-\frac{a+b}{2}\right)^2\right]\|f'\|_\infty.$$

The inequality is sharp in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

Later, many authors have introduced some results on the extensions, generalizations, and applications of the Ostrowski type inequality. For example, the inequality can be used to estimate the error of approximation to integration, which depends on the theory of integration inequality, in studying the stability and reliability of numerical computation, see [1].

The development of the theory of time scales was initiated by Hilger [2] in 1988 as a theory capable to contain both difference and differential calculus in a consistent way. Since then, many researchers have studied the theory of certain integral inequalities or dynamic equations on time scales (cf. [3–5]). In particular, Bohner and Matthews [5, Theorem 3.5] established the following Ostrowski type inequality on time scales:

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Suppose that $a,\ b,\ s,\ t \in T,\ a < b$, and that $f : [a,b] \to R$ is differentiable. Then the following inequality holds:

$$\left| f(t) - \frac{1}{b-a}\int_a^b f(s)\triangle s \right| \leqslant \frac{M}{b-a}(h_2(t,a)+h_2(t,b)), \tag{1.1}$$

where T is a time scale set, h_2 is a function defined in Section 2, and $M = \sup\limits_{a<t<b} |f^{\triangle}(t)| < \infty$. This inequality is sharp in the sense that the right-hand side of (1.1) cannot be replaced by a smaller one.

In addition, one can refer to [6-8] to see some studies on the weighted Ostrowski type inequalities on time scales. For example, Liu, Tuna and Jiang [6] established some weighted Ostrowski type inequalities based on a weighted Montgomery identity on time scales. Liu and Tuna [8] obtained another weighted Ostrowski type inequalities on time scales by using the recent theory of combined dynamic derivatives on time scales. Meanwhile, many authors have studied multivariate Ostrowski type inequalities on time scales, we provide the reader to the literature [9–11].

For the studies on Ostrowski type inequality including parameters, Liu and Ngô [12, Theorem 4] discussed inequality (1.1) with a parameter λ and gave a more detailed result as follows:

Suppose that $a,\ b,\ s,\ t \in T,\ a < b$, and that $f : [a,b] \to R$ is a differentiable function. Then the following inequality holds:

$$\left|(1-\lambda)f(t)+\lambda\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f^{\sigma}(s)\triangle s\right| \leqslant \frac{M}{b-a}\left[h_2\left(a,a+\lambda\frac{b-a}{2}\right)\right.$$

$$\left. +h_2\left(t,a+\lambda\frac{b-a}{2}\right)+h_2\left(t,b-\lambda\frac{b-a}{2}\right)+h_2\left(b,b-\lambda\frac{b-a}{2}\right)\right], \tag{1.2}$$

for all $\lambda \in [0,1]$ such that $a+\lambda\frac{b-a}{2}$ and $b-\lambda\frac{b-a}{2}$ are in T, and $t \in [a+\lambda\frac{b-a}{2}, b-\lambda\frac{b-a}{2}]\bigcap T$, where $M = \sup_{a<t<b}|f^{\triangle}(t)| < \infty$. This inequality is sharp provided

$$\frac{\lambda}{2}a(b-a)+\frac{\lambda^2}{4}(b-a)^2 \leqslant \int_a^{a+\lambda\frac{b-a}{2}} s\triangle s.$$

Motivated by the mentioned works above, we introduce a parameter function to generalize the Ostrowski type inequality and then unify corresponding continuous and discrete versions. In the end, we also discuss some particular integral inequalities on time scales as special cases.

2. A new Ostrowski type inequality on time scales

Throughout this article, we assume that T is a time scale set, which is an arbitrary nonempty closed subset of real numbers. For more definitions and basic properties about the theory of time scales, one can refer to [2, 12, 13, 14]. Before introducing the generalized Ostrowski type inequalities, we firstly define the function $h_k(t,s)$ and give the generalized Montgomery identity for parameter functions.

DEFINITION 1. Let $h_k : T \times T \to R,\ k \in N$ be functions that are recursively defined as

$$h_0(s,t) = 1,$$

and

$$h_{k+1} = \int_s^t h_k(\tau,s)\triangle t \quad \text{for all } s,\ t \in T.$$

LEMMA 1. (Generalized Montgomery identity) *Suppose that* $a,\ b,\ s,\ t \in T$, $a < b,\ f : [a,b] \to R$ *is differentiable, and that* g *is a function of* $[0,1]$ *into* $[0,1]$. *We then have the equation*

$$\frac{1-g(1-\lambda)-g(\lambda)}{2}f(t)+\frac{g(\lambda)f(a)+(1-g(1-\lambda))f(b)}{2}$$
$$=\frac{1}{b-a}\int_a^b f^\sigma(s)\triangle s+\frac{1}{b-a}\int_a^b K(s,t)f^\triangle(s)\triangle s,$$

where

$$K(s,t)=\begin{cases} s-\left(a+g(\lambda)\frac{b-a}{2}\right), & s\in[a,t),\\ s-\left(a+\left(1+g(1-\lambda)\right)\frac{b-a}{2}\right), & s\in[t,b]. \end{cases}$$

Proof. Integrating by parts, we have

$$\begin{aligned}
&\int_a^b K(s,t)f^\triangle(s)\triangle s\\
&=\int_a^t\left[s-\left(a+g(\lambda)\frac{b-a}{2}\right)\right]f^\triangle(s)\triangle s+\int_t^b\left[s-\left(a+\left(1+g(1-\lambda)\right)\frac{b-a}{2}\right)\right]f^\triangle(s)\triangle s\\
&=\left[t-\left(a+g(\lambda)\frac{b-a}{2}\right)\right]f(t)+g(\lambda)\frac{b-a}{2}f(a)-\int_a^t f^\sigma(s)\triangle s-\int_t^b f^\sigma(s)\triangle s\\
&\quad-\left(t-a-(1+g(1-\lambda))\frac{b-a}{2}\right)f(t)+\left(b-a-(1+g(1-\lambda))\frac{b-a}{2}\right)f(b)\\
&=(b-a)\left(\frac{1+g(1-\lambda)-g(\lambda)}{2}f(t)+\frac{g(\lambda)f(a)+(1-g(1-\lambda))f(b)}{2}\right)-\int_a^b f^\sigma(s)\triangle s,
\end{aligned}$$

from which we get the desired equality. □

We now establish our main result in the following theorem:

THEOREM 1. *Suppose that* $a,\ b,\ s,\ t \in T,\ a < b,\ f : [a,b] \to R$ *is differentiable, and that* g *is a function of* $[0,1]$ *into* $[0,1]$. *We then have the inequality*

$$\begin{aligned}
&\left|\frac{1+g(1-\lambda)-g(\lambda)}{2}f(t)+\frac{g(\lambda)f(a)+(1-g(1-\lambda))f(b)}{2}-\frac{1}{b-a}\int_a^b f^\sigma(s)\triangle s\right|\\
&\leqslant\frac{M}{b-a}\left[h_2\left(a,a+g(\lambda)\frac{b-a}{2}\right)+h_2\left(t,a+g(\lambda)\frac{b-a}{2}\right)+h_2\left(t,a+(1+g(1-\lambda))\frac{b-a}{2}\right)\right.\\
&\quad\left.+h_2\left(b,a+(1+g(1-\lambda))\frac{b-a}{2}\right)\right], \qquad (2.1)
\end{aligned}$$

for all $\lambda \in [0,1]$ *such that* $a+g(\lambda)\frac{b-a}{2}$ *and* $a+(1+g(1-\lambda))\frac{b-a}{2}$ *are in* T, *and* $t \in [a+g(\lambda)\frac{b-a}{2}, a+(1+g(1-\lambda))\frac{b-a}{2}]\bigcap T$, *where* $M=\sup_{a<t<b}|f^{\triangle}(t)|<\infty$. *This inequality is sharp provided*

$$\frac{g^2(\lambda)-2g(\lambda)}{2}a-\frac{g^2(\lambda)}{2}b \geqslant \int_{a+g(\lambda)\frac{b-a}{2}}^{a} s\triangle s. \tag{2.2}$$

Proof. By applying Lemma 1, we get

$$\begin{aligned}
&\left|\frac{1+g(1-\lambda)-g(\lambda)}{2}f(t)+\frac{g(\lambda)f(a)+(1-g(1-\lambda))f(b)}{2}-\frac{1}{b-a}\int_a^b f^{\sigma}(s)\triangle s\right| \\
&= \frac{1}{b-a}\left|\int_a^b K(s,t)f^{\triangle}(s)\triangle s\right| \\
&\leqslant \frac{M}{b-a}\left[\int_a^t |K(s,t)|\triangle s+\int_t^b |K(s,t)|\triangle s\right] \\
&= \frac{M}{b-a}\left[\int_a^t \left|s-\left(a+g(\lambda)\frac{b-a}{2}\right)\right|\triangle s+\int_t^b \left|s-\left(a+\left(1+g(1-\lambda)\right)\frac{b-a}{2}\right)\right|\triangle s\right] \\
&= \frac{M}{b-a}\left[\int_a^{a+g(\lambda)\frac{b-a}{2}} \left|s-\left(a+g(\lambda)\frac{b-a}{2}\right)\right|\triangle s+\int_{a+g(\lambda)\frac{b-a}{2}}^{t} \left|s-\left(a+g(\lambda)\frac{b-a}{2}\right)\right|\triangle s\right. \\
&\quad +\int_t^{a+\left(1+g(1-\lambda)\right)\frac{b-a}{2}} \left|s-\left(a+\left(1+g(1-\lambda)\right)\frac{b-a}{2}\right)\right|\triangle s \\
&\quad \left.+\int_{a+\left(1+g(1-\lambda)\right)\frac{b-a}{2}}^{b} \left|s-\left(a+\left(1+g(1-\lambda)\right)\frac{b-a}{2}\right)\right|\triangle s\right] \\
&= \frac{M}{b-a}\left[\int_{a+g(\lambda)\frac{b-a}{2}}^{a} \left(s-\left(a+g(\lambda)\frac{b-a}{2}\right)\right)\triangle s+\int_{a+g(\lambda)\frac{b-a}{2}}^{t} \left(s-\left(a+g(\lambda)\frac{b-a}{2}\right)\right)\triangle s\right. \\
&\quad +\int_{a+\left(1+g(1-\lambda)\right)\frac{b-a}{2}}^{t} \left(s-\left(a+\left(1+g(1-\lambda)\right)\frac{b-a}{2}\right)\right)\triangle s \\
&\quad \left.+\int_{a+\left(1+g(1-\lambda)\right)\frac{b-a}{2}}^{b} \left(s-\left(a+\left(1+g(1-\lambda)\right)\frac{b-a}{2}\right)\right)\triangle s\right] \\
&= \frac{M}{b-a}\left[h_2\left(a,a+g(\lambda)\frac{b-a}{2}\right)+h_2\left(t,a+g(\lambda)\frac{b-a}{2}\right)+h_2\left(t,a+(1+g(1-\lambda))\frac{b-a}{2}\right)\right. \\
&\quad \left.+h_2\left(b,a+(1+g(1-\lambda))\frac{b-a}{2}\right)\right],
\end{aligned}$$

from which inequality (2.1) can be obtained.

To prove the sharpness of this inequality, let $f(t)=t$ and $t=a+(1+g(1-\lambda))\frac{b-a}{2}$. Then one can see that $M=1$. Starting with the right-hand side of inequality

(2.1), we have the equation

$$\begin{aligned}
&\frac{M}{b-a}\Bigg[h_2\left(a,a+g(\lambda)\frac{b-a}{2}\right)+h_2\left(t,a+g(\lambda)\frac{b-a}{2}\right)\\
&+h_2\left(t,a+(1+g(1-\lambda))\frac{b-a}{2}\right)+h_2\left(b,a+(1+g(1-\lambda))\frac{b-a}{2}\right)\\
=&\frac{1}{b-a}\Bigg[h_2\left(a,a+g(\lambda)\frac{b-a}{2}\right)+h_2\left(a+(1+g(1-\lambda))\frac{b-a}{2},a+g(\lambda)\frac{b-a}{2}\right)\\
&+h_2\left(b,a+(1+g(1-\lambda))\frac{b-a}{2}\right)\Bigg].
\end{aligned}$$

Moreover, we have that

$$\begin{aligned}
&h_2\left(a,a+g(\lambda)\frac{b-a}{2}\right)\\
=&\int_{a+g(\lambda)\frac{b-a}{2}}^{a}\left(s-\left(a+g(\lambda)\frac{b-a}{2}\right)\right)\triangle s\\
=&\int_{a+g(\lambda)\frac{b-a}{2}}^{a}s\triangle s-\left(a+g(\lambda)\frac{b-a}{2}\right)\left(a-\left(a+g(\lambda)\frac{b-a}{2}\right)\right)\\
=&\int_{a+g(\lambda)\frac{b-a}{2}}^{a}s\triangle s+\left(a+g(\lambda)\frac{b-a}{2}\right)+g(\lambda)\frac{b-a}{2},
\end{aligned}$$

$$\begin{aligned}
&h_2\left(a+(1+g(1-\lambda)\frac{b-a}{2},a+g(\lambda)\frac{b-a}{2}\right)\\
=&\int_{a+g(\lambda)\frac{b-a}{2}}^{a+(1+g(1-\lambda)\frac{b-a}{2}}\left(s-\left(a+g(\lambda)\frac{b-a}{2}\right)\right)\triangle s\\
=&\int_{a+g(\lambda)\frac{b-a}{2}}^{a+(1+g(1-\lambda)\frac{b-a}{2}}s\triangle s-\left(a+g(\lambda)\frac{b-a}{2}\right)\left(a+(1+g(1-\lambda))\frac{b-a}{2}-\left(a+g(\lambda)\frac{b-a}{2}\right)\right)\\
=&\int_{a+g(\lambda)\frac{b-a}{2}}^{a+(1+g(1-\lambda)\frac{b-a}{2}}s\triangle s-\left(a+g(\lambda)\frac{b-a}{2}\right)(1+g(1-\lambda)-g(\lambda))\frac{b-a}{2},
\end{aligned}$$

and

$$\begin{aligned}
&h_2\left(b,a+(1+g(1-\lambda))\frac{b-a}{2}\right)\\
=&\int_{a+\left(1+g(1-\lambda)\right)\frac{b-a}{2}}^{b}\left(s-\left(a+\left(1+g(1-\lambda)\right)\frac{b-a}{2}\right)\right)\triangle s\\
=&\int_{a+\left(1+g(1-\lambda)\right)\frac{b-a}{2}}^{b}s\triangle s-\left(a+\left(1+g(1-\lambda)\right)\frac{b-a}{2}\right)\left(b-a-\left(1+g(1-\lambda)\right)\frac{b-a}{2}\right).
\end{aligned}$$

Thus, the right-hand side of inequality (2.1) equals to

$$
\begin{aligned}
&\frac{1}{b-a}\Bigg[\int_{a+g(\lambda)\frac{b-a}{2}}^{a} s\triangle s+\int_{a+g(\lambda)\frac{b-a}{2}}^{a+(1+g(1-\lambda)\frac{b-a}{2}} s\triangle s+\int_{a+\left(1+g(1-\lambda)\right)\frac{b-a}{2}}^{b} s\triangle s\\
&+\left(a+g(\lambda)\frac{b-a}{2}\right)+g(\lambda)\frac{b-a}{2}-\left(a+g(\lambda)\frac{b-a}{2}\right)(1+g(1-\lambda)-g(\lambda))\frac{b-a}{2}\\
&-\left(a+\left(1+g(1-\lambda)\right)\frac{b-a}{2}\right)\left(b-a-\left(1+g(1-\lambda)\right)\frac{b-a}{2}\right)\Bigg]\\
=&\frac{5g(\lambda)-2g^2(\lambda)+g(\lambda)g(1-\lambda)-g^2(1-\lambda)-3}{4}a\\
&+\frac{2g^2(\lambda)-g(\lambda)-g(\lambda)g(1-\lambda)+g^2(1-\lambda)-1}{4}b\\
&+\frac{1}{b-a}\left(2\int_{a+g(\lambda)\frac{b-a}{2}}^{a} s\triangle s+\int_a^b s\triangle s\right).
\end{aligned}
$$

Starting with the left-hand side of (2.1), we have

$$
\begin{aligned}
&\left|\frac{1+g(1-\lambda)-g(\lambda)}{2}f(t)+\frac{g(\lambda)f(a)+(1-g(1-\lambda))f(b)}{2}-\frac{1}{b-a}\int_a^b f^{\sigma}(s)\triangle s\right|\\
=&\Bigg|\frac{1+g(1-\lambda)-g(\lambda)}{2}\left(a+(1+g(1-\lambda))\frac{b-a}{2}\right)\\
&+\frac{ag(\lambda)+(1-g(1-\lambda))b}{2}-\frac{1}{b-a}\int_a^b \sigma(s)\triangle s\Bigg|\\
=&\Bigg|\frac{1+g(1-\lambda)-g(\lambda)}{2}\left(a+(1+g(1-\lambda))\frac{b-a}{2}\right)+\frac{ag(\lambda)+(1-g(1-\lambda))b}{2}\\
&+\frac{1}{b-a}\int_a^b s\triangle s-a-b\Bigg|\\
=&\Bigg|\frac{g(\lambda)-g^2(1-\lambda)+g(\lambda)g(1-\lambda)-3}{4}a\\
&+\frac{g^2(1-\lambda)-g(\lambda)g(1-\lambda)-g(\lambda)-1}{4}b+\int_a^b s\triangle s\Bigg|,
\end{aligned}
$$

where we have used

$$
\int_a^b \sigma(s)\triangle s=\int_a^b(\sigma(s)+s)\triangle s-\int_a^b s\triangle s=\int_a^b(s^2)^{\triangle}\triangle s-\int_a^b s\triangle s=b^2-a^2-\int_a^b s\triangle s.
$$

Hence, if inequality (2.2) holds, then we obtain the inequalities

$$
\begin{aligned}
&\left|\frac{g(\lambda)-g^2(1-\lambda)+g(\lambda)g(1-\lambda)-3}{4}a+\frac{g^2(1-\lambda)-g(\lambda)g(1-\lambda)-g(\lambda)-1}{4}b+\int_a^b s\triangle s\right|\\
\geqslant&\frac{g(\lambda)-g^2(1-\lambda)+g(\lambda)g(1-\lambda)-3}{4}a+\frac{g^2(1-\lambda)-g(\lambda)g(1-\lambda)-g(\lambda)-1}{4}b\\
&+\int_a^b s\triangle s
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant \frac{5g(\lambda)-2g^2(\lambda)+g(\lambda)g(1-\lambda)-g^2(1-\lambda)-3}{4}a \\
&\quad+\frac{2g^2(\lambda)-g(\lambda)-g(\lambda)g(1-\lambda)+g^2(1-\lambda)-1}{4}b \\
&\quad+\frac{1}{b-a}\left(2\int_{a+g(\lambda)\frac{b-a}{2}}^{a} s\triangle s+\int_a^b s\triangle s\right),
\end{aligned}
$$

which completes our proof. $\square$

REMARK 1. Taking $g(\lambda)=\lambda$ in Theorem 1, one can have inequality (1.2), and hence, Theorem 1 covers the result in [12].

REMARK 2. In fact, with different functions of $K(s,t)$, one can obtain some new Ostrowski type inequalities including parameter functions and the inequalities can be summarized as follows:

(a) If

$$
K(s,t)=\begin{cases} s-\left(a+g(\lambda)\frac{b-a}{2}\right), & s\in[a,t),\\ s-\left(b-g(\lambda)\frac{b-a}{2}\right), & s\in[t,b],\end{cases}
$$

then we have the inequality

$$
\begin{aligned}
&\left|(1-g(\lambda))f(t)+g(\lambda)\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f^{\sigma}(s)\triangle s\right| \\
&\leqslant \frac{M}{b-a}\left[h_2\left(a,a+g(\lambda)\frac{b-a}{2}\right)+h_2\left(t,a+g(\lambda)\frac{b-a}{2}\right)+h_2\left(t,b-g(\lambda)\frac{b-a}{2}\right)\right.\\
&\quad\left.+h_2\left(b,b-g(\lambda)\frac{b-a}{2}\right)\right].
\end{aligned}
$$

(b) If

$$
K(s,t)=\begin{cases} s-\left(a+g(\lambda)\frac{b-a}{2}\right), & s\in[a,t),\\ s-\left(a+\left(2+g(1-\lambda)\right)\frac{b-a}{2}\right), & s\in[t,b],\end{cases}
$$

then we have the inequality

$$
\begin{aligned}
&\left|\frac{2+g(1-\lambda)-g(\lambda)}{2}f(t)+\frac{g(\lambda)f(a)-g(1-\lambda)f(b)}{2}-\frac{1}{b-a}\int_a^b f^{\sigma}(s)\triangle s\right| \\
&\leqslant \frac{M}{b-a}\left[h_2\left(a,a+g(\lambda)\frac{b-a}{2}\right)+h_2\left(t,a+(2+g(1-\lambda))\frac{b-a}{2}\right)\right.\\
&\quad\left.+h_2\left(t,a+g(\lambda)\frac{b-a}{2}\right)+h_2\left(b,a+(2+g(1-\lambda))\frac{b-a}{2}\right)\right].
\end{aligned}
$$

Furthermore, if we apply inequalities in Theorem 1 and Remark 1 to different time scales, one can obtain some new results. For a brief explanation, we only give the relevant results to inequality (2,1).

COROLLARY 1. (Continuous case) *If $T=R$, we have the inequality*

$$\left|\frac{1+g(1-\lambda)-g(\lambda)}{2}f(t)+\frac{g(\lambda)f(a)+(1-g(1-\lambda))f(b)}{2}-\frac{1}{b-a}\int_a^b f(s)ds\right|$$
$$\leqslant \frac{M}{b-a}\left[\frac{g^2(\lambda)(b-a)^2}{8}+\frac{\left(t-\left(a+g(\lambda)\frac{b-a}{2}\right)\right)^2+\left(t-\left(a+(1+g(1-\lambda))\frac{b-a}{2}\right)\right)^2}{2}\right.$$
$$\left.+\frac{\left(b-\left(a+(1+g(1-\lambda))\frac{b-a}{2}\right)\right)^2}{2}\right],$$

for all $\lambda\in[0,1]$ *and* $t\in\left[a+g(\lambda)\frac{b-a}{2},a+(1+g(1-\lambda))\frac{b-a}{2}\right]\bigcap T$, *where* $M=\sup\limits_{a<t<b}|f^{\triangle}(t)|<\infty$.

Proof. If $T=R$, then our delta integral is the usual Riemann integral, and hence,

$$h_2(t,s)=\frac{(t-s)^2}{2}\quad \text{for all } s,\ t\in R,$$

from which the result follows. □

COROLLARY 2. (Discrete case) *If $T=Z$, $a=0$, $b=n$, $s=j$, $t=i$, and $f(k)=x_k$, we have the inequality*

$$\left|\frac{1+g(1-\lambda)-g(\lambda)}{2}x_i+\frac{g(\lambda)x_0+(1-g(1-\lambda))x_n}{2}-\frac{1}{n}\sum_{j=1}^{n}\right|$$
$$\leqslant \frac{M}{2}\left[2i^2-i(ng(\lambda)+ng(1-\lambda)+n+2)+\frac{n^2\left(g^2(\lambda)+(1+g(1-\lambda))^2\right)}{2}\right.$$
$$\left.+ng(\lambda)+(n-n^2)g(1-\lambda)-n^2\right],$$

for all $\lambda\in[0,1]$, $\frac{ng(\lambda)}{2}$, $\frac{n(1+g(1-\lambda))}{2}\in T$, *and* $i\in\left[\frac{ng(\lambda)}{2},\frac{n(1+g(1-\lambda))}{2}\right]\bigcap Z$, *where* $M=\sup_{0<i<n}|\triangle x_i|<\infty$.

Proof. It is known that

$$h_k(t,s)=\binom{t-s}{k}\quad \text{for all } s,\ t\in Z.$$

Taking $k=2$ in the equation above, the result follows. □

3. Some particular integral inequalities on time scales

In this section, we discuss some particular integral inequalities on time scales as special cases. Throughout this section, we assume that $a,b\in T$ with $a<b$, $f:[a,b]\to R$ is differentiable, and that g is a function of $[0,1]$ into $[0,1]$. We set $M=\sup_{a<t<b}|f^{\triangle}(t)|<\infty$.

PROPOSITION 1. Under the assumptions of Theorem 1 with $t=\frac{a+b}{2}$, we have the sharp rectangle inequality on a time scale set T

$$\left|\frac{1+g(1-\lambda)-g(\lambda)}{2}f\left(\frac{a+b}{2}\right)+\frac{g(\lambda)f(a)+(1-g(1-\lambda))f(b)}{2}-\frac{1}{b-a}\int_a^b f^\sigma(s)\triangle s\right|$$
$$\leqslant \frac{M}{b-a}\left[h_2\left(a,a+g(\lambda)\frac{b-a}{2}\right)+h_2\left(\frac{a+b}{2},a+(1+g(1-\lambda))\frac{b-a}{2}\right)\right.$$
$$\left.+h_2\left(\frac{a+b}{2},a+g(\lambda)\frac{b-a}{2}\right)+h_2\left(b,a+(1+g(1-\lambda))\frac{b-a}{2}\right)\right].$$

PROPOSITION 2. Under the assumptions of Theorem 1 with $g(\lambda)=\lambda^2$, we have the inequality on a time scale set T

$$\left|(1-\lambda)f(t)+\frac{\lambda^2 f(a)+\left(2\lambda-\lambda^2\right)f(b)}{2}-\frac{1}{b-a}\int_a^b f^\sigma(s)\triangle s\right|$$
$$\leqslant \frac{M}{b-a}\left[h_2\left(a,a+\lambda^2\frac{b-a}{2}\right)+h_2\left(\frac{a+b}{2},a+(\lambda^2-2\lambda+2)\frac{b-a}{2}\right)\right.$$
$$\left.+h_2\left(\frac{a+b}{2},a+\lambda^2\frac{b-a}{2}\right)+h_2\left(b,a+(\lambda^2-2\lambda+2))\frac{b-a}{2}\right)\right].$$

REMARK 3.

(a) If we take $\lambda=0$ in Proposition 2, then one can have the sharp rectangle inequality on a time scale set T

$$\left|f(t)-\frac{1}{b-a}\int_a^b f^\sigma(s)\triangle s\right|\leqslant \frac{M}{b-a}\left[h_2(t,a)+h_2(t,b)\right].$$

(b) If we take $\lambda=1/2$ in Proposition 2, then one can obtain the inequality on a time scale set T

$$\left|f(t)+\frac{f(a)+3f(b)}{4}-\frac{2}{b-a}\int_a^b f^\sigma(s)\triangle s\right|$$
$$\leqslant \frac{2M}{b-a}\left[h_2\left(a,\frac{b+7a}{8}\right)+h_2\left(t,\frac{b+7a}{8}\right)+h_2\left(t,\frac{5b+3a}{8}\right)+h_2\left(b,\frac{5b+3a}{8}\right)\right].$$

(c) If we take $\lambda=1$ in Proposition 2, then one can have the inequality on a time scale set T

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f^\sigma(s)\triangle s\right|$$
$$\leqslant \frac{M}{b-a}\left[h_2\left(a,\frac{b+a}{2}\right)+2h_2\left(t,\frac{b+a}{2}\right)+h_2\left(b,\frac{b+a}{2}\right)\right], \qquad (3.1)$$

and if we take $t = \frac{a+b}{2}$, inequality (3.1) becomes the trapezoid inequality on a time scale set T

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f^\sigma(s)\triangle s\right| \leqslant \frac{M}{b-a}\left[h_2\left(a,\frac{b+a}{2}\right)+h_2\left(b,\frac{b+a}{2}\right)\right].$$

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