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A POINCARÉ–TYPE INEQUALITY ON THE EUCLIDEAN UNIT SPHERE

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Abstract. We consider the second variation for the volume of convex bodies associated with the L_p Minkowski-Firey combination and obtain a Poincaré-type inequality on the Euclidean unit sphere S^{n-1}.

1. Introduction

Throughout this paper, a convex body K (compact convex set with non-empty interior) in $\mathbb{R}^n$ is assumed to containing the origin in its interior. Let $\mathscr{K}_0^n$ denote the set of such convex bodies equipped with the Hausdorff metric. The unit sphere of Euclidean space of $\mathbb{R}^n$ is denoted by S^{n-1}.

In the early 1960s, the L_p Minkowski-Firey combination (or L_p addition) of convex bodies was introduced and studied by Firey [13]: Let $1 \leqslant p < \infty$ and $0 < t_1, t_2 < \infty$. If $K, L \in \mathscr{K}_0^n$, then the L_p *Minkowski-Firey combination* $t_1 \cdot K +_p t_2 \cdot L$ is defined by

$$h_{t_1 \cdot K +_p t_2 \cdot L}(\cdot)^p = t_1 h_K(\cdot)^p + t_2 h_L(\cdot)^p,$$

where h is the support function of convex bodies. The L_p combination is the generalization of the classic *Minkowski combination* defined by

$$K + tL = \{x + ty : x \in K, y \in L\}.$$

In the mid 1990s, Lutwak in his profound papers [22, 23] investigated the L_p addition and introduced the L_p-surface area measure by the first variational formula of the volume associated with L_p addition.

It is the starting point towards many other inequalities involving volumes and L_p mixed volumes, such as the L_p Minkowski inequality and L_p Brunn-Minkowski inequality. The first variation, together with the L_p-surface area measure, also leads to an embryonic L_p Brunn-Minkowski theory. A good reference is the book by Schneider [30], in particular Chapter 9, for a detailed presentation of the L_p Brunn-Minkowski theory. For the various elements of the L_p Brunn-Minkowski theory, please see, for example, [6, 7], [14]–[29], [31].

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It is a motivation for us to investigate the second variation of volume for the L_p addition. Note that the second variation of the volume for the Minkowski combination was studied by Colesanti [10]. He then used it to lead from the Brunn-Minkowski inequality to a Poincaré type inequality on the smooth boundary of a convex body.

In this paper, we will adopt a new approach to consider the second variation for the volume of the convex body for the L_p addition. We use a selfadjoint operator (see Section 3) developed by Cheng and Yau [8] concerning the regularity of the solution of the Minkowski problem. Consider the convex body Ω_t associated with the L_p addition defined by

$$\Omega_t = \bigcap_{u\in S^{n-1}} \left\{x\in\mathbb{R}^n : \langle x,u\rangle \leqslant \left(h_K(u)^p + t\varphi(u)^p\right)^{\frac{1}{p}}\right\},$$

where $K\in\mathscr{K}_0^n$ is of class $C_+^2(S^{n-1})$, and $\varphi\in C^2(S^{n-1})$ is positive such that $\left(h_K(u)^p + t\varphi(u)^p\right)^{\frac{1}{p}} \in C_+^2(S^{n-1})$ for sufficient small $t>0$.

Let $\mathscr{H}^{n-1}$ denote the $(n-1)$-dimensional Hasusforff measure. A Poincaré-type inequality on the unit sphere is deduced.

THEOREM 1.1. *Let $1\leqslant p<\infty$. For every positive function $\psi\in C^1(S^{n-1})$, we have*

$$\frac{p-n}{n\omega_n}\Big(\int_{S^{n-1}}\psi(u)d\mathscr{H}^{n-1}(u)\Big)^2 + (n-p)\int_{S^{n-1}}\psi(u)^2 d\mathscr{H}^{n-1}(u) \leqslant \int_{S^{n-1}}\left|\nabla\psi(u)\right|^2 d\mathscr{H}^{n-1}(u), \tag{1.1}$$

where ω_n is the volume of unit ball of $\mathbb{R}^n$.

If $p=1$ and $\int_{S^{n-1}}\psi(u)d\mathscr{H}^{n-1}(u)=0$, then the inequality (1.1) implies the classic Poincaré inequality on S^{n-1} with the sharp constant:

$$(n-1)\int_{S^{n-1}}\psi(u)^2 d\mathscr{H}^{n-1}(u) \leqslant \int_{S^{n-1}}\left|\nabla\psi(u)\right|^2 d\mathscr{H}^{n-1}(u).$$

There have been a lot of literature about the Poincaré-type inequalities and related topic, see for example, [2]–[5], [9], [11], [12], [18], [25], [32] and the references therein.

2. Notations and preliminaries

We shall work in $\mathbb{R}^n$ equipped with the canonical scalar product $\langle\cdot,\cdot\rangle$ and write $|\cdot|$ for the corresponding Euclidean norm. The support function of a convex body K, $h(\cdot):\mathbb{R}^n\to(0,\infty)$, is defined for $x\in\mathbb{R}^n$ by

$$h(x) := h_K(x) = \max\{\langle x,y\rangle : y\in K\}.$$

Obviously, h is positively homogeneous of order 1. The set $\mathscr{K}_0^n$ will be viewed as equipped with the Hausdorff metric and thus for $K_i\in\mathscr{K}_0^n$, we have $K_i\to K\in\mathscr{K}_0^n$

provided that

$$\|h_{K_i}-h_K\|_\infty := \max_{u\in S^{n-1}} |h_{K_i}(u)-h_K(u)| \to 0.$$

A convex body $K \in \mathscr{K}_0^n$ is said to be of class C_+^2 if $\partial K \in C^2$ and the Gauss curvature is strictly positive at each point of ∂K. If K is of class C_+^2 we denote its Gauss map by ν. Then the support function of K can be written as

$$h(x) = \langle x, \nu(x)\rangle, \quad x \in \partial K. \tag{2.1}$$

Let h^* denote the support function of K^*, where K^* is the polar of K defined by

$$K^* = \{x \in \mathbb{R}^n : \langle x,y\rangle \leqslant 1 \text{ for all } y \in K\}.$$

Note that

$$h^*(x) = 1, \quad \text{for each } x \in \partial K. \tag{2.2}$$

Then the Gauss map can be defined on ∂K as

$$\nu = \frac{\nabla h^*}{|\nabla h^*|}. \tag{2.3}$$

Since $h(\nabla h^*(x)) = 1$, it follows that

$$h(\nu(x)) = \frac{1}{|\nabla h^*(x)|} \tag{2.4}$$

for all $x \in \partial K$. The Gauss map is a homeomorphism between a closed smooth convex hypersurface M in $\mathbb{R}^n$ and the unite sphere S^{n-1}. It assigns each point of the boundary of M to its outer normal. Then the Gauss curvature H of M can be transplanted via the Gauss map to a function defined on S^{n-1}. If the closed smooth convex hypersurface M encloses a body K in $\mathbb{R}^n$, then

$$\frac{1}{H} d\mathscr{H}^{n-1}(u) = dS_K(u), \tag{2.5}$$

where $dS_K(u)$ is the surface area measure of K, which is defined on S^{n-1} by

$$S_K(\omega) = \mathscr{H}^{n-1}(\nu^{-1}(K,\omega)) \tag{2.6}$$

for each Borel set $\omega \subseteq S^{n-1}$, where ν^{-1} denotes the inverse Gauss map ν.

Let $K,L \in \mathscr{K}_0^n$. The L_p*-mixed volume* of K and L is defined by

$$\frac{n}{p} V_p(K,L) = \lim_{t\to 0^+} \frac{V(K +_p t\cdot L) - V(K)}{t}.$$

The L_p-surface area measure $S_p(K,\cdot)$ of K is a positive Borel measure on S^{n-1} such that the L_p-mixed volume has the following integral representation

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L^p(u)\, dS_p(K,u). \tag{2.7}$$

It generalizes the mixed volume $V_1(K,L)$ of K and L defined by

$$nV_1(K,L) = \lim_{t\to 0^+} \frac{V(K+tL)-V(K)}{t}.$$

A fundamental fact is that the mixed volume $V_1(K,L)$ can be expressed as

$$V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L(u)dS_K(u). \tag{2.8}$$

As showed in [22], if $K \in \mathscr{K}_0^n$, then the L_p-surface area measure $S_p(K,\cdot)$ of K defined on S^{n-1} is absolutely continuous with respect to its surface area measure and that the Radon- Nikodym derivative is

$$\frac{dS_p(K,\cdot)}{dS_K(\cdot)} = h_K^{1-p}(\cdot).$$

The L_p Brunn-Minkowski inequality says that if $K,L \in \mathscr{K}_0^n$, and $1 \leqslant p < \infty$, then for $0 \leqslant \lambda \leqslant 1$,

$$V((1-\lambda)\cdot K +_p \lambda \cdot L)^{\frac{p}{n}} \geqslant (1-\lambda)V(K)^{\frac{p}{n}} + \lambda V(L)^{\frac{p}{n}}, \tag{2.9}$$

with equality if and only if K and L are dilates.

3. A selfadjoint operator

Let $K \in \mathscr{K}_0^n$. If K is of class C_+^2, the differential $D\nu$ is the *Weingarten map* of ∂K. Let h be the support function of K and ν^{-1} be the inverse Gauss map. Then the matrix associated with the linear map $D(\nu^{-1})$ is $(h_{ij}+h\delta_{ij})$, $i,j = 1,\dots,n-1$, where h_i and h_{ij} is the first and second covariant derivatives of h with respect to an orthonormal frame $\{e_1,\dots,e_{n-1}\}$ on S^{n-1} and δ_{ij} is the standard Kronecker symbol. In other words, $(h_{ij}+h\delta_{ij})$ is the matrix of the reverse second fundamental form of ∂K. It follows that the reciprocal Gauss curvature has the following formula,

$$\frac{1}{H} = \det(h_{ij}+h\delta_{ij}),\ i,j = 1,\dots,n-1. \tag{3.1}$$

Define the coefficients c_{ij} of the cofactor matrix of $(h_{ij}+h\delta_{ij})$ by

$$\sum_{jl} c_{ij}(h_{jl}+h\delta_{jl}) = \delta_{il}\det(h_{pq}+h\delta_{pq}) = \frac{\delta_{il}}{H}. \tag{3.2}$$

Recall that

$$\sum_{i=1}^{n-1}(c_{ij})_i = 0. \tag{3.3}$$

It follows that

$$\sum_{ij} c_{ij}(h_{ij}+h\delta_{ij}) = \mathrm{tr}(c_{ij})h + \sum_{ij}(c_{ij}h_j)_i. \tag{3.4}$$

Let L_h be the linear operator of the operator $h \to \det(h_{ij} + h\delta_{ij})$ defined by

$$L_h(g) = \sum_{ij} c_{ij}(g_{ij} + g\delta_{ij}) \tag{3.5}$$

for each $g \in C^2(S^{n-1})$. Cheng and Yau [8] obtained the following result.

LEMMA 3.1. *The operator L_h is selfadjoint, i.e.,*

$$\int_{S^{n-1}} gL_h(w)d\mathscr{H}^{n-1}(u) = \int_{S^{n-1}} wL_h(g)d\mathscr{H}^{n-1}(u), \tag{3.6}$$

where g,w are functions in $C^2(S^{n-1})$.

Define the set $\mathscr{C}$ of functions by

$$\mathscr{C} = \{f \in C^2(S^{n-1}) : (f_{ij} + f\delta_{ij}) > 0 \text{ on } S^{n-1}\}.$$

Obviously, the set $\mathscr{C}$ consists of support functions of convex bodies (containing the origin in its interior) of class C^2_+.

4. The first and second variational formula

Let $K \in \mathscr{K}^n_0$ be of class C^2_+, $\varphi \in C^2(S^{n-1})$ be positive and $1 \leqslant p < \infty$. For $t > 0$ sufficient small such that $(h_K(u)^p + t\varphi(u)^p)^{\frac{1}{p}} \in \mathscr{C}$. Define a convex body Ω_t by

$$\Omega_t = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : \langle x,u\rangle \leqslant \big(h_K(u)^p + t\varphi(u)^p\big)^{\frac{1}{p}}\}.$$

It follows that Ω_t contains the origin in its interior. The critical observations of this body are

$$h_{\Omega_t} \leqslant (h_K^p + t\varphi^p)^{\frac{1}{p}}$$

and

$$h_{\Omega_t} = (h_K^p + t\varphi^p)^{\frac{1}{p}}, \ \text{ a.e. with respect to } S_{\Omega_t}.$$

In fact, the inverse Gauss map of Ω of the set

$$\omega = \{u \in S^{n-1} : h_\Omega(u) < \big(h_K(u)^p + t\varphi(u)^p\big)^{\frac{1}{p}}\},$$

which was shown by Aleksandrov [1] (see [22] also), must be a singular boundary point of Ω_t. Since the set of singular boundary points of a convex body has $\mathscr{H}^{n-1}$-measure zero, we conclude from Reidemeister's theorem [30] that $S_{\Omega_t}(\omega) = 0$ and $h_{\Omega_t} = \big(h_K^p + t\varphi^p\big)^{\frac{1}{p}}$ almost everywhere with respect to the surface measure S_{Ω_t}. Denote

$g_t(u)$ by the function $\left(h_K(u)^p + t\varphi(u)^p\right)^{\frac{1}{p}}$. From this, (2.8), (2.5) and (3.1) we have

$$\begin{aligned} V(\Omega_t) &= \frac{1}{n}\int_{S^{n-1}} h_{\Omega_t}(u)dS_{\Omega_t}(u) \\ &= \frac{1}{n}\int_{S^{n-1}} \left(h_K(u)^p + t\varphi(u)^p\right)^{\frac{1}{p}} dS_{\Omega_t}(u) \\ &= \frac{1}{n}\int_{S^{n-1}} g_t(u)\det\left((g_t(u))_{ij} + g_t(u)\delta_{ij}\right)d\mathscr{H}^{n-1}(u). \end{aligned} \tag{4.1}$$

LEMMA 4.1. *If $K \in \mathscr{K}_0^n$ be of class C_+^2 and $\varphi \in C^2(S^{n-1})$ be positive such that $(h_K(u)^p + t\varphi(u)^p)^{\frac{1}{p}} \in \mathscr{C}$ for sufficient small $t > 0$, then, for $1 \leqslant p < \infty$,*

$$\frac{d}{dt}V(\Omega_t)\big|_{t=0} = \frac{1}{p}\int_{S^{n-1}} \varphi(u)^p h_K(u)^{1-p}\det\left((h_K(u))_{ij} + h_K(u)\delta_{ij}\right)d\mathscr{H}^{n-1}(u). \tag{4.2}$$

Proof. For every $u \in S^{n-1}$, from (3.2), we have

$$\begin{aligned} &\frac{d}{dt}\left[g_t(u)\det\left((g_t(u))_{ij} + g_t(u)\delta_{ij}\right)\right] \\ &= g_t'(u)\det\left((g_t(u))_{ij} + g_t(u)\delta_{ij}\right) + g_t(u)\sum_{i,j=1}^{n-1} c_{ij}^t(u)((g_t'(u))_{ij} + g_t'(u)\delta_{ij}), \end{aligned}$$

where (c_{ij}^t) denotes the cofactor matrix of $((g_t)_{ij} + g_t\delta_{ij})$.

Differentiating under the integral sign we obtain

$$\begin{aligned} \frac{d}{dt}V(\Omega_t) &= \frac{1}{n}\int_{S^{n-1}} g_t'(u)\det\left((g_t(u))_{ij} + g_t(u)\delta_{ij}\right)d\mathscr{H}^{n-1}(u) \\ &\quad + \frac{1}{n}\int_{S^{n-1}} g_t(u)\sum_{i,j=1}^{n-1} c_{ij}^t(u)((g_t'(u))_{ij} + g_t'(u)\delta_{ij})d\mathscr{H}^{n-1}(u) \\ &= \frac{1}{n}\int_{S^{n-1}} g_t'(u)\det\left((g_t(u))_{ij} + g_t(u)\delta_{ij}\right)d\mathscr{H}^{n-1}(u) \\ &\quad + \frac{1}{n}\int_{S^{n-1}} g_t(u)L_{g_t}\left(g_t'(u)\right)d\mathscr{H}^{n-1}(u), \end{aligned} \tag{4.3}$$

where L_{g_t} is a linear operator given by (3.5).

It is easy to check from (3.2) and (3.1) that

$$L_{g_t}(g_t) = (n-1)\det((g_t)_{ij} + g_t\delta_{ij}). \tag{4.4}$$

Then, using Lemma 3.1, we have

$$\begin{aligned} &\frac{1}{n}\int_{S^{n-1}} g_t(u)L_{g_t}\left(g_t'(u)\right)d\mathscr{H}^{n-1}(u) \\ &= \frac{1}{n}\int_{S^{n-1}} g_t'(u)L_{g_t}(g_t(u))d\mathscr{H}^{n-1}(u) \\ &= \frac{n-1}{n}\int_{S^{n-1}} g_t'(u)\det((g_t(u))_{ij} + g_t(u)\delta_{ij})d\mathscr{H}^{n-1}(u). \end{aligned}$$

Inserting the above equation into (4.3) gives that

$$\frac{d}{dt}V(\Omega_t) = \int_{S^{n-1}} g_t'(u)\det\big((g_t(u))_{ij} + g_t(u)\delta_{ij}\big)\,d\mathscr{H}^{n-1}(u). \tag{4.5}$$

Then (4.2) follows by letting $t = 0$. □

Analogously, differentiating the function $t \mapsto \frac{d}{dt}V(\Omega_t)$ (4.5) again gives

$$\begin{aligned}\frac{d^2}{dt^2}V(\Omega_t) &= \int_{S^{n-1}} g_t''(u)\det\big((g_t(u))_{ij} + g_t(u)\delta_{ij}\big)\,du\\ &\quad + \int_{S^{n-1}} g_t'(u)\sum_{i,j=1}^{n-1} c_{ij}^t(u)((g_t'(u))_{ij} + g_t'(u)\delta_{ij})\,du,\end{aligned}$$

where (c_{ij}^t) denotes the cofactor matrix of $((g_t)_{ij} + g_t\delta_{ij})$. Therefore, we obtain the second variational formula.

LEMMA 4.2. *If $K \in \mathscr{K}_0^n$ is of class C_+^2 and $\varphi \in C^2(S^{n-1})$ positive such that $(h_K(u)^p + t\varphi(u)^p)^{\frac{1}{p}} \in \mathscr{C}$ for sufficient small $t > 0$, then, for $1 \leqslant p < \infty$,*

$$\begin{aligned}\frac{d^2}{dt^2}V(\Omega_t)\Big|_{t=0} &= \frac{1-p}{p^2}\int_{S^{n-1}} h_K(u)^{1-2p}\varphi(u)^{2p}\det\big((h_K(u))_{ij} + h_K(u)\delta_{ij}\big)\,d\mathscr{H}^{n-1}(u)\\ &\quad + \frac{1}{p^2}\int_{S^{n-1}} h_K^{1-p}(u)\varphi(u)^p\sum_{i,j=1}^{n-1} c_{ij}(u)\big((h_K(u)^{1-p}\varphi(u)^p)_{ij}\\ &\quad + h_K(u)^{1-p}\varphi(u)^p\delta_{ij}\big)\,d\mathscr{H}^{n-1}(u),\end{aligned}$$

where (c_{ij}) is the cofactor matrix of $((h_K)_{ij} + h_K\delta_{ij})$.

5. The Poincaré-type inequalities

THEOREM 5.1. *Let $K \in \mathscr{K}_0^n$ be of class C_+^2. For every positive function $\phi \in C^1(\partial K)$, we have*

$$\begin{aligned}&\frac{p-n}{nV(K)}\Big(\int_{\partial K}\frac{\phi(x)^p}{|\nabla h_K^*(x)|^{1-p}}\,d\mathscr{H}^{n-1}(x)\Big)^2 + (1-p)\int_{\partial K}\frac{\phi(x)^{2p}}{|\nabla h_K^*(x)|^{1-2p}}\,d\mathscr{H}^{n-1}(x)\\ &\quad + \int_{\partial K}\mathrm{tr}(D\nu_K(x))\frac{\phi(x)^{2p}}{|\nabla h_K^*(x)|^{2(1-p)}}\,d\mathscr{H}^{n-1}(x)\\ &\leqslant \int_{\partial K}\Big\langle (D\nu_K(x))^{-1}\nabla\Big(\frac{\phi(x)^p}{\big|\nabla h_K^*(x)\big|^{1-p}}\Big), \nabla\Big(\frac{\phi(x)^p}{\big|\nabla h_K^*(x)\big|^{1-p}}\Big)\Big\rangle\,d\mathscr{H}^{n-1}(x),\end{aligned}$$

where $D\nu_K$ is the differential of the Gauss map; i.e., the Weingarten map.

Proof. Assume that $\varphi \in C^2(S^{n-1})$ is positive. Let $t > 0$ be sufficient small such that $(h_K(u)^p + t\varphi(u)^p)^{\frac{1}{p}} \in \mathscr{C}$. It follows from the L_p Brunn-Minkowski inequality (2.9) that the function $V(\Omega_t)^{\frac{p}{n}}$ is concave, so that

$$\big(V(\Omega_0)^{\frac{p}{n}}\big)'' = \frac{p(p-n)}{n^2}V(\Omega_0)^{\frac{p}{n}-2}(V'(\Omega_0))^2 + \frac{p}{n}V(\Omega_0)^{\frac{p}{n}-1}V''(\Omega_0) \leqslant 0.$$

By Lemma 4.1 and 4.2 we obtain

$$\begin{aligned}
&\frac{p-n}{nV(K)}\Big(\int_{S^{n-1}} h_K(u)^{1-p}\varphi(u)^p \det\big((h_K(u))_{ij} + h_K(u)\delta_{ij}\big)d\mathscr{H}^{n-1}(u)\Big)^2 \\
&\qquad + (1-p)\int_{S^{n-1}} h_K(u)^{1-2p}\varphi(u)^{2p}\det\big((h_K(u))_{ij} + h_K(u)\delta_{ij}\big)d\mathscr{H}^{n-1}(u) \\
&\leqslant -\int_{S^{n-1}} h_K(u)^{1-p}\varphi(u)^p \sum_{i,j=1}^{n-1} c_{ij}(u)\big((h_K(u)^{1-p}\varphi(u)^p)_{ij} \qquad (5.1)\\
&\qquad + h_K(u)^{1-p}\varphi(u)^p\delta_{ij}\big)d\mathscr{H}^{n-1}(u), \qquad (5.2)
\end{aligned}$$

where (c_{ij}) is the cofactor matrix of $((h_K)_{ij} + h_K\delta_{ij})$. Using (3.4), integrating by parts and using (3.3), the integral in the right hand-side is equal to

$$\begin{aligned}
&-\int_{S^{n-1}} \operatorname{tr}(c_{ij})h_K(u)^{2(1-p)}\varphi(u)^{2p}d\mathscr{H}^{n-1}(u) \\
&\qquad + \int_{S^{n-1}} \sum_{i,j=1}^{n-1} c_{ij}(u)(h_K(u)^{1-p}\varphi(u)^p)_i(h_K(u)^{1-p}\varphi(u)^p)_j d\mathscr{H}^{n-1}(u). \qquad (5.3)
\end{aligned}$$

By a standard approximation argument, the above equation can fulfills for positive $\varphi \in C^1(S^{n-1})$. So let $\varphi(u) = \phi(\nu_K^{-1}(u))$, $u \in S^{n-1}$, where $\phi \in C^1(\partial K)$ is positive and ν_K^{-1} is the inverse Gauss map. Note that

$$(c_{ij}) = \det((h_K)_{ij} + h_K\delta_{ij})((h_K)_{ij} + h_K\delta_{ij})^{-1}. \qquad (5.4)$$

Making the change of variable $u = \nu(x)$ and using (2.4) we have

$$\begin{aligned}
&\int_{S^{n-1}} \varphi(u)^p h_K(u)^{1-p}\det\big((h_K(u))_{ij} + h_K(u)\delta_{ij}\big)d\mathscr{H}^{n-1}(u) \\
&= \int_{\partial K} \frac{\phi(x)^p}{|\nabla h_K^*(x)|^{1-p}}d\mathscr{H}^{n-1}(x), \qquad (5.5)
\end{aligned}$$

and from (5.4) we obtain

$$\begin{aligned}
&\int_{S^{n-1}} \operatorname{tr}(c_{ij})h_K(u)^{2(1-p)}\varphi(u)^{2p}d\mathscr{H}^{n-1}(u) \\
&= \int_{\partial K} \operatorname{tr}(D\nu_K(x))\frac{\phi(x)^{2p}}{|\nabla h_K^*(x)|^{2(1-p)}}d\mathscr{H}^{n-1}(x). \qquad (5.6)
\end{aligned}$$

Moreover, by (5.4) and (2.4) we have

$$\begin{aligned}
&\sum_{i,j=1}^{n-1} c_{ij}(u)(h_K(u)^{1-p}\varphi(u)^p)_i(h_K(u)^{1-p}\varphi(u)^p)_j \\
&= \det\big((h_K(u))_{ij}+h_K(u)\delta_{ij}\big)\big\langle D\nu_K^{-1}(u)\nabla(h_K(u)^{1-p}\varphi(u)^p),\nabla(h_K(u)^{1-p}\varphi(u)^p)\big\rangle \\
&= \det\big((h_K(u))_{ij}+h_K(u)\delta_{ij}\big) \\
&\quad\times\left\langle D\nu_K^{-1}(u)\nabla\left(\frac{\phi(\nu_K^{-1}(u))^p}{\left|\nabla h_K^*(\nu_K^{-1}(u))\right|^{1-p}}\right),\nabla\left(\frac{\phi(\nu_K^{-1}(u))^p}{\left|\nabla h_K^*(\nu_K^{-1}(u))\right|^{1-p}}\right)\right\rangle. \qquad (5.7)
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{S^{n-1}}\sum_{i,j=1}^{n-1} c_{ij}(u)(h_K(u)^{1-p}\varphi(u)^p)_i(h_K(u)^{1-p}\varphi(u)^p)_j d\mathscr{H}^{n-1}(u) \\
&= \int_{\partial K}\left\langle (D\nu_K(x))^{-1}\nabla\left(\frac{\phi(x)^p}{\left|\nabla h_K^*(x)\right|^{1-p}}\right),\nabla\left(\frac{\phi(x)^p}{\left|\nabla h_K^*(x)\right|^{1-p}}\right)\right\rangle d\mathscr{H}^{n-1}(x). \qquad (5.8)
\end{aligned}$$

Finally, combining (5.1), (5.3), and (5.5), (5.6) and (5.8) we obtain the desired result. □

The case of $p=1$ of Theorem 5.1 was proved by Colesanti [10]. Moreover, Colesanti and Saorin-Gomez [11] used Brunn-Minkowski inequalities for quermassintegrals to deduce a family of Poincaré type inequalities.

If we choose K to be the unit ball, then $\partial K = S^{n-1}$, ν_K is the identity map on S^{n-1} and $|\nabla h_K^*(x)| = 1$ for $x\in\partial K$. In this case, let $\varphi(u)^p = \psi(u)$, we obtain the Poincaré-type inequality on S^{n-1} as follows.

THEOREM 5.2. *Let $1\leqslant p<\infty$. For every positive function $\psi\in C^1(S^{n-1})$, we have*

$$\begin{aligned}
&\frac{p-n}{n\omega_n}\left(\int_{S^{n-1}}\psi(u)d\mathscr{H}^{n-1}(u)\right)^2+(n-p)\int_{S^{n-1}}\psi(u)^2 d\mathscr{H}^{n-1}(u) \\
&\leqslant \int_{S^{n-1}}\left|\nabla\psi(u)\right|^2 d\mathscr{H}^{n-1}(u). \qquad (5.9)
\end{aligned}$$

Note that when $p=1$, the inequality (5.9) can be obtained from Colesanti' result [10] by replacing ψ by $\psi-\frac{1}{n\omega_n}\int_{S^{n-1}}\psi(u)d\mathscr{H}^{n-1}(u)$. From (5.9) it immediately yields the following Poincaré-type inequality on S^{n-1}.

COROLLARY 5.3. *Let $1\leqslant p<\infty$. For every positive function $\psi\in C^1(S^{n-1})$, if*

$$\int_{S^{n-1}}\psi(u)d\mathscr{H}^{n-1}(u)=0,$$

then we have

$$(n-p)\int_{S^{n-1}}\psi(u)^2 d\mathscr{H}^{n-1}(u)\leqslant\int_{S^{n-1}}\left|\nabla\psi(u)\right|^2 d\mathscr{H}^{n-1}(u).$$

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