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ON AN APPLICATION OF VIETORIS'S INEQUALITY

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Abstract. The radius of starlikeness for polynomials with zeroes distributed at certain curves in the unit disc as well as the case in which zeroes are concentrated at a single point are considered and sharp bounds are obtained.

1. Introduction

Let $\mathscr{H}$ be the class of analytic functions in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let us recall a concept of the radius for a certain property in a certain set. Given a set of functions $\mathscr{M}$ and a property $\mathscr{P}$ which functions may or may not have in a disc $|z| < r$, the radius for the property $\mathscr{P}$ in the set $\mathscr{M}$, denoted by $R_{\mathscr{P}}(\mathscr{M})$, is the largest R such that every function in the set $\mathscr{M}$ has the property $\mathscr{P}$ in each disc $\mathbb{D}_r = \{z : |z| < r\}$ for every $r < R$. Let us consider the property $\mathscr{S}\mathscr{T}$ that $f(\mathbb{D}_r)$ is a starlike region. Recall that a set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if and only if every linear segment joining w_0 to an arbitrary point $w \in E$ lies entirely in E. Let a function $f \in \mathscr{H}$ be univalent in the unit disc $\mathbb{D}$ with the normalization $f(0) = 0$. Then f maps $\mathbb{D}$ onto a starlike domain with respect to $w_0 = 0$ if and only if [7]

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \tag{1.1}$$

for all $z \in \mathbb{D}$. Such function f is said to be starlike in $\mathbb{D}$ with respect to $w_0 = 0$ (or briefly starlike). It is well known that if an analytic function f satisfies (1.1) for all $z \in \mathbb{D}_R$, $f(0) = 0$ and $f'(0) \neq 0$ then f is univalent and starlike in $\mathbb{D}_r$ for every $r < R$. The subclass of $\mathscr{H}$ consisting of functions normalized by $f(0) = 0$ and $f'(0) = 1$ that are univalent functions in $\mathbb{D}$ will be denoted by $\mathscr{S}$. The set of all functions $f \in \mathscr{S}$ that are starlike univalent in $\mathbb{D}$ will be denoted by $\mathscr{S}^*$. The radius of starlikeness in the class $\mathscr{S}$ is [5]

$$R_{\mathscr{S}\mathscr{T}}(\mathscr{S}) = \tanh(\pi/4) \approx 0.65579.$$

The question about the starlikeness radius of functions was first considered in the paper [1] in which the author primarily dealt with polynomial mappings. Sometimes the value of r can be expressed in terms of the zeroes and critical points of a polynomial. The problem of the starlikeness of polynomials and finite Blaschke products was considered in the recent papers [6] and [4].

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2. Radius of starlikeness for polynomials

In this work we shall consider the radius of the starlikeness in a set of polynomials. We assume that $\mathscr{P}_{n,\rho}$ consists of the polynomials P having n zeroes located in the ring $0<\rho\leqslant|z|<1$ and a single zero at the origin and write $P(z)=z\prod_{k=1}^{n}(1-z/z_k)$, $0<\rho\leqslant|z_k|<1$. As we have written above (1.1), the polynomial P maps $\mathbb{D}_r$ bijectively to a starlike region if and only if

$$\mathfrak{Re}\frac{zP'(z)}{P(z)}>0 \quad \text{for} \quad z\in\mathbb{D}_r.$$

Using logarithmic differentiation one has

$$\mathfrak{Re}\frac{zP'(z)}{P(z)}=1+\mathfrak{Re}\left\{\sum_{k=1}^{n}\frac{z}{z-z_k}\right\}. \tag{2.1}$$

THEOREM 2.1. *The radius of starlikeness in the set $\mathscr{P}_{n,\rho}$ is*

$$R_{\mathscr{S}\mathscr{T}}\left(\mathscr{P}_{n,\rho}\right)=\frac{\rho}{1+n}. \tag{2.2}$$

Proof. Let $P(z)=z\prod_{k=1}^{n}(1-z/z_k)\in\mathscr{P}_{n,\rho}$. Denote $\widetilde{\rho}=\min\{|z_k|:k\in\{1,\dots,n\}\}$ and $z=re^{it}$, $r\in(0,\rho)$; $z_k=r_ke^{i\theta_k}$, $0<\rho\leqslant|z_k|<1$. From [4], we have that $P(z)$ maps $|z|<r$ univalently onto a starlike region when

$$r<\frac{\widetilde{\rho}}{1+\sum_{k=1}^{n}\widetilde{\rho}/|z_k|}.$$

Hence

$$R_{\mathscr{S}\mathscr{T}}\left(\mathscr{P}_{n,\rho}\right)\geqslant\frac{\widetilde{\rho}}{1+\sum_{k=1}^{n}\widetilde{\rho}/|z_k|}\geqslant\frac{\rho}{1+n}. \tag{2.3}$$

The polynomial $P_\rho(z)=z(1-z/\rho)^n$ is in $\mathscr{P}_{n,\rho}$ and

$$\mathfrak{Re}\frac{zP_\rho'(z)}{P_\rho(z)}=1-n\mathfrak{Re}\frac{z/\rho}{1-z/\rho}>0 \quad \text{for} \quad z\in\mathbb{D}_r$$

if and only if $r\leqslant\rho/(1+n)$, thus applying (2.3) we obtain $R_{\mathscr{S}\mathscr{T}}\left(P_\rho\right)=\rho/(1+n)$, that is (2.2). $\square$

THEOREM 2.2. [2] *Let $\beta_0=0.308443\ldots$ denote the Littlewood-Salem-Izumi number, i.e. the unique solution of the equation*

$$\int_0^{3\pi/2}\frac{\cos t}{t^\beta}\,\mathrm{d}t=0. \tag{2.4}$$

Assume also that $\beta_0\leqslant\beta\leqslant1$ and that $\{a_k\}_{k=0}^{n}$ is a sequence of real numbers satisfying

$$a_0\geqslant a_1\geqslant\ldots\geqslant a_n>0 \tag{2.5}$$

and

$$a_{2k} \leqslant \left(1 - \frac{\beta}{k}\right) a_{2k-1}, \;\; 2k \leqslant n. \tag{2.6}$$

Then for all $0 < \theta < \pi$ *and* $n \in \mathbb{N}$

$$\sum_{k=0}^{n} a_k \cos(k\theta) > 0. \tag{2.7}$$

Moreover, if either n *is odd and* $0 < \theta < \pi$ *or* n *is even and* $0 < \theta < \pi - \pi/n$*, then*

$$\sum_{k=1}^{n} a_k \sin(k\theta) > 0. \tag{2.8}$$

For $\beta = 1/2$ the above theorem is close to the following Vietoris' result.

THEOREM 2.3. [8] *Let* $\{a_k\}_{k=0}^{n}$ *be a sequence of real numbers satisfying*

$$a_0 \geqslant a_1 \geqslant \ldots \geqslant a_n > 0 \;\; \textit{and} \;\; a_{2k} \leqslant \left(1 - \frac{1}{2k}\right) a_{2k-1}, \;\; 2k \leqslant n. \tag{2.9}$$

Then for all $0 < \theta < \pi$ *and* $n \in \mathbb{N}$

$$\sum_{k=0}^{n} a_k \cos(k\theta) > 0 \tag{2.10}$$

and

$$\sum_{k=1}^{n} a_k \sin(k\theta) > 0. \tag{2.11}$$

THEOREM 2.4. *Let* $\beta_0 = 0.308443\ldots$ *denote the Littlewood-Salem-Izumi number that is the solution of* (2.4). *Assume that* n *is odd and that the coefficients of the polynomial* $p_n(z) = z + b_2 z^2 + \cdots + b_n z^n$ *satisfy*

$$1 = b_1 \geqslant b_2 \geqslant b_3 \geqslant \ldots \geqslant b_n > 0. \tag{2.12}$$

Let us denote

$$r_1 = \min\left\{\left(1 - \frac{\beta_0}{k}\right)\left(\frac{2k-1}{2k}\right)\frac{b_{2k-1}}{b_{2k}} : 2k \leqslant n\right\}, \tag{2.13}$$

where $\beta_0 \leqslant \beta \leqslant 1$*. Furthermore,*

$$r_2 = \min\left\{\frac{kb_k}{(k+1)b_{k+1}} : k \in \{1, 2, \ldots, n-1\}\right\}. \tag{2.14}$$

Then p_n *is starlike in* $|z| < r$*, where*

$$r = \min\{r_1, r_2\}. \tag{2.15}$$

Proof. Let n be odd and $p_n(z) = z + b_2z^2 + b_3z^3 + \cdots + b_nz^n$. Let the coefficients of $p_n(z)$ satisfy,

$$1 = b_1 \geqslant b_2 \geqslant b_3 \geqslant \cdots \geqslant b_n > 0.$$

Using Eneström-Kakeya theorem [3], $p_n(z)$ does not vanish in $\mathbb{D} \setminus \{0\}$. So $p_n(z)/z$ has no zeros in $\mathbb{D}$, and so $zp_n'(z)/p_n(z)$ is an analytic function in $\mathbb{D}$. We have to find radius $r > 0$ such that

$$\mathfrak{Re}\left\{\frac{zp_n'(z)}{p_n(z)}\right\} > 0 \ \text{ for } \ |z| < r.$$

Let $z = re^{i\theta}$, then

$$\begin{aligned}
&\mathfrak{Re}\frac{re^{i\theta}p_n'(re^{i\theta})}{p_n(re^{i\theta})} \\
&= \mathfrak{Re}\frac{re^{i\theta}\sum\limits_{k=1}^{n} kb_kr^{k-1}e^{i(k-1)\theta}}{\sum\limits_{k=1}^{n} b_kr^ke^{i\theta k}} \\
&= \mathfrak{Re}\frac{\sum\limits_{k=1}^{n} kb_kr^ke^{i\theta k}}{\sum\limits_{k=1}^{n} b_kr^ke^{i\theta k}} \\
&= \mathfrak{Re}\frac{\sum\limits_{k=1}^{n} kb_kr^k(\cos(k\theta) + i\sin(k\theta))}{\sum\limits_{k=1}^{n} b_kr^k(\cos(k\theta) + i\sin(k\theta))} \\
&= \frac{\left(\sum\limits_{k=1}^{n} kb_kr^k\cos(k\theta)\right)\left(\sum\limits_{k=1}^{n} b_kr^k\cos(k\theta)\right) + \left(\sum\limits_{k=1}^{n} kb_kr^k\sin(k\theta)\right)\left(\sum\limits_{k=1}^{n} b_kr^k\sin(k\theta)\right)}{\left(\sum\limits_{k=1}^{n} b_kr^k\cos(k\theta)\right)^2 + \left(\sum\limits_{k=1}^{n} b_kr^k\sin(k\theta)\right)^2}.
\end{aligned}$$

Now we will prove that all trigonometric sums inside the brackets are positive. Since the coefficients of $p_n(z)$ are real, so $p_n(\overline{z}) = \overline{p_n(z)}$ i.e. $p_n(z)$ is symmetric with respect to real axis, so we will prove it for $0 < \theta < \pi$. To find the condition on b_k, we will use Theorem 2.2. Let

$$c_k(r) = kb_kr^k, \ \ k \in \{1, 2, \ldots, n\}, \tag{2.16}$$

$$d_k(r) = b_kr^k, \ \ k \in \{1, 2, \ldots, n\}. \tag{2.17}$$

The sequence $c_k(r)$ satisfies the condition (2.6) if

$$\begin{aligned}
c_{2k}(r) &\leqslant \left(1 - \frac{\beta_0}{k}\right)c_{2k-1}(r) \\
2kb_{2k}r^{2k} &\leqslant \left(1 - \frac{\beta_0}{k}\right)(2k-1)b_{2k-1}r^{2k-1} \\
r &\leqslant \left(1 - \frac{\beta_0}{k}\right)\left(\frac{2k-1}{2k}\right)\frac{b_{2k-1}}{b_{2k}}, \ \ 2k \leqslant n.
\end{aligned}$$

Let

$$r_1 = \min\left\{\left(1-\frac{\beta_0}{k}\right)\left(\frac{2k-1}{2k}\right)\frac{b_{2k-1}}{b_{2k}}; 2k \leqslant n\right\}.$$

For such r_1, $d_k(r)$ also satisfies condition (2.6). The sequence $c_k(r)$ satisfies condition (2.5) if

$$\begin{aligned} c_{k+1}(r) &\leqslant c_k(r) \\ (k+1)b_{k+1}r^{k+1} &\leqslant kb_k r^k \\ r &\leqslant \frac{kb_k}{(k+1)b_{k+1}}, \quad k \in \{1,2,\dots,n-1\}. \end{aligned}$$

Let

$$r_2 = \min\left\{\frac{kb_k}{(k+1)b_{k+1}}; k \in \{1,2,\dots,n-1\}\right\}.$$

For such r_2, $d_k(r)$ also satisfies condition (2.5) because $k < k+1$. Let $r = \min\{r_1, r_2\}$, for such r, the trigonometric sums are positive. Hence $p_n(z)$ is starlike in $|z| < r$. □

THEOREM 2.5. *Assume that n is even and that the coefficients of the polynomial $p_n(z) = z + b_2z^2 + \cdots + b_nz^n$ satisfy*

$$1 = b_1 \geqslant b_2 \geqslant b_3 \geqslant \ldots \geqslant b_n > 0. \tag{2.18}$$

Let us denote

$$r_1 = \min\left\{\frac{(2k-1)^2 b_{2k-1}}{(2k)^2 b_{2k}} : 2k \leqslant n\right\}. \tag{2.19}$$

Furthermore,

$$r_2 = \min\left\{\frac{kb_k}{(k+1)b_{k+1}} : k \in \{1,2,\dots,n-1\}\right\}. \tag{2.20}$$

Then p_n is starlike in $|z| < r$, where

$$r = \min\{r_1, r_2\}. \tag{2.21}$$

Proof. We do the proof in the same manner as the proof of Theorem 2.4 with one exception: instead of Theorem 2.2 we apply Theorem 2.3. □

Let n be odd and $b_k = q^{k-1}$, $0 < q < 1$ and $p_n(z) = z + q^2z^2 + q^3z^3 + \cdots + q^{n-1}z^n$. Then

$$\begin{aligned} r_1 &= \min\left\{\left(1-\frac{\beta_0}{k}\right)\left(\frac{2k-1}{2k}\right)\frac{b_{2k-1}}{b_{2k}} : 2k \leqslant n\right\} \\ &= \min\left\{\left(1-\frac{\beta_0}{k}\right)\left(\frac{2k-1}{2k}\right)\frac{1}{q} : 2k \leqslant n\right\} \\ &= \min\left\{\frac{1-\beta_0}{2q}, \frac{(2-\beta_0)3}{8q}, \dots, \frac{(n-2\beta_0)(n-1)}{n^2q}\right\} \\ &= \frac{1-\beta_0}{2q} \end{aligned}$$

and

$$\begin{aligned} r_2 &= \min\left\{\frac{kb_k}{(k+1)b_{k+1}}; k \in \{1,2,\ldots,n-1\}\right\} \\ &= \min\left\{\frac{k}{(k+1)q}; k \in \{1,2,\ldots,n-1\}\right\} \\ &= \min\left\{\frac{1}{2q}, \frac{2}{3q}, \ldots, \frac{n-1}{nq}\right\} \\ &= \frac{1}{2q}. \end{aligned}$$

So, $r = \min\{r_1, r_2\} = (1-\beta_0)/(2q)$. Hence $p_n(z)$ is starlike in $|z| < (1-\beta_0)/(2q)$. We have

$$\frac{1-\beta_0}{2q} \geqslant 1 \Leftrightarrow q \leqslant \frac{1-\beta_0}{2} = 0.345778\ldots.$$

So if $0 < q \leqslant 0.345778\ldots$, then $p_n(z)$ is starlike in $|z| < 1$.

COROLLARY 2.6. *Let n be an odd positive integer. If $p_n(z) = z + b_2 z^2 + \cdots + b_n z^n$, $b_k = q^{k-1}$. If $0 < q \leqslant \frac{1-\beta_0}{2} = 0.345778\ldots$, then p_n is starlike in $|z| < 1$.*

Here we have some interesting particular cases for $q = 1/3$, $q = 1/2$, $q = 1/4$ contained in the following three corollaries, respectively.

COROLLARY 2.7. *Let n be an odd positive integer. If*

$$p_n(z) = z + \frac{z^2}{3} + \frac{z^3}{9} + \frac{z^4}{27} + \cdots + \frac{z^n}{3^{n-1}}$$

then p_n is starlike in $|z| < 3(1-\beta_0)/2 = 1.037334\ldots$.

COROLLARY 2.8. *Let n be an odd positive integer. If*

$$p_n(z) = z + \frac{z^2}{2} + \frac{z^3}{4} + \frac{z^4}{8} + \cdots + \frac{z^n}{2^{n-1}}$$

then p_n is starlike in $|z| < 1-\beta_0 = 0.691556\ldots$.

COROLLARY 2.9. *Let n be an odd positive integer. If*

$$p_n(z) = z + \frac{z^2}{4} + \frac{z^3}{16} + \frac{z^4}{64} + \cdots + \frac{z^n}{4^{n-1}}$$

then p_n is starlike in $|z| < 4(1-\beta_0)/2 = 1.383112\ldots$.

If n is even and $p_n(z) = z + b_2z^2 + \cdots + b_nz^n$, $b_n = q^{n-1}$, $0 < q < 1$, then

$$r_1 = \min\left\{\frac{(2k-1)^2}{(2k)^2q} : 2k \leqslant n\right\} = \frac{1}{4q} \tag{2.22}$$

and

$$r_2 = \min\left\{\frac{k}{(k+1)q} : k \in \{1,2,\ldots,n-1\}\right\} = \frac{1}{2q}. \tag{2.23}$$

Therefore,

$$r = \min\{r_1, r_2\} = \frac{1}{4q}. \tag{2.24}$$

For $q \leqslant 1/4$ we obtain $r \geqslant 1$. For $q = 1/2$ we obtain $r = 1/2$. Hence we get the following corollaries.

COROLLARY 2.10. *Let n be an even positive integer. If $p_n(z) = z + b_2z^2 + \cdots + b_nz^n$, $b_n = q^{n-1}$, $q \leqslant 1/4$, then p_n is starlike in $|z| < 1$.*

COROLLARY 2.11. *Let n be an even positive integer. If*

$$p_n(z) = z + \frac{z^2}{4} + \frac{z^3}{16} + \frac{z^4}{64} + \cdots + \frac{z^n}{4^{n-1}}$$

then p_n is starlike in $|z| < 1$.

COROLLARY 2.12. *Let n be an even positive integer. If*

$$p_n(z) = z + \frac{z^2}{2} + \frac{z^3}{4} + \frac{z^4}{8} + \cdots + \frac{z^n}{2^{n-1}}$$

then p_n is starlike in $|z| < 1/2$.

If in Theorems 2.4 and 2.5 instead of the assumption $r < r_2$ we write the condition $1 = b_1 \geqslant 2b_2 \geqslant 3b_3 \geqslant \ldots \geqslant nb_n > 0$ then we obtain the following theorems which proofs run as the proofs of Theorems 2.4 and 2.5.

THEOREM 2.13. *Let $\beta_0 = 0.308443\ldots$ denote the Littlewood-Salem-Izumi number which is the solution of (2.4). Assume that n is odd and that the coefficient of the polynomial $p_n(z) = z + b_2z^2 + \cdots + b_nz^n$ satisfy*

$$1 = b_1 \geqslant 2b_2 \geqslant 3b_3 \geqslant \ldots \geqslant nb_n > 0. \tag{2.25}$$

Let us denote

$$r_1 = \min\left\{\left(1 - \frac{\beta}{k}\right)\left(\frac{2k-1}{2k}\right)\frac{b_{2k-1}}{b_{2k}} : 2k \leqslant n\right\}, \tag{2.26}$$

where $\beta_0 \leqslant \beta \leqslant 1$. Then p_n is starlike in $|z| < r_1$.

THEOREM 2.14. *Assume that n is even and that the coefficient of the polynomial $p_n(z) = z + b_2z^2 + \cdots + b_nz^n$ satisfy*

$$1 = b_1 \geqslant 2b_2 \geqslant 3b_3 \geqslant \ldots \geqslant nb_n > 0. \tag{2.27}$$

Let us denote

$$r_1 = \min\left\{\frac{(2k-1)^2 b_{2k-1}}{(2k)^2 b_{2k}} : 2k \leqslant n\right\}. \tag{2.28}$$

Then p_n is starlike in $|z| < r_1$.

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