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GENERALIZED MARCINKIEWICZ–ZYGMUND TYPE INEQUALITIES FOR RANDOM VARIABLES AND APPLICATIONS

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Abstract. It is shown that if the higher order Marcinkiewicz-Zygmund type inequality holds, then some generalized Marcinkiewicz-Zygmund type inequality holds, in particular, the lower order Marcinkiewicz-Zygmund type inequality also holds. No additional assumptions are made on the random variables. As applications, a generalized C_r-inequality and a weak law of large numbers for pairwise independent random variables are obtained.

1. Introduction

Probability and moment inequalities play an important role in the properties of sums of random variables. A number of inequalities have been established for independent random variables. One of the most interesting inequalities is the Marcinkiewicz-Zygmund inequalities. For a sequence $\{X_i, 1 \leqslant i \leqslant n\}$ of independent random variables with mean 0 and $E|X_i|^p < \infty, 1 \leqslant i \leqslant n$, for some $p > 1$, there exist positive constants A_p and B_p depending only on p such that

$$A_p E\left(\sum_{i=1}^n X_i^2\right)^{p/2} \leqslant E\left|\sum_{i=1}^n X_i\right|^p \leqslant B_p E\left(\sum_{i=1}^n X_i^2\right)^{p/2}. \tag{1.1}$$

If $1 < p \leqslant 2$, then we can immediately obtain from (1.1) that

$$E\left|\sum_{i=1}^n X_i\right|^p \leqslant B_p \sum_{i=1}^n E|X_i|^p. \tag{1.2}$$

The above inequalities (1.1) and (1.2) were extended to a sequence of martingale differences by Burkholder [3] and von Bahr and Esseen [2], respectively. Hadjikyriakou [6] extended the Marcinkiewicz-Zygmund inequalities to the case of nonnegative N-demimartingales. Asadian et al. [1] proved that (1.2) holds for a sequence of negatively orthant dependent mean zero random variables. Burkholder [3] also proved that the Marcinkiewicz-Zygmund inequalities holds for the maximum of partial sums of martingale differences. Therefore, (1.1) also holds for the maximum of partial sums of

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independent random variables with mean 0. That is, there exist positive constants C_p and D_p depending only on p such that

$$C_pE\left(\sum_{i=1}^{n}X_i^2\right)^{p/2} \leqslant E\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}X_i\right|^p \leqslant D_pE\left(\sum_{i=1}^{n}X_i^2\right)^{p/2}. \tag{1.3}$$

If $1<p\leqslant 2$, then

$$E\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}X_i\right|^p \leqslant D_p\sum_{i=1}^{n}E|X_i|^p. \tag{1.4}$$

Shao [12] proved that (1.4) holds for a sequence of negatively associated mean zero random variables. The inequality (1.2) is called the p-th order Marcinkiewicz-Zygmund type inequality for the sum of random variables. The inequality (1.4) is called the p-th order Marcinkiewicz-Zygmund type inequality for the maximum of partial sums of random variables.

Let $f(x)=x^p$, $x\geqslant 0$, where $1<p\leqslant 2$. Then we can rewrite (1.2) and (1.4) as

$$Ef\left(\left|\sum_{i=1}^{n}X_i\right|\right) \leqslant C_f\sum_{i=1}^{n}Ef(|X_i|) \tag{1.5}$$

and

$$Ef\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}X_i\right|\right) \leqslant C_f\sum_{i=1}^{n}Ef(|X_i|) \tag{1.6}$$

respectively, where the first $C_f=B_p$, and the second $C_f=D_p$. The inequalities (1.5) and (1.6) are called the generalized Marcinkiewicz-Zygmund type inequalities if the function $f(x)$ is more general than $f(x)=x^p$.

In this paper, we prove that if the q-th order Marcinkiewicz-Zygmund type inequality holds for the sum of random variables, then the generalized Marcinkiewicz-Zygmund type inequality (1.5) with $f\in\Phi_q$ (Φ_q is defined below) holds, in particular, the $p(1<p<q)$-th order Marcinkiewicz-Zygmund type inequality also holds. No additional assumptions are made on the random variables. We also prove the same result for the maximum of partial sums of random variables. As applications, we obtain a generalized C_r-inequality for real numbers and a weak law of large numbers for pairwise independent random variables.

Now we introduce two notions and one lemma.

For any $q>0$, let Φ_q be the set of all continuous and strictly increasing functions $f(x)$ from $[0,\infty)$ to $[0,\infty)$ satisfying

(i) $f(0)=0$,

(ii) $f(x)\leqslant C'_f xf'(x)$ for almost all $x\in(0,\infty)$,

(iii) $\int_x^\infty[g(s)]^{-q}ds\leqslant C''_f x[g(x)]^{-q}$ for all $x\in(0,\infty)$,

(iv) $\int_0^x[g(s)]^{-1}ds\leqslant C'''_f x[g(x)]^{-1}$ for all $x\in(0,\infty)$,

where C'_f, C''_f, C'''_f are positive constants depending only on $f(x)$, and $g(x)$ is the inverse function of $f(x)$.

For any $q > 0$, let Ψ_q be the set of all continuous and strictly increasing functions $f(x)$ from $[0,\infty)$ to $[0,\infty)$ satisfying the above (i), (ii) and (iii).

Obviously, $\Phi_q \subset \Psi_q$, $\Phi_q \subset \Phi_{q'}$ and $\Psi_q \subset \Psi_{q'}$ if $q < q'$. We present some examples of functions in Φ_q and Ψ_q.

EXAMPLES. (1) For any $1 < p < q$ and $\alpha \in \mathbb{R}$, set $f(x) = x^p$ if $x \in [0,1]$ and $f(x) = x^p(\ln x + 1)^\alpha$ if $x \in (1,\infty)$. Then $f \in \Phi_q$.

(2) For any $1 < p < q$ and $\alpha \in (1-p, q-p)$, set $f(x) = x^p \ln^\alpha(x+1)$. Then $f \in \Phi_q$.

(3) For any $1 < p < q$, $\alpha \in \mathbb{R}$ and $a > 1$, set $f(x) = x^p \ln^\alpha(x+a)$. Then $f \in \Phi_q$.

(4) For any $1 < p < q$, $\alpha \in \mathbb{R}$, $a > 1$ and $\ln a > -\alpha/p$, set $f(x) = x^p \ln^\alpha a$ if $x \in [0,a]$ and $f(x) = x^p \ln^\alpha x$ if $x \in (a,\infty)$. Then $f \in \Phi_q$.

(5) For any $q > p > 0$ and $\alpha \in \mathbb{R}$, set $f(x) = x^p$ if $x \in [0,1]$ and $f(x) = x^p(\ln x + 1)^\alpha$ if $x \in (1,\infty)$. Then $f \in \Psi_q$.

(6) For any $q > p > 0$ and $\alpha \in (-p, q-p)$, set $f(x) = x^p \ln^\alpha(x+1)$. Then $f \in \Psi_q$.

(7) For any $q > p > 0$, $\alpha \in \mathbb{R}$ and $a > 1$, set $f(x) = x^p \ln^\alpha(x+a)$. Then $f \in \Psi_q$.

(8) For any $q > p > 0$, $\alpha \in \mathbb{R}$, $a > 1$ and $\ln a > -\alpha/p$, set $f(x) = x^p \ln^\alpha a$ if $x \in [0,a]$ and $f(x) = x^p \ln^\alpha x$ if $x \in (a,\infty)$. Then $f \in \Psi_q$.

The following lemma is well-known. It can be proved by the Fubini theorem.

LEMMA 1.1. *Let Y be a random variable and let r and b be positive constants. Then the following statements hold.*

(1) $E|Y|^r = r\int_0^\infty s^{r-1}P(|Y| > s)ds$.

(2) $E|Y|^r I(|Y| \leqslant b) = r\int_0^b s^{r-1}P(|Y| > s)ds - b^r P(|Y| > b)$.

(3) $E|Y|^r I(|Y| > b) = r\int_b^\infty s^{r-1}P(|Y| > s)ds + b^r P(|Y| > b)$.

Throughout this paper, $I(A)$ denotes the indicator function of the event A.

2. Generalized Marcinkiewicz-Zygmund type inequalities

The following theorem shows that if the higher order Marcinkiewicz-Zygmund type inequality holds for the sum of random variables, then some generalized Marcinkiewicz-Zygmund type inequality holds, in particular, the lower order Marcinkiewicz-Zygmund type inequality also holds.

THEOREM 2.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables with finite means. Assume that for some $q > 1$, there exists a positive function $\alpha_q(x)$ such that*

$$E\left|\sum_{i=1}^n (X_i(x) - EX_i(x))\right|^q \leqslant \alpha_q(n) \sum_{i=1}^n E|X_i(x)|^q \quad \text{for all } n \geqslant 1 \text{ and } x > 0, \tag{2.1}$$

where $X_i(x) = X_i I(|X_i| \leqslant x) + xI(X_i > x) - xI(X_i < -x)$. Then for $f \in \Phi_q$,

$$Ef\left(\left|\sum_{i=1}^n (X_i - EX_i)\right|\right) \leqslant \left(2^q \alpha_q(n) \cdot qC'_f C''_f + 4C'_f C'''_f\right) \sum_{i=1}^n Ef(|X_i|).$$

In particular, for $1 < p < q$,

$$E\left|\sum_{i=1}^{n}(X_i - EX_i)\right|^p \leqslant \left(2^q \alpha_q(n)\frac{q}{q-p} + \frac{4}{p-1}\right)\sum_{i=1}^{n} E|X_i|^p.$$

Proof. By Markov's inequality and (2.1), we have that

$$\begin{aligned}
&Ef\left(\left|\sum_{i=1}^{n}(X_i - EX_i)\right|\right)\\
&= \int_0^\infty P\left(\left|\sum_{i=1}^{n}(X_i - EX_i)\right| > g(x)\right)dx\\
&\leqslant \int_0^\infty P\left(\left|\sum_{i=1}^{n}(X_i(g(x)) - EX_i(g(x)))\right| > g(x)/2\right)dx\\
&\quad + \int_0^\infty P\left(\left|\sum_{i=1}^{n}(X_i - X_i(g(x)) - E(X_i - X_i(g(x))))\right| > g(x)/2\right)dx\\
&\leqslant 2^q \int_0^\infty [g(x)]^{-q} E\left|\sum_{i=1}^{n}(X_i(g(x)) - EX_i(g(x)))\right|^q dx\\
&\quad + 2\int_0^\infty [g(x)]^{-1} E\left|\sum_{i=1}^{n}(X_i - X_i(g(x)) - E(X_i - X_i(g(x))))\right| dx\\
&\leqslant 2^q \alpha_q(n) \sum_{i=1}^{n} \int_0^\infty [g(x)]^{-q} E\,|X_i(g(x))|^q\,dx\\
&\quad + 4\sum_{i=1}^{n}\int_0^\infty [g(x)]^{-1} E\,|X_i - X_i(g(x))|\,dx.
\end{aligned} \tag{2.2}$$

By Lemma 1.1 and the Fubini theorem, we get

$$\begin{aligned}
&\int_0^\infty [g(x)]^{-q} E\,|X_i(g(x))|^q\,dx\\
&= \int_0^\infty [g(x)]^{-q} dx \int_0^{[g(x)]^q} P(|X_i| > y^{1/q})dy\\
&= \int_0^\infty P(|X_i| > y^{1/q})dy \int_{f(y^{1/q})}^\infty [g(x)]^{-q} dx\\
&\leqslant C_f'' \int_0^\infty y^{-1} f(y^{1/q}) P(|X_i| > y^{1/q})dy \quad \text{(by (iii))}\\
&= qC_f'' \int_0^\infty x^{-1} f(x) P(|X_i| > x)dx \quad \text{(by taking } y = x^q\text{)}\\
&\leqslant qC_f' C_f'' \int_0^\infty f'(x) P(|X_i| > x)dx \quad \text{(by (ii))}\\
&= qC_f' C_f'' Ef(|X_i|).
\end{aligned} \tag{2.3}$$

We also get

$$\begin{aligned}
&\int_0^\infty [g(x)]^{-1} E\,|X_i - X_i(g(x))|\,dx \\
&= \int_0^\infty [g(x)]^{-1} dx \int_{g(x)}^\infty P(|X_i| > y)dy \\
&= \int_0^\infty P(|X_i| > y)dy \int_0^{f(y)} [g(x)]^{-1} dx \\
&\leqslant C_f''' \int_0^\infty y^{-1} f(y) P(|X_i| > y)dy \ \ \text{(by (iv))} \\
&\leqslant C_f' C_f''' \int_0^\infty f'(y) P(|X_i| > y)dy \ \ \text{(by (ii))} \\
&= C_f' C_f''' E f(|X_i|).
\end{aligned} \tag{2.4}$$

Substituting (2.3) and (2.4) into (2.2), we obtain the desired result. □

The following theorem shows that if the higher order Marcinkiewicz-Zygmund type inequality holds for the maximum of partial sums of random variables, then some generalized Marcinkiewicz-Zygmund type inequality holds, in particular, the lower order Marcinkiewicz-Zygmund type inequality also holds.

THEOREM 2.2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables with finite means. Assume that for some $q > 1$, there exists a positive function $\beta_q(x)$ such that*

$$E \max_{1\leqslant k\leqslant n} \left| \sum_{i=1}^k (X_i(x) - EX_i(x)) \right|^q \leqslant \beta_q(n) \sum_{i=1}^n E\,|X_i(x)|^q \quad \text{for all } n \geqslant 1 \text{ and } x > 0, \tag{2.5}$$

where $X_i(x) = X_i I(|X_i| \leqslant x) + xI(X_i > x) - xI(X_i < -x)$. Then for $f \in \Phi_q$,

$$Ef\left(\max_{1\leqslant k\leqslant n} \left| \sum_{i=1}^k (X_i - EX_i) \right| \right) \leqslant \left(2^q \beta_q(n) \cdot qC_f' C_f'' + 4C_f' C_f''' \right) \sum_{i=1}^n Ef(|X_i|).$$

In particular, for $1 < p < q$,

$$E \max_{1\leqslant k\leqslant n} \left| \sum_{i=1}^k (X_i - EX_i) \right|^p \leqslant \left(2^q \beta_q(n) \frac{q}{q-p} + \frac{4}{p-1} \right) \sum_{i=1}^n E|X_i|^p.$$

Proof. The proof is similar to that of Theorem 2.1. By Markov's inequality and

(2.5), we have that

$$
\begin{aligned}
&Ef\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}(X_i-EX_i)\right|\right)\\
&=\int_0^\infty P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}(X_i-EX_i)\right|>g(x)\right)dx\\
&\leqslant\int_0^\infty P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}(X_i(g(x))-EX_i(g(x)))\right|>g(x)/2\right)dx\\
&\quad+\int_0^\infty P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}(X_i-X_i(g(x))-E(X_i-X_i(g(x))))\right|>g(x)/2\right)dx\\
&\leqslant 2^q\int_0^\infty[g(x)]^{-q}E\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}(X_i(g(x))-EX_i(g(x)))\right|^q dx\\
&\quad+2\int_0^\infty[g(x)]^{-1}E\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}(X_i-X_i(g(x))-E(X_i-X_i(g(x))))\right|dx\\
&\leqslant 2^q\beta_q(n)\sum_{i=1}^{n}\int_0^\infty[g(x)]^{-q}E\,|X_i(g(x))|^q\,dx\\
&\quad+4\sum_{i=1}^{n}\int_0^\infty[g(x)]^{-1}E\,|X_i-X_i(g(x))|\,dx.
\end{aligned}
$$

The rest of the proof is the same as that of Theorem 2.1 and is omitted. □

REMARK 2.1. The 2nd order Marcinkiewicz-Zygmund type inequalities ((2.1) and (2.5) with $q=2$) hold for dependent random variables as well as independent random variables.

(1) Let $\{X_n, n\geqslant 1\}$ be a sequence of pairwise independent random variables. Then (2.1) holds trivially for $q=2$ and $\alpha_q(n)=1$. By Theorem 3 of Móricz [9], (2.5) holds for $q=2$ and $\beta_q(n)=(\ln 2n/\ln 2)^2$.

(2) Let $\{X_n, n\geqslant 1\}$ be a sequence of pairwise negative quadrant dependent random variables (for the definition of negative quadrant dependence, see Lehmann [7]). Then (2.1) holds trivially for $q=2$ and $\alpha_q(n)=1$. By Theorem 3 of Móricz [9], (2.5) holds for $q=2$ and $\beta_q(n)=(\ln 2n/\ln 2)^2$.

(3) Let $\{X_n, n\geqslant 1\}$ be a sequence of negatively associated random variables. Then (2.1) holds trivially for $q=2$ and $\alpha_q(n)=1$, and (2.5) holds for $q=2$ and $\beta_q(n)=2$ (see Matula [8]).

(4) Let $\{X_n, n\geqslant 1\}$ be a sequence of φ-mixing random variables. Then (2.5) holds for $q=2$ and a constant function $\beta_q(x)$ if $\sum_{n=1}^{\infty}\varphi^{1/2}(n)<\infty$ (see Yang [14]).

(5) Let $\{X_n, n\geqslant 1\}$ be a sequence of identically distributed φ-mixing random variables. Then (2.5) holds for $q=2$ and a slowly varying function $\beta_q(x)$ (see Shao [10]). In particular, if $\sum_{n=1}^{\infty}\varphi^{1/2}(2^n)<\infty$, then (2.5) holds for $q=2$ and a constant function $\beta_q(x)$.

(6) Let $\{X_n, n \geqslant 1\}$ be a sequence of ρ-mixing random variables. Then (2.1) holds for $q = 2$ and a constant function $\beta_q(x)$ if $\sum_{n=1}^{\infty} \rho(n) < \infty$ (see Yang [14]).

(7) Let $\{X_n, n \geqslant 1\}$ be a sequence of identically distributed ρ-mixing random variables. Then (2.5) holds for $q = 2$ and a slowly varying functions $\beta_q(x)$ (see Shao [11]). In particular, if $\sum_{n=1}^{\infty} \rho(2^n) < \infty$, then (2.5) holds for $q = 2$ and a constant function $\beta_q(x)$.

(8) Let $\{X_n, n \geqslant 1\}$ be a sequence of ρ^*-mixing random variables. Then (2.5) holds for $q = 2$ and a constant function $\beta_q(x)$ (see Utev and Peligrad [13]).

COROLLARY 2.1. *Let $1 < p < 2$ and let $\{X_n, n \geqslant 1\}$ be a sequence of pairwise independent random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for $n \geqslant 1$. Then*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leqslant \left(\frac{8}{2-p} + \frac{4}{p-1}\right) \sum_{i=1}^{n} E|X_i|^p,$$

$$E\max_{1 \leqslant k \leqslant n}\left|\sum_{i=1}^{k} X_i\right|^p \leqslant \left(\left(\frac{\ln 2n}{\ln 2}\right)^2 \frac{8}{2-p} + \frac{4}{p-1}\right) \sum_{i=1}^{n} E|X_i|^p.$$

Proof. The result follows from Theorems 2.1 and 2.2 together with Remark 2.1 (1). □

REMARK 2.2. Under the same conditions as Corollary 2.1, Chen et al. [5] proved that

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leqslant C_p \sum_{i=1}^{n} E|X_i|^p,$$

where $C_p = \inf_{0<\varepsilon<\infty} f(\varepsilon)$ and $f(\varepsilon) = 2 + \varepsilon + \left(\frac{1+\varepsilon}{\varepsilon}\right)^2 \left(\frac{2}{2-p}\right)^2$. We compare the coefficient C_p with that of Corollary 2.1. Since $8/(2-p) + 4/(p-1)$ diverges as $p > 1$ goes to 1, $8/(2-p) + 4/(p-1) > C_p$ for all $p > 1$ sufficiently close to 1. Since $(2-p)f(\varepsilon) > 4/(2-p) > 8 + 4(2-p)/(p-1)$ for all $p < 2$ sufficiently close to 2, $8/(2-p) + 4/(p-1) < C_p$ for all $p < 2$ sufficiently close to 2. Hence, we cannot compare the coefficients.

COROLLARY 2.2. *Let $1 < p < 2$ and let $\{X_n, n \geqslant 1\}$ be a sequence of ρ^*-mixing random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for $n \geqslant 1$. Then*

$$E\max_{1 \leqslant k \leqslant n}\left|\sum_{i=1}^{k} X_i\right|^p \leqslant C_p \sum_{i=1}^{n} E|X_i|^p,$$

where C_p is a positive constant independent of n.

Proof. The result follows from Theorem 2.2 together with Remark 2.1 (8). □

REMARK 2.3. When $\{X_n, n \geqslant 1\}$ is a sequence of asymptotically linear negative quadrant dependent (ALNQD) random variables with mean 0, Zhang [15] proved that for $p > 1$,

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leqslant C_p E\left(\sum_{i=1}^{n} X_i^2\right)^{p/2}, \tag{2.6}$$

where C_p is a positive constant independent of n. From (2.6), we get that for $1 < p < 2$,

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leqslant C_p \sum_{i=1}^{n} E|X_i|^p. \tag{2.7}$$

If ALNQD condition is strengthen to ρ^*-mixing, then (2.7) holds for the maximum of partial sums (see Corollary 2.2).

In the symmetric case, the condition $f \in \Phi_q$ can be weaken to $f \in \Psi_q$ (see the following theorem).

THEOREM 2.3. *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables. Assume that for some $q > 1$, there exists a positive function $\alpha_q(x)$ such that*

$$E\left|\sum_{i=1}^{n} X_i(x)\right|^q \leqslant \alpha_q(n) \sum_{i=1}^{n} E\left|X_i(x)\right|^q \quad \text{for all } n \geqslant 1 \text{ and } x > 0, \tag{2.8}$$

where $X_i(x) = X_i I(|X_i| \leqslant x) + xI(X_i > x) - xI(X_i < -x)$. Then for $f \in \Psi_q$,

$$Ef\left(\left|\sum_{i=1}^{n} X_i\right|\right) \leqslant \left(1 + \alpha_q(n) \cdot qC'_f C''_f\right) \sum_{i=1}^{n} Ef(|X_i|).$$

In particular, for $1 < p < q$,

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leqslant \left(1 + \alpha_q(n) \cdot \frac{q}{q-p}\right) \sum_{i=1}^{n} E|X_i|^p.$$

Proof. Note that

$$\begin{aligned}
&Ef\left(\left|\sum_{i=1}^{n} X_i\right|\right) \\
&= \int_0^\infty P\left(\left|\sum_{i=1}^{n} X_i\right| > g(x)\right) dx \\
&\leqslant \sum_{i=1}^{n} \int_0^\infty P(|X_i| > g(x))dx + \int_0^\infty P\left(\left|\sum_{i=1}^{n} X_i(g(x))\right| > g(x)\right) dx \\
&\leqslant \sum_{i=1}^{n} Ef(|X_i|) + \alpha_q(n) \sum_{i=1}^{n} \int_0^\infty [g(x)]^{-q} E|X_i(g(x))|^q dx.
\end{aligned}$$

By (2.3), we complete the proof. □

When $f \in \Psi_1$, the generalized Marcinkiewicz-Zygmund type inequality holds without any assumptions.

THEOREM 2.4. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of random variables. If* $f \in \Psi_1$, *then*

$$Ef\left(\left|\sum_{i=1}^{n} X_i\right|\right) \leqslant C'_f C''_f \sum_{i=1}^{n} Ef(|X_i|)$$

and

$$Ef\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} X_i\right|\right) \leqslant C'_f C''_f \sum_{i=1}^{n} Ef(|X_i|).$$

Proof. Note that

$$\begin{aligned}
&Ef\left(\left|\sum_{i=1}^{n} X_i\right|\right)\\
&= \int_0^\infty P\left(\left|\sum_{i=1}^{n} X_i\right| > g(x)\right) dx\\
&\leqslant \sum_{i=1}^{n}\int_0^\infty P(|X_i| > g(x))dx + \int_0^\infty P\left(\left|\sum_{i=1}^{n} X_i I(|X_i| \leqslant g(x))\right| > g(x)\right) dx\\
&\leqslant \sum_{i=1}^{n} Ef(|X_i|) + \sum_{i=1}^{n}\int_0^\infty [g(x)]^{-1} E|X_i| I(|X_i| \leqslant g(x))dx.
\end{aligned}$$

By Lemma 1.1, we have that for any $1 \leqslant i \leqslant n$,

$$\begin{aligned}
&\int_0^\infty [g(x)]^{-1} E|X_i| I(|X_i| \leqslant g(x))dx\\
&= \int_0^\infty [g(x)]^{-1}\left(\int_0^{g(x)} P(|X_i| > y)dy - g(x)P(|X_i| > g(x))\right) dx\\
&= \int_0^\infty P(|X_i| > y)dy \int_{f(y)}^\infty [g(x)]^{-1}dx - Ef(|X_i|)\\
&\leqslant C''_f \int_0^\infty y^{-1} f(y) P(|X_i| > y)dy - Ef(|X_i|)\\
&\leqslant C'_f C''_f \int_0^\infty f'(y) P(|X_i| > y)dy - Ef(|X_i|)\\
&= (C'_f C''_f - 1)Ef(|X_i|).
\end{aligned}$$

So the first result is proved. Observing that

$$f\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} X_i\right|\right) \leqslant f\left(\sum_{i=1}^{n} |X_i|\right),$$

the second result follows from the first. $\square$

3. Applications

Generalized Marcinkiewicz-Zygmund inequalities can be applied to estimate the moments and tail probabilities of sums of random variables. In this section, we present only two of many such applications. As a simple application of Theorem 2.4, we can obtain a generalized version of the C_r-inequality.

THEOREM 3.1. *Let $\{x_n, n \geqslant 1\}$ be a sequence of real numbers. If $f \in \Psi_1$, then*

$$f\left(\left|\sum_{i=1}^{n} x_i\right|\right) \leqslant C'_f C''_f \sum_{i=1}^{n} f(|x_i|).$$

As an application of Theorem 2.1, we can obtain a weak law of large numbers for pairwise independent random variables.

THEOREM 3.2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of pairwise independent random variables with $EX_n = 0$ for all $n \geqslant 1$ and satisfying $f\,(f \in \Phi_2)$-uniform integrability in the Cesàro sense, i.e.,*

$$\lim_{x\to\infty} \sup_{n\geqslant 1} n^{-1} \sum_{i=1}^{n} Ef(|X_i|)I(|X_i| > x) = 0.$$

Assume that for any $\varepsilon > 0$,

$$\frac{n}{f(\varepsilon g(n))} = O(1) \quad \text{and} \quad \frac{n}{g^2(n)} \to 0,$$

where g is the inverse function of f. Then $g^{-1}(n)\sum_{i=1}^{n} X_i \to 0$ in probability as $n \to \infty$.

Proof. Let $\varepsilon > 0$ be given. Then we get by Theorem 2.1 that for any $x > 0$,

$$\begin{aligned}
&P\left(g^{-1}(n)\left|\sum_{i=1}^{n} X_i\right| > \varepsilon\right)\\
&\leqslant P\left(\left|\sum_{i=1}^{n}(X_iI(|X_i| \leqslant x) - EX_iI(|X_i| \leqslant x))\right| > \varepsilon g(n)/2\right)\\
&\quad + P\left(\left|\sum_{i=1}^{n}(X_iI(|X_i| > x) - EX_iI(|X_i| > x))\right| > \varepsilon g(n)/2\right)\\
&\leqslant 4\varepsilon^{-2}g^{-2}(n)E\left|\sum_{i=1}^{n}(X_iI(|X_i| \leqslant x) - EX_iI(|X_i| \leqslant x))\right|^2\\
&\quad + f^{-1}(\varepsilon g(n)/2)Ef\left(\left|\sum_{i=1}^{n}(X_iI(|X_i| > x) - EX_iI(|X_i| > x))\right|\right)\\
&\leqslant 4\varepsilon^{-2}g^{-2}(n)nx^2 + \left(8C'_fC''_f + 4C'_fC'''_f\right) f^{-1}(\varepsilon g(n)/2)\sum_{i=1}^{n} Ef\left(|X_i|I(|X_i| > x)\right)\\
&\leqslant 4\varepsilon^{-2}g^{-2}(n)nx^2 + \left(8C'_fC''_f + 4C'_fC'''_f\right) f^{-1}(\varepsilon g(n)/2)n\sup_{m\geqslant 1} m^{-1}\sum_{i=1}^{m} Ef\left(|X_i|\right)I(|X_i| > x).
\end{aligned}$$

It follows by the assumptions that

$$\begin{aligned}
&\limsup_{n\to\infty} P\left(g^{-1}(n)\left|\sum_{i=1}^{n} X_i\right| > \varepsilon\right) \\
&\leqslant O(1)\left(8C'_f C''_f + 4C'_f C'''_f\right)\sup_{m\geqslant 1} m^{-1}\sum_{i=1}^{m} Ef\left(|X_i|\right)I(|X_i|>x) \to 0,
\end{aligned}$$

as $x\to\infty$. Hence the result is proved. □

COROLLARY 3.1. *Let* $1<p<2$ *and* $\alpha\in\mathbb{R}$. *Let* $\{X_n, n\geqslant 1\}$ *be a sequence of pairwise independent random variables with* $EX_n=0$ *for all* $n\geqslant 1$ *and satisfying*

$$\lim_{x\to\infty}\sup_{n\geqslant 1} n^{-1}\sum_{i=1}^{n} E|X_i|^p \ln^{\alpha}|X_i| I(|X_i|>x) = 0.$$

Then $n^{-1/p}(\ln n)^{\alpha/p}\sum_{i=1}^{n} X_i \to 0$ *in probability as* $n\to\infty$.

Proof. Let $a>1$ and $\ln a > -\alpha/p$. Set $f(x)=x^p\ln^{\alpha} a$ if $x\in[0,a]$ and $f(x)=x^p\ln^{\alpha} x$ if $x\in(a,\infty)$. Then $f\in\Phi_2$ by Example (4). Note that

$$\lim_{x\to\infty}\sup_{n\geqslant 1} n^{-1}\sum_{i=1}^{n} Ef(|X_i|)I(|X_i|>x) = \lim_{x\to\infty}\sup_{n\geqslant 1} n^{-1}\sum_{i=1}^{n} E|X_i|^p\ln^{\alpha}|X_i| I(|X_i|>x) = 0.$$

Since

$$g(x)\sim p^{\alpha/p}\frac{x^{1/p}}{(\ln x)^{\alpha/p}} \quad \text{as } x\to\infty,$$

$n/f(\varepsilon g(n)) = O(1)$ for any $\varepsilon>0$ and $n/g^2(n)\to 0$ as $n\to\infty$, where g is the inverse function of f. Hence the result follows from Theorem 3.2. □

REMARK 3.1. When $\alpha=0$, Corollary 3.1 ($p=1$ and $1<p<2$, respectively) were proved by Chandra [4] and Chen et al. [5].

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