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A METHOD FOR PROVING SOME INEQUALITIES ON MIXED TRIGONOMETRIC POLYNOMIAL FUNCTIONS

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Abstract. In this article we present a method for proving a class of inequalities of the form (1). The method is based on the precise approximations of the sine and cosine functions by Maclaurin polynomials of given order. By using this method we present new proofs of some inequalities of C.-P. Chen, W.-S. Cheung [J. Inequal. Appl. 2012:72 (2012)] and Z.-J. Sun, L. Zhu [ISRN Math. Anal. (2011)].

1. Introduction

In this article we consider a method for proving trigonometric inequalities of the form

$$f(x)=\sum_{i=1}^{n}\alpha_i x^{p_i}\cos^{q_i}x\sin^{r_i}x>0, \tag{1}$$

for $x\in(\delta_1,\delta_2)$, $\delta_1\leqslant 0\leqslant\delta_2$ and $\delta_1<\delta_2$; where $\alpha_i\in\mathbb{R}\setminus\{0\}$, $p_i,q_i,r_i\in\mathbb{N}_0$ and $n\in\mathbb{N}$. The function $f(x)$ is a mixed trigonometric polynomial function, see [10]. These functions appear in the theory of analytic inequalities [1]–[9], [13]–[19], [21], [23]–[45].

In the article [26] a natural approach for proving some concrete examples of inequalities of the form (1) has been shown. This method is based on the direct comparison of the sine and cosine functions with the corresponding Maclaurin polynomials. However, the above-mentioned method is not appliable to the function $\cos^2 x$ in the whole interval $[0,\frac{\pi}{2}]$ and to the function $\sin^2 x$ in the whole interval $[0,\pi]$. Based on that fact, note that it is not advisable to make comparisons of $\cos^{q_i}x\cdot\sin^{r_i}x$ with the product of the corresponding Maclaurin approximations of the cosine and sine functions raised to the powers q_i and r_i respectively. Therefore, one of the possibilities is to make a transformation of $\cos^{q_i}x\cdot\sin^{r_i}x$ into the sum of sines and cosines of multiple angles. In the continuation of the article, we have explained a method for proving inequalities of the form (1) by transforming the function $f(x)$ into an equivalent form in which sines and cosines of multiple angles appear.

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Let the function $\varphi(x)$ be approximated by Taylor polynomial $T_k(x)$ of degree k in the neighbourhood of a point a. If there is $\eta > 0$ such that in the interval $(a-\eta, a+\eta)$ it holds:

$$T_k(x) \geqslant \varphi(x),$$

then we introduce the symbol $\overline{T}_k^{\varphi,a}(x) = T_k(x)$ and we call $\overline{T}_k^{\varphi,a}(x)$ the upward approximation of the function $\varphi(x)$ in the neighbourhood of the point a. Analogously, if there is $\eta > 0$ such that in the interval $(a-\eta, a+\eta)$ it holds:

$$T_k(x) \leqslant \varphi(x),$$

then we introduce the symbol $\underline{T}_k^{\varphi,a}(x) = T_k(x)$ and we call $\underline{T}_k^{\varphi,a}(x)$ the downward approximation of the function $\varphi(x)$ in the neighbourhood of the point a. Further on, we observe the function $\varphi(x)$ as a function from the set $\{\sin x, \cos x\}$.

Observing Maclaurin approximations of the sine and cosine functions, we notice that $\overline{T}_1^{\sin,0}(x)$ is above and $\underline{T}_3^{\sin,0}(x)$ is below the graph of the function $\sin x$ for $x > 0$ and $\underline{T}_1^{\sin,0}(x)$ is below and $\overline{T}_3^{\sin,0}(x)$ is above the graph of the function $\sin x$ for $x < 0$ as well as that $\overline{T}_0^{\cos,0}(x)$ is above and $\underline{T}_2^{\cos,0}(x)$ is below the graph of the function $\cos x$. The previous facts are stated precisely and generalized through the following Lemmas:

LEMMA 1.1. **(i)** *For the polynomial* $T_n(t) = \displaystyle\sum_{i=0}^{(n-1)/2} \frac{(-1)^i t^{2i+1}}{(2i+1)!}$, *where* $n = 4k+1$ $k \in \mathbb{N}_0$, *it is valid:*

$$\left(\forall t \in \left[0, \sqrt{(n+3)(n+4)}\,\right]\right) \overline{T}_n(t) \geqslant \overline{T}_{n+4}(t) \geqslant \sin t, \tag{2}$$

$$\left(\forall t \in \left[-\sqrt{(n+3)(n+4)}, 0\right]\right) \underline{T}_n(t) \leqslant \underline{T}_{n+4}(t) \leqslant \sin t. \tag{3}$$

For the value $t = 0$ *the inequalities in* (2) *and* (3) *turn into equalities. For the values* $t = \pm\sqrt{(n+3)(n+4)}$ *the equalities* $\overline{T}_n(t) = \overline{T}_{n+4}(t)$ *and* $\underline{T}_n(t) = \underline{T}_{n+4}(t)$ *are true, respectively.*

(ii) *For the polynomial* $T_n(t) = \displaystyle\sum_{i=0}^{(n-1)/2} \frac{(-1)^i t^{2i+1}}{(2i+1)!}$, *where* $n = 4k+3$, $k \in \mathbb{N}_0$, *it is valid:*

$$\left(\forall t \in \left[0, \sqrt{(n+3)(n+4)}\,\right]\right) \underline{T}_n(t) \leqslant \underline{T}_{n+4}(t) \leqslant \sin t, \tag{4}$$

$$\left(\forall t \in \left[-\sqrt{(n+3)(n+4)}, 0\right]\right) \overline{T}_n(t) \geqslant \overline{T}_{n+4}(t) \geqslant \sin t. \tag{5}$$

For the value $t = 0$ *the inequalities in* (4) *and* (5) *turn into equalities. For the values* $t = \pm\sqrt{(n+3)(n+4)}$ *the equalities* $\underline{T}_n(t) = \underline{T}_{n+4}(t)$ *and* $\overline{T}_n(t) = \overline{T}_{n+4}(t)$ *are true, respectively.*

Proof. **(i)** Let $0 < t \leqslant \sqrt{(n+3)(n+4)}$. Then:

$$\overline{T}_n(t) = \sum_{i=0}^{2k} \frac{(-1)^i t^{2i+1}}{(2i+1)!} > \sum_{i=0}^{\infty} \frac{(-1)^i t^{2i+1}}{(2i+1)!} = \sin t$$

$$\iff \sum_{j=1}^{\infty} \frac{t^{4(k+j)-1}}{\big(4(k+j)-1\big)!} \underbrace{\left(1 - \frac{t^2}{\big(4(k+j)\big)\big(4(k+j)+1\big)}\right)}_{(\geqslant 0)} > 0.$$

Thus

$$\overline{T}_{n+4}(t) = \overline{T}_n(t) - \frac{t^{n+2}}{(n+2)!} \underbrace{\left(1 - \frac{t^2}{(n+3)(n+4)}\right)}_{(\geqslant 0)} \leqslant \overline{T}_n(t)$$

and

$$\overline{T}_{n+4}(t) = \sum_{i=0}^{2k+2} \frac{(-1)^i t^{2i+1}}{(2i+1)!} > \sum_{i=0}^{\infty} \frac{(-1)^i t^{2i+1}}{(2i+1)!} = \sin t$$

$$\iff \sum_{j=1}^{\infty} \frac{t^{4(k+j)+3}}{\big(4(k+j)+3\big)!} \underbrace{\left(1 - \frac{t^2}{\big(4(k+j)+4\big)\big(4(k+j)+5\big)}\right)}_{(>0)} > 0.$$

The equalities at the endpoints of the segment $\left[0, \sqrt{(n+3)(n+4)}\,\right]$ are also true. Overall, (2) has been proved. For $t \in \left[-\sqrt{(n+3)(n+4)}, 0\right)$, (3) is valid on the basis of the odd property of the function $\sin x$. Overall, (3) has been proved.

(ii) Let $0 < t \leqslant \sqrt{(n+3)(n+4)}$. Then:

$$\underline{T}_n(t) = \sum_{i=0}^{2k+1} \frac{(-1)^i t^{2i+1}}{(2i+1)!} < \sum_{i=0}^{\infty} \frac{(-1)^i t^{2i+1}}{(2i+1)!} = \sin t$$

$$\iff \sum_{j=1}^{\infty} -\frac{t^{4(k+j)+1}}{\big(4(k+j)+1\big)!} \underbrace{\left(1 - \frac{t^2}{\big(4(k+j)+2\big)\big(4(k+j)+3\big)}\right)}_{(\geqslant 0)} < 0.$$

Thus

$$\underline{T}_{n+4}(t) = \underline{T}_n(t) + \frac{t^{n+2}}{(n+2)!} \underbrace{\left(1 - \frac{t^2}{(n+3)(n+4)}\right)}_{(\geqslant 0)} \geqslant \underline{T}_n(t)$$

and

$$\underline{T}_{n+4}(t) = \sum_{i=0}^{2k+3} \frac{(-1)^i t^{2i+1}}{(2i+1)!} < \sum_{i=0}^{\infty} \frac{(-1)^i t^{2i+1}}{(2i+1)!} = \sin t$$

$$\iff \sum_{j=1}^{\infty} \frac{-t^{4(k+j)+5}}{(4(k+j)+5)!} \underbrace{\left(1 - \frac{t^2}{\big(4(k+j)+6\big)\big(4(k+j)+7\big)}\right)}_{(>0)} < 0.$$

The equalities at the endpoints of the segment $\left[0, \sqrt{(n+3)(n+4)}\,\right]$ are true. Overall, (4) has been proved. For $t \in \left[-\sqrt{(n+3)(n+4)}, 0\right)$, (5) is valid on the basis of the odd property of the function $\sin x$. Overall, (5) has been proved. □

LEMMA 1.2. **(i)** *For the polynomial* $T_n(t) = \sum_{i=0}^{n/2} \frac{(-1)^i t^{2i}}{(2i)!}$, *where* $n = 4k,\ k \in \mathbb{N}_0$, *it is valid:*

$$\left(\forall t \in \left[-\sqrt{(n+3)(n+4)}, \sqrt{(n+3)(n+4)}\,\right]\right)\ \overline{T}_n(t) \geqslant \overline{T}_{n+4}(t) \geqslant \cos t. \tag{6}$$

For the value $t = 0$ *the inequalities in* (6) *turn into equalities. For the values* $t = \pm\sqrt{(n+3)(n+4)}$ *the equality* $\overline{T}_n(t) = \overline{T}_{n+4}(t)$ *is true.*

(ii) *For the polynomial* $T_n(t) = \sum_{i=0}^{n/2} \frac{(-1)^i t^{2i}}{(2i)!}$, *where* $n = 4k+2,\ k \in \mathbb{N}_0$, *it is valid:*

$$\left(\forall t \in \left[-\sqrt{(n+3)(n+4)}, \sqrt{(n+3)(n+4)}\,\right]\right)\ \underline{T}_n(t) \leqslant \underline{T}_{n+4}(t) \leqslant \cos t. \tag{7}$$

For the value $t = 0$ *the inequalities in* (7) *turn into equalities. For the values* $t = \pm\sqrt{(n+3)(n+4)}$ *the equality* $\underline{T}_n(t) = \underline{T}_{n+4}(t)$ *is true.*

Proof. **(i)** Let $0 < t \leqslant \sqrt{(n+3)(n+4)}$. Then:

$$\overline{T}_n(t) = \sum_{i=0}^{2k} \frac{(-1)^i t^{2i}}{(2i)!} > \sum_{i=0}^{\infty} \frac{(-1)^i t^{2i}}{(2i)!} = \cos t$$

$$\iff \sum_{j=1}^{\infty} \frac{t^{4(k+j)-2}}{(4(k+j)-2)!} \underbrace{\left(1 - \frac{t^2}{\big(4(k+j)-1\big)\big(4(k+j)\big)}\right)}_{(\geqslant 0)} > 0.$$

Thus

$$\overline{T}_{n+4}(t) = \overline{T}_n(t) - \frac{t^{n+2}}{(n+2)!} \underbrace{\left(1 - \frac{t^2}{(n+3)(n+4)}\right)}_{(\geqslant 0)} \leqslant \overline{T}_n(t)$$

and

$$\overline{T}_{n+4}(t) = \sum_{i=0}^{2k+2} \frac{(-1)^i t^{2i}}{(2i)!} > \sum_{i=0}^{\infty} \frac{(-1)^i t^{2i}}{(2i)!} = \cos t$$

$$\iff \sum_{j=1}^{\infty} \frac{t^{4(k+j)+2}}{\big(4(k+j)+2\big)!} \underbrace{\left(1 - \frac{t^2}{\big(4(k+j)+3\big)\big(4(k+j)+4\big)}\right)}_{(>0)} > 0.$$

The equalities at the endpoints of the segment $\left[0, \sqrt{(n+3)(n+4)}\,\right]$ are true. Therefore, the inequalities in (6) hold true for $t \in \left[0, \sqrt{(n+3)(n+4)}\,\right]$. For $t \in \left[-\sqrt{(n+3)(n+4)}, 0\right)$ the inequalities in (6) are valid on the basis of the even property of the function $\cos x$. Overall, (6) has been proved.

(ii) Let $0 < t \leqslant \sqrt{(n+3)(n+4)}$. Then:

$$\underline{T}_n(t) = \sum_{i=0}^{2k+1} \frac{(-1)^i t^{2i}}{(2i)!} < \sum_{i=0}^{\infty} \frac{(-1)^i t^{2i}}{(2i)!} = \cos t$$

$$\iff \sum_{j=1}^{\infty} - \frac{t^{4(k+j)}}{\big(4(k+j)\big)!} \underbrace{\left(1 - \frac{t^2}{\big(4(k+j)+1\big)\big(4(k+j)+2\big)}\right)}_{(\geqslant 0)} < 0.$$

Thus

$$\underline{T}_{n+4}(t) = \underline{T}_n(t) + \frac{t^{n+2}}{(n+2)!} \underbrace{\left(1 - \frac{t^2}{(n+3)(n+4)}\right)}_{(\geqslant 0)} \geqslant \underline{T}_n(t)$$

and

$$\underline{T}_{n+4}(t) = \sum_{i=0}^{2k+3} \frac{(-1)^i t^{2i}}{(2i)!} < \sum_{i=0}^{\infty} \frac{(-1)^i t^{2i}}{(2i)!} = \cos t$$

$$\iff \sum_{j=1}^{\infty} \frac{-t^{4(k+j)+4}}{\big(4(k+j)+4\big)!} \underbrace{\left(1 - \frac{t^2}{\big(4(k+j)+5\big)\big(4(k+j)+6\big)}\right)}_{(>0)} < 0.$$

The equalities at the endpoints of the segment $\left[0, \sqrt{(n+3)(n+4)}\,\right]$ are also true. Therefore, the inequalities in (7) hold true for $t \in \left[0, \sqrt{(n+3)(n+4)}\,\right]$. For $t \in \left[-\sqrt{(n+3)(n+4)}, 0\right)$ the inequalities in (7) are valid on the basis of the even property of the function $\cos x$. Overall, (7) has been proved. □

Let us consider a complex number $z = e^{\mathrm{i}x}$ $\left(x \in \mathbb{R}, \mathrm{i} = \sqrt{-1} \text{ – imaginary unit}\right)$. Then it holds:

$$\cos x = \frac{1}{2}\left(z + \frac{1}{z}\right) \quad \text{and} \quad \sin x = \frac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right). \tag{8}$$

Let us introduce the following functions:

$$R_k(z) = z^k + \frac{1}{z^k} \quad \text{and} \quad Q_k(z) = z^k - \frac{1}{z^k}, \tag{9}$$

for $k = 1, 2, \ldots$. Then it is:

$$R_k(z) = 2\cos(kx) \quad \text{and} \quad Q_k(z) = 2\imath\sin(kx), \tag{10}$$

for $z = e^{\imath x}$ and $k = 1, 2, \ldots$. Hence, we may come to the conclusion that it holds:

$$R_n(z) \cdot R_m(z) = R_{n+m}(z) + R_{|n-m|}(z) \tag{11}$$

and

$$R_n(z) \cdot Q_m(z) = Q_{n+m}(z) + \nu \cdot Q_{|n-m|}(z), \tag{12}$$

where $\nu = \operatorname{sgn}\big(m-n\big)$. Specifically, $R_0(z) = 2$ and $Q_0(z) = 0$.

In the following auxiliary proposition we show that $\sin^n x$ can be presented as a sum of sines of multiple angels or sum of cosines of multiple angels depending on the parity of degree n.

LEMMA 1.3. *For $n \in \mathbb{N}$ the following formulas are valid:*
(i) *if n is odd, then:*

$$\sin^n x = \frac{2}{2^n} \sum_{k=0}^{\frac{n-1}{2}} (-1)^{\frac{n-1}{2}+k} \binom{n}{k} \sin\big((n-2k)x\big), \tag{13}$$

(ii) *if n is even, then:*

$$\sin^n x = \frac{1}{2^n} \binom{n}{\frac{n}{2}} + \frac{2}{2^n} \sum_{k=0}^{\frac{n}{2}-1} (-1)^{\frac{n}{2}+k} \binom{n}{k} \cos\big((n-2k)x\big). \tag{14}$$

Proof. See Ex. 17, 18, Chapter IX [11] and the method of proving from [22]. □

For the function $\cos^n x$ the following proposition analogously holds:

LEMMA 1.4. *For $n \in \mathbb{N}$ the following formulas are valid:*
(i) *if n is odd, then:*

$$\cos^n x = \frac{2}{2^n} \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \cos\big((n-2k)x\big), \tag{15}$$

(ii) *if n is even, then:*

$$\cos^n x = \frac{1}{2^n} \binom{n}{\frac{n}{2}} + \frac{2}{2^n} \sum_{k=0}^{\frac{n}{2}-1} \binom{n}{k} \cos\big((n-2k)x\big). \tag{16}$$

Proof. See Ex. 15, 16, Chapter IX [11] and the method of proving from [22]. □

Based on the previous two Lemmas we give a proof of the following statement:

THEOREM 1.5. *For $n,m \in \mathbb{N}$ we have the following cases:*
(i) *if both n and m are odd*

$$\cos^n x \cdot \sin^m x = \frac{1}{2^{n+m-1}} \sum_{k=0}^{\frac{n+m}{2}-1} (-1)^{\frac{m-1}{2}+k} \left(\sum_{r=0}^{k} (-1)^r \binom{n}{r} \binom{m}{k-r} \sin\left((n+m-2k)x\right) \right), \tag{17}$$

(ii) *if n is even and m is odd*

$$\cos^n x \cdot \sin^m x = \frac{1}{2^{n+m-1}} \sum_{k=0}^{\frac{n+m-1}{2}} (-1)^{\frac{m-1}{2}+k} \left(\sum_{r=0}^{k} (-1)^r \binom{n}{r} \binom{m}{k-r} \sin\left((n+m-2k)x\right) \right), \tag{18}$$

(iii) *if n is odd and m is even*

$$\cos^n x \cdot \sin^m x = \frac{1}{2^{n+m-1}} \sum_{k=0}^{\frac{n+m-1}{2}} (-1)^{\frac{m}{2}+k} \left(\sum_{r=0}^{k} (-1)^r \binom{n}{r} \binom{m}{k-r} \cos\left((n+m-2k)x\right) \right), \tag{19}$$

(iv) *if both n and m are even*

$$\cos^n x \cdot \sin^m x = \frac{1}{2^{n+m-1}} \left(\sum_{k=0}^{\frac{n+m}{2}-1} (-1)^{\frac{m}{2}+k} \sum_{r=0}^{k} (-1)^r \binom{n}{r} \binom{m}{k-r} \cos\left((n+m-2k)x\right) + \frac{1}{2}(-1)^{\frac{2m+n}{2}} \sum_{r=0}^{\frac{n+m}{2}} (-1)^r \binom{n}{r} \binom{m}{\frac{n+m}{2}-r} \right). \tag{20}$$

Proof. **(i)** Let us suppose that n and m are both odd, then:

$$\begin{aligned}
&\cos^n x \cdot \sin^m x \\
&= \left(\frac{2}{2^n} \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{i} \cos\left((n-2i)x\right) \right) \cdot \left(\frac{2}{2^m} \sum_{j=0}^{\frac{m-1}{2}} (-1)^{\frac{m-1}{2}+j} \binom{m}{j} \sin\left((m-2j)x\right) \right) \\
&= \frac{1}{2^{n+m-1}} \left(\sum_{i=0}^{\frac{n-1}{2}} \binom{n}{i} R_{n-2i}(z) \right) \cdot \left(\sum_{j=0}^{\frac{m-1}{2}} (-1)^{\frac{m-1}{2}+j} \binom{m}{j} Q_{m-2j}(z) \right) \\
&= \frac{1}{2^{n+m-1}} \sum_{i=0}^{\frac{n-1}{2}} \sum_{j=0}^{\frac{m-1}{2}} (-1)^{\frac{m-1}{2}+j} \binom{n}{i} \binom{m}{j} R_{n-2i}(z) \cdot Q_{m-2j}(z) \\
&= \frac{1}{2^{n+m-1}} \sum_{i=0}^{\frac{n-1}{2}} \sum_{j=0}^{\frac{m-1}{2}} (-1)^{\frac{m-1}{2}+j} \binom{n}{i} \binom{m}{j} \left(Q_{n+m-2(i+j)}(z) + \nu Q_{|n-m-2(i-j)|} \right) \\
&= \frac{1}{2^{n+m-1}} \left(\sum_{i=0}^{\frac{n-1}{2}} \sum_{j=0}^{\frac{m-1}{2}} (-1)^{\frac{m-1}{2}+j} \binom{n}{i} \binom{m}{j} Q_{n+m-2(i+j)}(z) \right. \\
&\qquad \left. + \sum_{i=0}^{\frac{n-1}{2}} \sum_{j=0}^{\frac{m-1}{2}} (-1)^{\frac{m-1}{2}+j} \binom{n}{i} \binom{m}{j} \nu Q_{|n-m-2(i-j)|}(z) \right),
\end{aligned}$$

where $\nu = \operatorname{sgn}\big(m-n-2(j-i)\big)$. Now we observe the binomial coefficients next to $Q_{|n-m-2(i-j)|}(z)$. The products of binomial coefficients can be written in the following way:

$$\binom{n}{i}\binom{m}{j} = \binom{n}{n-i}\binom{m}{j} = \binom{n}{i}\binom{m}{m-j} = \binom{n}{n-i}\binom{m}{m-j}. \tag{21}$$

We may notice that the sums of the lower numbers in the products of the binomial coefficients of the previous equalities in (21) are $i+j$, $n-i+j$, $i+m-j$ and $n-i+m-j$. Let us mark the index $|n-m-2(i-j)|$ with d. Our aim is to determine k in such a way that $n+m-2k=d$. Then we see two possibilities:

1) when $n-m-2(i-j)>0$, then

$$n+m-2k = n-m-2(i-j) \Longrightarrow k = m+i-j, \tag{22}$$

2) when $n-m-2(i-j)<0$, then

$$n+m-2k = -n+m-2(j-i) \Longrightarrow k = n-i+j. \tag{23}$$

Therefore, while calculating $\displaystyle\sum_{i=0}^{\frac{n-1}{2}}\sum_{j=0}^{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}+j}\binom{n}{i}\binom{m}{j}\nu Q_{|n-m-2(i-j)|}(z)$, on the basis of the implication (22), we chose the product of the binomial coefficients $\binom{n}{i}\binom{m}{m-j}$, and on the basis of the implication (23), we chose $\binom{n}{n-i}\binom{m}{j}$, i.e. we chose that product of binomial coefficients whose sum of the lower numbers equals to k.

Finally, we get the requested result:

$$\cos^n x\cdot\sin^m x = \frac{1}{2^{n+m_{\mathrm{i}}}}\sum_{k=0}^{\frac{n+m}{2}-1}(-1)^{\frac{m-1}{2}+k}\left(\sum_{r=0}^{k}(-1)^r\binom{n}{r}\binom{m}{k-r}Q_{n+m-2k}(z)\right)$$
$$= \frac{1}{2^{n+m-1}}\sum_{k=0}^{\frac{n+m}{2}-1}(-1)^{\frac{m-1}{2}+k}\left(\sum_{r=0}^{k}(-1)^r\binom{n}{r}\binom{m}{k-r}\sin\big((n+m-2k)x\big)\right).$$

(ii) Let n be even and m odd, then:

$$\cos^n x\cdot\sin^m x = \left(\frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{2}{2^n}\sum_{i=0}^{\frac{n}{2}-1}\binom{n}{i}\cos\big((n-2i)x\big)\right)$$
$$\times\left(\frac{2}{2^m}\sum_{j=0}^{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}+j}\binom{m}{j}\sin\big((m-2j)x\big)\right)$$
$$= \frac{1}{2^{n+m_{\mathrm{i}}}}\Bigg(\binom{n}{\frac{n}{2}}\sum_{j=0}^{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}+j}\binom{m}{j}Q_{m-2j}(z)$$
$$+\left(\sum_{i=0}^{\frac{n}{2}-1}\binom{n}{i}R_{n-2i}(z)\right)\cdot\left(\sum_{j=0}^{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}+j}\binom{m}{j}Q_{m-2j}(z)\right)\Bigg)$$

$$
\begin{aligned}
&= \frac{1}{2^{n+m_{\mathrm{i}}}}\Bigg(\binom{n}{\frac{n}{2}}\sum_{j=0}^{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}+j}\binom{m}{j}Q_{m-2j}(z) \\
&\quad + \sum_{i=0}^{\frac{n}{2}-1}\sum_{j=0}^{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}+j}\binom{n}{i}\binom{m}{j}R_{n-2i}(z)\cdot Q_{m-2j}(z)\Bigg) \\
&= \frac{1}{2^{n+m_{\mathrm{i}}}}\Bigg(\binom{n}{\frac{n}{2}}\sum_{j=0}^{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}+j}\binom{m}{j}Q_{m-2j}(z) \\
&\quad + \sum_{i=0}^{\frac{n}{2}-1}\sum_{j=0}^{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}+j}\binom{n}{i}\binom{m}{j}\Big(Q_{n+m-2(i+j)}(z)+\nu Q_{|n-m-2(i-j)|}(z)\Big)\Bigg) \\
&= \frac{1}{2^{n+m_{\mathrm{i}}}}\Bigg(\binom{n}{\frac{n}{2}}\sum_{j=0}^{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}+j}\binom{m}{j}Q_{m-2j}(z) \\
&\quad + \sum_{i=0}^{\frac{n}{2}-1}\sum_{j=0}^{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}+j}\binom{n}{i}\binom{m}{j}Q_{n+m-2(i+j)}(z) \\
&\quad + \sum_{i=0}^{\frac{n}{2}-1}\sum_{j=0}^{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}+j}\binom{n}{i}\binom{m}{j}\nu Q_{|n-m-2(i-j)|}(z)\Bigg),
\end{aligned}
$$

where $\nu = \operatorname{sgn}\big(m-n-2(j-i)\big)$. Looking at the products of the binomial coefficients next to $Q_{|n-m-2(i-j)|}(z)$, analogously to the equalities (21) and the procedure with the implications (22) and (23), we may conclude that it is valid:

$$
\begin{aligned}
\cos^n x\cdot\sin^m x &= \frac{1}{2^{n+m_{\mathrm{i}}}}\sum_{k=0}^{\frac{n+m-1}{2}}(-1)^{\frac{m-1}{2}+k}\left(\sum_{r=0}^{k}(-1)^r\binom{n}{r}\binom{m}{k-r}Q_{n+m-2k}(z)\right) \\
&= \frac{1}{2^{n+m-1}}\sum_{k=0}^{\frac{n+m-1}{2}}(-1)^{\frac{m-1}{2}+k}\left(\sum_{r=0}^{k}(-1)^r\binom{n}{r}\binom{m}{k-r}\sin\big((n+m-2k)x\big)\right),
\end{aligned}
$$

(iii) Replacing x by $\frac{\pi}{2}-x$ in formula (18), we get formula (19).

(iv) If n and m are both even, then:

$$
\begin{aligned}
&\cos^n x\cdot\sin^m x \\
&= \left(\frac{1}{2^n}\binom{n}{\frac{n}{2}}+\frac{2}{2^n}\sum_{i=0}^{\frac{n}{2}-1}\binom{n}{i}\cos\big((n-2i)x\big)\right) \\
&\quad\times\left(\frac{1}{2^m}\binom{m}{\frac{m}{2}}+\frac{2}{2^m}\sum_{j=0}^{\frac{m}{2}-1}(-1)^{\frac{m}{2}+j}\binom{m}{j}\cos\big((m-2j)x\big)\right) \\
&= \frac{1}{2^{n+m}}\Bigg(\binom{n}{\frac{n}{2}}\binom{m}{\frac{m}{2}}+\binom{n}{\frac{n}{2}}\sum_{j=0}^{\frac{m}{2}-1}(-1)^{\frac{m}{2}+j}\binom{m}{j}R_{m-2j}(z)
\end{aligned}
$$

$$+\binom{m}{\frac{m}{2}}\sum_{i=0}^{\frac{n}{2}-1}\binom{n}{i}R_{n-2i}(z)+\left(\sum_{i=0}^{\frac{n}{2}-1}\binom{n}{i}R_{n-2i}(z)\right)\cdot\left(\sum_{j=0}^{\frac{m}{2}-1}(-1)^{\frac{m}{2}+j}\binom{m}{j}R_{m-2j}(z)\right)\Bigg)$$

$$=\frac{1}{2^{n+m}}\Bigg(\binom{n}{\frac{n}{2}}\binom{m}{\frac{m}{2}}+\binom{n}{\frac{n}{2}}\sum_{j=0}^{\frac{m}{2}-1}(-1)^{\frac{m}{2}+j}\binom{m}{j}R_{m-2j}(z)+\binom{m}{\frac{m}{2}}\sum_{i=0}^{\frac{n}{2}-1}\binom{n}{i}R_{n-2i}(z)$$

$$+\sum_{i=0}^{\frac{n}{2}-1}\sum_{j=0}^{\frac{m}{2}-1}(-1)^{\frac{m}{2}+j}\binom{n}{i}\binom{m}{j}R_{n-2i}(z)\cdot R_{m-2j}(z)\Bigg)$$

$$=\frac{1}{2^{n+m}}\Bigg(\binom{n}{\frac{n}{2}}\binom{m}{\frac{m}{2}}+\binom{n}{\frac{n}{2}}\sum_{j=0}^{\frac{m}{2}-1}(-1)^{\frac{m}{2}+j}\binom{m}{j}R_{m-2j}(z)+\binom{m}{\frac{m}{2}}\sum_{i=0}^{\frac{n}{2}-1}\binom{n}{i}R_{n-2i}(z)$$

$$+\sum_{i=0}^{\frac{n}{2}-1}\sum_{j=0}^{\frac{m}{2}-1}(-1)^{\frac{m}{2}+j}\binom{n}{i}\binom{m}{j}\left(R_{n+m-2(i+j)}(z)+R_{|n-m-2(i-j)|}(z)\right)\Bigg)$$

$$=\frac{1}{2^{n+m}}\Bigg(\binom{n}{\frac{n}{2}}\binom{m}{\frac{m}{2}}+\binom{n}{\frac{n}{2}}\sum_{j=0}^{\frac{m}{2}-1}(-1)^{\frac{m}{2}+j}\binom{m}{j}R_{m-2j}(z)+\binom{m}{\frac{m}{2}}\sum_{i=0}^{\frac{n}{2}-1}\binom{n}{i}R_{n-2i}(z)$$

$$+\sum_{i=0}^{\frac{n}{2}-1}\sum_{j=0}^{\frac{m}{2}-1}(-1)^{\frac{m}{2}+j}\binom{n}{i}\binom{m}{j}R_{n+m-2(i+j)}(z)$$

$$+\sum_{i=0}^{\frac{n}{2}-1}\sum_{j=0}^{\frac{m}{2}-1}(-1)^{\frac{m}{2}+j}\binom{n}{i}\binom{m}{j}R_{|n-m-2(i-j)|}(z)\Bigg).$$

Looking at the products of the binomial coefficients next to $R_{|n-m-2(i-j)|}(z)$, analogously to the equalities (21) and the procedure with the implications (22) and (23), we may conclude that it is valid:

$$\cos^n x\cdot\sin^m x=\tfrac{1}{2^{n+m}}\Bigg(\sum_{k=0}^{\frac{n+m}{2}-1}(-1)^{\frac{m}{2}+k}\sum_{r=0}^{k}(-1)^r\binom{n}{r}\binom{m}{k-r}R_{n+m-2k}(z)$$

$$+\tfrac{1}{2}(-1)^{\frac{2m+n}{2}}\sum_{r=0}^{\frac{n+m}{2}}(-1)^r\binom{n}{r}\binom{m}{\frac{n+m}{2}-r}R_0\Bigg)$$

$$=\tfrac{1}{2^{n+m-1}}\Bigg(\sum_{k=0}^{\frac{n+m}{2}-1}(-1)^{\frac{m}{2}+k}\sum_{r=0}^{k}(-1)^r\binom{n}{r}\binom{m}{k-r}\cos\left((n+m-2k)x\right)$$

$$+\tfrac{1}{2}(-1)^{\frac{2m+n}{2}}\sum_{r=0}^{\frac{n+m}{2}}(-1)^r\binom{n}{r}\binom{m}{\frac{n+m}{2}-r}\Bigg),$$

with the note that $\binom{n}{i}\binom{m}{j}\cdot R_0$ $(R_0=2)$ is written as a sum of two products of binomial coefficients equal to $\binom{n}{i}\binom{m}{j}$, analogously to (21), whose sum of the lower numbers equals to $\frac{n+m}{2}$. $\square$

2. The description of the method

I Our aim is to present a method for proving inequalities of the type (1) for $x \in (0,\delta)$ and $\delta = \delta_2 > 0$. We will use the upward and downward Maclaurin approximations of the sine and cosine functions determined in the Lemmas 1.1. and 1.2.

Let us observe the addend of the sum (1): $s_i(x) = \alpha_i x^{p_i} \cos^{q_i} x \sin^{r_i} x$, where $\alpha_i \neq 0$ for $i = 1,\dots,n$. Let us introduce the symbol

$$\mathfrak{m}_i = \begin{cases} \frac{q_i+r_i}{2} - 1, & \text{when } q_i \text{ and } r_i \text{ are both even or both odd,} \\ \frac{q_i+r_i-1}{2}, & \text{when } q_i \text{ and } r_i \text{ have different parity.} \end{cases} \tag{24}$$

According to the Theorem 1.5. the addends $s_i(x)$ $(i = 1,2,\dots,n)$ are represented in four different ways depending on the cases, so the following possibilities are given in the description of the method:

1. Let q_i and r_i be odd or let q_i be even and r_i odd. In both cases, it holds:

$$\begin{aligned} s_i(x) &= \alpha_i x^{p_i} \cos^{q_i} x \sin^{r_i} x \\ &= \frac{\alpha_i x^{p_i}}{2^{q_i+r_i-1}} \sum_{k=0}^{\mathfrak{m}_i} (-1)^{\frac{r_i-1}{2}+k} \sum_{r=0}^{k} (-1)^r \binom{q_i}{r}\binom{r_i}{k-r} \sin\big((q_i+r_i-2k)x\big) \\ &= \frac{x^{p_i}}{2^{q_i+r_i-1}} \sum_{k=0}^{\mathfrak{m}_i} \left(\sum_{r=0}^{k} \alpha_i (-1)^{\frac{r_i-1}{2}+k+r} \binom{q_i}{r}\binom{r_i}{k-r} \right) \sin\big((q_i+r_i-2k)x\big). \end{aligned} \tag{25}$$

Let us mark with $\beta_k = \sum_{r=0}^{k} \alpha_i (-1)^{\frac{r_i-1}{2}+k+r} \binom{q_i}{r}\binom{r_i}{k-r}$. Then, for every sub-addend $\beta_k \sin\big((q_i+r_i-2k)x\big)$, depending on the sign of β_k, two cases are possible:

1) if $\beta_k > 0$:

$$\beta_k \sin\big((q_i+r_i-2k)x\big) > \beta_k \underline{T}^{\sin,0}_{4l_k^{(i)}+3}\big((q_i+r_i-2k)x\big), \tag{26}$$

2) if $\beta_k < 0$:

$$\beta_k \sin\big((q_i+r_i-2k)x\big) > \beta_k \overline{T}^{\sin,0}_{4l_k^{(i)}+1}\big((q_i+r_i-2k)x\big). \tag{27}$$

Let the addend $s_i(x)$ be written in the form:

$$s_i(x) = \frac{x^{p_i}}{2^{q_i+r_i-1}} \sum_{k=0}^{\mathfrak{m}_i} \beta_k \sin\big((q_i+r_i-2k)x\big). \tag{28}$$

Then it holds:

$$s_i(x) > \tau_i(x) = \frac{x^{p_i}}{2^{q_i+r_i-1}} \sum_{k=0}^{\mathfrak{m}_i} \beta_k T^{\sin,0}_{4l_k^{(i)}+u}\big((q_i+r_i-2k)x\big), \tag{29}$$

where $u = \begin{cases} 3, \ \beta_k > 0, \\ 1, \ \beta_k < 0 \end{cases}$, $l_k^{(i)} \in \mathbb{N}_0$ and $T \in \{\overline{T}, \underline{T}\}$.

2. Let q_i be odd and r_i even, then it holds:

$$\begin{aligned} s_i(x) &= \alpha_i x^{p_i} \cos^{q_i} x \sin^{r_i} x \\ &= \frac{\alpha_i x^{p_i}}{2^{q_i+r_i-1}} \sum_{k=0}^{\mathfrak{m}_i} (-1)^{\frac{r_i}{2}+k} \sum_{r=0}^{k} (-1)^r \binom{q_i}{r} \binom{r_i}{k-r} \cos\big((q_i+r_i-2k)x\big) \\ &= \frac{x^{p_i}}{2^{q_i+r_i-1}} \sum_{k=0}^{\mathfrak{m}_i} \left(\sum_{r=0}^{k} \alpha_i (-1)^{\frac{r_i}{2}+k+r} \binom{q_i}{r} \binom{r_i}{k-r} \right) \cos\big((q_i+r_i-2k)x\big). \end{aligned} \tag{30}$$

Let us mark with $\gamma_k = \sum_{r=0}^{k} \alpha_i (-1)^{\frac{r_i}{2}+k+r} \binom{q_i}{r} \binom{r_i}{k-r}$. Then, for every sub-addend $\gamma_k \cos\big((q_i+r_i-2k)x\big)$, depending on the sign of γ_k, two cases are possible:

1) if $\gamma_k > 0$:

$$\gamma_k \cos\big((q_i+r_i-2k)x\big) > \gamma_k \underline{T}^{\cos,0}_{4l_k^{(i)}+2}\big((q_i+r_i-2k)x\big), \tag{31}$$

2) if $\gamma_k < 0$:

$$\gamma_k \cos\big((q_i+r_i-2k)x\big) > \gamma_k \overline{T}^{\cos,0}_{4l_k^{(i)}+0}\big((q_i+r_i-2k)x\big). \tag{32}$$

Let the addend $s_i(x)$ be written in the form:

$$s_i(x) = \frac{x^{p_i}}{2^{q_i+r_i-1}} \sum_{k=0}^{\mathfrak{m}_i} \gamma_k \cos\big((q_i+r_i-2k)x\big). \tag{33}$$

Then it holds:

$$s_i(x) > \tau_i(x) = \frac{x^{p_i}}{2^{q_i+r_i-1}} \sum_{k=0}^{\mathfrak{m}_i} \gamma_k T^{\cos,0}_{4l_k^{(i)}+v}\big((q_i+r_i-2k)x\big), \tag{34}$$

where $v = \begin{cases} 2, & \gamma_k > 0, \\ 0, & \gamma_k < 0 \end{cases}$, $l_k^{(i)} \in \mathbb{N}_0$ and $T \in \{\overline{T}, \underline{T}\}$.

3. Let q_i and r_i be even, then based on the previous case (under **2.**) it holds:

$$\begin{aligned} s_i(x) &= \frac{x^{p_i}}{2^{q_i+r_i-1}} \left(\sum_{k=0}^{\mathfrak{m}_i} \left(\sum_{r=0}^{k} \alpha_i (-1)^{\frac{r_i}{2}+k+r} \binom{q_i}{r} \binom{r_i}{k-r} \right) \cos\big((q_i+r_i-2k)x\big) \right. \\ &\qquad \left. + \frac{1}{2}(-1)^{\frac{2r_i+q_i}{2}} \sum_{r=0}^{\frac{q_i+r_i}{2}} (-1)^r \binom{q_i}{r} \binom{r_i}{\frac{q_i+r_i}{2}-r} \right) \\ &> \tau_i(x) = \frac{x^{p_i}}{2^{q_i+r_i-1}} \left(\sum_{k=0}^{\mathfrak{m}_i} \gamma_k T^{\cos,0}_{4l_k^{(i)}+v}\big((q_i+r_i-2k)x\big) \right. \\ &\qquad \left. + \frac{1}{2}(-1)^{\frac{2r_i+q_i}{2}} \sum_{r=0}^{\frac{q_i+r_i}{2}} (-1)^r \binom{q_i}{r} \binom{r_i}{\frac{q_i+r_i}{2}-r} \right), \end{aligned} \tag{35}$$

where $v = \begin{cases} 2, \ \gamma_k > 0, \\ 0, \ \gamma_k < 0 \end{cases}$, $l_k^{(i)} \in \mathbb{N}_0$ and $T \in \{\overline{T}, \underline{T}\}$.

Comparing all the addends $s_i(x)$ $(i = 1, 2, \ldots, n)$ that appear in the sum (1), according to the above stated cases, we get the polynomial

$$P(x) = \sum_{i=1}^{n} \tau_i(x) \tag{36}$$

(downward approximation of the function $f(x)$ in (1)); i.e. it holds:

$$f(x) > P(x). \tag{37}$$

On the basis of the previous consideration, the following statement ensues:

THEOREM 2.1. *Let the following properties of the polynomial* $P(x) = \sum_{i=1}^{n} \tau_i(x)$ *be true:*

(i) there is at least one positive real root of the polynomial $P(x)$*;*
(ii) $P(x) > 0$ *for* $x \in (0, x^*)$*, where* x^* *is the least positive real root of the polynomial* $P(x)$*;*

then it is valid

$$f(x) > 0$$

for $x \in (0, x^*) \subseteq (0, \delta)$.

REMARK 2.2. Let us notice that hereby the proof of the inequality $f(x) > 0$ has been obtained for $x \in (0, \delta_2)$, where $\delta_2 = x^*$. The previous Theorem can be applied in the interval $(\delta_1, 0)$ by introducing the substitute $t = -x$.

REMARK 2.3. If there is not at least one positive real root of the polynomial $P(x)$ and $P(x) > 0$ for $x \in (0, \infty)$, then it is valid $f(x) > 0$ for $x \in (0, \infty)$.

The previous Theorem determines a method of proving a class of trigonometric inequalities based on approximations of the sine and cosine functions by Maclaurin polynomials.

II We will consider completeness of the given method for the function $f(x)$, of the mixed trigonometric polynomial, which is not a classical polynomial. Let us start from the following auxiliary statement.

LEMMA 2.4. *Let* $f : (\delta_1, \delta_2) \longrightarrow \mathbb{R}$, $\delta_1 \leqslant 0 \leqslant \delta_2$ *and* $\delta_1 < \delta_2$*, be a real, non-constant, analytic function such that domain* (δ_1, δ_2) *belongs to the interval of convergence of the function* $f(x)$*.*

If $f(0) \neq 0$*, then it holds:*

1.

$$f(0) > 0 \Longleftrightarrow \big(\exists x^+ \in (0, \delta_2]\big)\big(\forall x \in (0, x^+)\big) f(x) > 0, \tag{38}$$

2.
$$f(0) < 0 \Longleftrightarrow \big(\exists x^+ \in (0,\delta_2]\big)\big(\forall x \in (0,x^+)\big) f(x) < 0. \tag{39}$$

If $f(0) = \ldots = f^{(n-1)}(0) = 0 \,\wedge\, f^{(n)}(0) \neq 0$*, for some* $n \in \mathbb{N}$*, then it holds:*

3.
$$f^{(n)}(0) > 0 \Longleftrightarrow \big(\exists x^+ \in (0,\delta_2]\big)\big(\forall x \in (0,x^+)\big) f(x) > 0, \tag{40}$$

4.
$$f^{(n)}(0) < 0 \Longleftrightarrow \big(\exists x^+ \in (0,\delta_2]\big)\big(\forall x \in (0,x^+)\big) f(x) < 0. \tag{41}$$

Proof. Let $f(x)$ be a non-constant function with Maclaurin series expansion

$$f(h) = f(0) + \frac{f'(0)}{1!}h + \frac{f''(0)}{2!}h^2 + \ldots, \qquad (h > 0). \tag{42}$$

In the proof we use the method from [12] (pages 157, 158), by which it has been shown that zeros of non-constant analytic function are isolated.

1. $(\Rightarrow)$ Let $f(0) > 0$. Let us note (42) in the form

$$f(h) = f(0)\big(1 + g(h)\big), \tag{43}$$

where $g(h)$ is the real analytical function. Then there exist $x^+ > 0$ and $M > 0$ such that $|g(h)| < Mh$ and $Mh < 1/2$ for every $h \in (0,x^+)$. Hence, we conclude that $f(h) = f(0) + f(0)g(h) > f(0) - f(0)Mh > f(0)/2 > 0$ for $h \in (0,x^+)$. $(\Leftarrow)$ Let us suppose that there exists $x^+ \in (0,\delta_2]$ such that for every $x \in (0,x^+)$ it holds $f(x) > 0$. Consequently, it ensues that $f(x)$ is a positive function in arbitrarily small right-hand neighbourhood of the point $x = 0$. Let $g(x)$ be the function considered in the previous part of the proof. Then there exist $M > 0$ and $x_1 \in (0,x^+]$ such that for every $x \in (0,x_1)$ it holds $|g(x)| < Mx < 1/2$. If it holds $f(0) < 0$, then for $x \in (0,x_1) \subseteq (0,x^+)$ we have the contradiction $f(x) = f(0)\big(1+g(x)\big) < 0$. Hereby it has been proved $f(0) > 0$.

2. It is sufficient to consider function $-f(x)$ instead of function $f(x)$ and to apply **1.**

In the case $f(0) = \ldots = f^{(n-1)}(0) = 0 \,\wedge\, f^{(n)}(0) \neq 0$, for some $n \in \mathbb{N}$, the point $x = 0$ has been isolated zero of order n. Let us note further (42) in the form

$$f(h) = \frac{f^{(n)}(0)}{n!}h^n + \frac{f^{(n+1)}(0)}{(n+1)!}h^{n+1} + \ldots, \qquad (h > 0). \tag{44}$$

3. $(\Rightarrow)$ Let $f^{(n)}(0) > 0$. Let us note (44) in the form

$$f(h) = \frac{f^{(n)}(0)}{n!}h^n\big(1 + g(h)\big), \tag{45}$$

where $g(h)$ is the real analytical function. Then there exist $x^+ > 0$ and $M > 0$ such that $|g(h)| < Mh$ and $Mh < 1/2$ for every $h \in (0,x^+)$. Hence, we conclude that $f(h) = \frac{f^{(n)}(0)}{n!}h^n + \frac{f^{(n)}(0)}{n!}h^n g(h) > \frac{f^{(n)}(0)}{n!}h^n - \frac{f^{(n)}(0)}{n!}h^n Mh > \frac{1}{2}\frac{f^{(n)}(0)}{n!}h^n > 0$ for $h \in (0,x^+)$. $(\Leftarrow)$ Let us suppose that there exists $x^+ \in (0,\delta_2]$ such that for every $x \in (0,x^+)$ it holds $f(x) > 0$. Consequently, it ensues that $f(x)$ is a positive function in arbitrarily small right-hand neighbourhood of the point $x = 0$. Let $g(x)$ be the function considered in the previous part of the proof. Then there exist $M > 0$ and $x_1 \in (0, x^+]$ such that for every $x \in (0, x_1)$ it holds $|g(x)| < Mx < 1/2$. If it holds $f^{(n)}(0) < 0$, then for $x \in (0, x_1) \subseteq (0, x^+)$ we have the contradiction $f(x) = \frac{f^{(n)}(0)}{n!} x^n \big(1 + g(x)\big) < 0$. Hereby it has been proved $f^{(n)}(0) > 0$.

4. It is sufficient to consider function $-f(x)$ instead of function $f(x)$ and to apply **3.** □

Based on the previous statement it follows:

THEOREM 2.5. *Let $f : (\delta_1, \delta_2) \longrightarrow \mathbb{R}$, $\delta_1 \leqslant 0 \leqslant \delta_2$ and $\delta_1 < \delta_2$, be a real, non-constant, analytic function such that domain (δ_1, δ_2) belongs to the interval of convergence of the function $f(x)$. Then the equivalences*

$$\begin{gathered} \big(\exists x^+ \in (0, \delta_2]\big) \big(\forall x \in (0, x^+)\big) f(x) > 0 \\ \iff \\ f(0) > 0 \;\vee\; \Big(\big(\exists n \in \mathbb{N}\big) f(0) = \ldots = f^{(n-1)}(0) = 0 \wedge f^{(n)}(0) > 0\Big) \end{gathered} \tag{46}$$

or

$$\begin{gathered} \big(\exists x^+ \in (0, \delta_2]\big) \big(\forall x \in (0, x^+)\big) f(x) < 0 \\ \iff \\ f(0) < 0 \;\vee\; \Big(\big(\exists n \in \mathbb{N}\big) f(0) = \ldots = f^{(n-1)}(0) = 0 \wedge f^{(n)}(0) < 0\Big) \end{gathered} \tag{47}$$

are true.

REMARK 2.6. In the following consideration we observe that $f(x)$ is a mixed trigonometric polynomial which is not a classical polynomial. Such functions are analytic with the interval of convergence which is determined as a set of real numbers. That is why the problem whether there is an interval $(0, x^+)$ for the mixed trigonometric polynomial $f(x)$, for some $x^+ > 0$, in which $f(x)$ is of the constant sign, represents a decidable problem based on the equivalences (46) and (47).

We will consider completeness of the given method for the function $f(x)$, of the mixed trigonometric polynomial, which is not a classical polynomial under the assumption

$$\big(\exists x^+ \in (0, \delta)\big) \big(\forall x \in (0, x^+)\big) f(x) > 0. \tag{48}$$

We will show that for the function $f(x)$ in every sub-interval $(a, b) \subset (0, x^+)$, where $0 < a < b < x^+$, there exists a positive downward polynomial approximation $P(x)$. Let all the indexes $l_k^{(i)}$ $\big(i \in \{1, \ldots, n\}$ and $k \in \{0, \ldots, m_i\}\big)$ have the same value $l_k^{(i)} = K \in$

N_0. As a function of index K, a polynomial $P^{[K]}(x) = \sum_{i=1}^{n} \tau_i^{[K]}(x)$ is formed. Previously formed polynomial $P^{[K]}(x)$ of index K is downward polynomial approximation of the function $f(x)$ such that it is valid

$$\lim_{K \longrightarrow \infty} P^{[K]}(x) = f(x), \tag{49}$$

where the previous convergence is uniform in $[0, x^+]$. The uniform property of the convergence follows based on the fact that for every $i = 1, \ldots, n$ the convergence

$$\lim_{K \longrightarrow \infty} \tau_i^{[K]}(x) = s_i(x), \tag{50}$$

is uniform in $[0, x^+]$. Based on $P^{[K]}(x) < f(x)$ and $P^{[K]}(x) \rightrightarrows f(x)$, in $[0, x^+]$, we get the following statement about the completeness of the discussed method.

THEOREM 2.7. *Let for the mixed trigonometric polynomial $f(x)$ which is not a classical polynomial, the condition (48) is valid. Then in every interval $(a,b) \subset (0, x^+)$, where $0<a<b<x^+$, there exists downward polynomial approximation $P^{[K]}(x)$ of the index K such that*

$$\big(\forall x \in (a,b)\big)\, f(x) > P^{[K]}(x) > 0. \tag{51}$$

REMARK 2.8. Under the assumptions of the previous Theorem for the function $f(x)$ it follows the completeness of the method in the sense that it is possible in every interval $(a,b) \subset (0, x^+)$, where $0 < a < b < x^+$, to prove the inequality $f(x) > 0$ by using some downward approximation $P^{[K]}(x)$.

2.1. Improving of the method

Let us emphasise that the previous method can be applied to the functions of the form $f(x) = \sum_{i=1}^{n} \alpha_i h_i(x) \cos^{q_i} x \sin^{r_i} x$ for $x \in (0, \delta)$, where $h_i(x)$ is a polynomial, in such a way that two possibilities exist. The first possibility is when the polynomial $h_i(x)$ is of the constant sign in the given interval and then we can see the cases $h_i(x) > 0$ or $h_i(x) < 0$, and we do that by analogy with the previously described procedure. On the other hand, we have a possibility that the polynomial $h_i(x)$ is not of the constant sign. Then, $\alpha_i h_i(x) \cos^{q_i} x \sin^{r_i} x$ can be written as a sum of addends of the form $s_i(x)$, and then we can apply the previously described method for each of those addends.

2.2. The end of the procedure

Let the indexes $l_k^{(i)}$ $\big(i \in \{1, \ldots, n\}$ and $k \in \{0, \ldots, \mathrm{m}_i\}\big)$, which appear in the polynomial $P(x)$, be aligned: $l_0, l_1, \ldots, l_m$; where $m+1$ is the overall number of sub-addends which come from every addend $s_i(x)$. The indexes $l_0, l_1, \ldots l_m$ have been determined in (29), (34) and (35). Let us notice that according to the index l_s it holds:

$$f(x) > P(x, l_0, l_1, \ldots, l_s + 1, \ldots, l_m) > P(x, l_0, l_1, \ldots, l_s, \ldots, l_m) \tag{52}$$

for every index $s \in \{0,1,2,\ldots,m\}$ and $l_s \in \mathbb{N}_0$. It should be noted that the interval in which the sharp inequality

$$P(x,l_0,l_1,\ldots,l_s+1,\ldots,l_m) > P(x,l_0,l_1,\ldots,l_s,\ldots,l_m) \tag{53}$$

is valid, can be determined according to the Lemmas 1.1. and 1.2. By increasing every index l_s, the intervals of validity of (52) are expanded based on the Lemmas 1.1 and 1.2. and we get even better and better downward approximations of the function $f(x)$. The previously described method defines a procedure which ends when at least one $(m+1)$-tuple of the indexes $(l_0,l_1,\ldots,l_m) = (\hat{l}_0,\hat{l}_1,\ldots,\hat{l}_m)$ has been determined for which it is valid:

$$P(x,\hat{l}_0,\hat{l}_1,\ldots,\hat{l}_m) > 0 \tag{54}$$

for $x \in (0,\delta)$. By completing the procedure, we get a proof of the initial inequality (1).

REMARK 2.9. This method represents a generalisation of the method that C. Mortici used for proving inequalities in the article [26]. The method comes down to proving polynomial inequalities of the form $P(x) > 0$ for $x \in (0,\delta)$ which is a decidable problem according to the results by Tarski [20].

By using this method it is our aim in this article to get some well-known results concerning the inequalities of the form (1) that have been considered in the lately published articles.

3. Some applications

In this section we consider some applications of the method based on the Theorem 2.1. in some concrete inequalities.

3.1. A proof of an inequality from the article [6]

In the article [6] C.-P. Chen and W.-S. Cheung have lately proved the following statement (Theorem 2):

THEOREM 3.1. **(i)** *For $0 < x < \pi/2$, we have*

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} < 2 + \frac{2}{45}x^3 \tan x. \tag{55}$$

The constant $\frac{2}{45}$ is best possible.
(ii) *For $0 < x < \pi/2$, we have*

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} < 2 + \frac{2}{45}x^4 + \frac{8}{945}x^5 \tan x. \tag{56}$$

The constant $\frac{8}{945}$ is best possible.

Now we present a proof of the inequality (55).

Proof. The requested inequality is equivalent to $f(x) > 0$ for $x \in (0, \pi/2)$, where

$$f(x) = 2\cos x \sin^2 x + \tfrac{2}{45} x^3 \sin^3 x - x\cos^2 x \sin x - x^2 \cos x, \tag{57}$$

which is a concrete mixed trigonometric polynomial. Let us notice that $x = 0$ is zero of the eighth order of the function $f(x)$. According to the Theorem 1.5. the function $f(x)$ can be written in the following way:

$$f(x) = \tfrac{1}{2}\cos x - x^2\cos x - \tfrac{1}{2}\cos 3x - \underbrace{\left(\tfrac{1}{90}x^3 + \tfrac{1}{4}x\right)}_{(>0)}\sin 3x + \underbrace{\left(\tfrac{1}{30}x^3 - \tfrac{1}{4}x\right)}_{(<0)}\sin x. \tag{58}$$

Then, according to the Lemmas 1.1. and 1.2. and the description of the method, the following inequalities are true: $\cos y > \underline{T}_k^{\cos,0}(y)$ $(k = 6)$, $\cos y < \overline{T}_k^{\cos,0}(y)$ $(k = 12)$ and $\sin y < \overline{T}_k^{\sin,0}(y)$ $(k = 13)$, for $y \in \left(0, \sqrt{(k+3)(k+4)}\right)$.

For $x \in (0, \pi/2)$ it is valid:

$$\begin{aligned} f(x) > \tfrac{1}{2}\underline{T}_6^{\cos,0}(x) - x^2\overline{T}_{12}^{\cos,0}(x) - \tfrac{1}{2}\overline{T}_{12}^{\cos,0}(3x) - \underbrace{\left(\tfrac{1}{90}x^3 + \tfrac{1}{4}x\right)}_{(>0)}\overline{T}_{13}^{\sin,0}(3x) \\ + \underbrace{\left(\tfrac{1}{30}x^3 - \tfrac{1}{4}x\right)}_{(<0)}\overline{T}_{13}^{\sin,0}(x) = P_{16}(x), \end{aligned} \tag{59}$$

where $P_{16}(x)$ is the polynomial

$$\begin{aligned} P_{16}(x) &= \frac{x^8}{186810624000}\big(-531440x^8 - 2746332x^6 - 8885955x^4 \\ &\quad -118584180x^2 + 1183782600\big) \\ &= \frac{x^8}{186810624000}P_8(x). \end{aligned} \tag{60}$$

Then we determine the sign of the polynomial $P_8(x)$ for $x \in (0, \pi/2)$. By introducing the substitute $z = x^2$, we get the fourth degree polynomial:

$$P_4(z) = -531440z^4 - 2746332z^3 - 8885955z^2 - 118584180z + 1183782600\,. \tag{61}$$

A real numerical factorization of the polynomial $P_4(z)$, has been determined via Matlab software, and given with

$$P_4(z) = \alpha(z - z_1)(z - z_2)(z^2 + pz + q), \tag{62}$$

where $\alpha = -531440$, $z_1 = 4.503628...$, $z_2 = -9.049...$, $p = 0.621...$, $q = 54.652...$; whereby the inequality $p^2 - 4q < 0$ is true. The polynomial $P_4(z)$ has exactly two simple real roots with a symbolic radical representation and the corresponding numerical values z_1 and z_2. Since $P_4(0) > 0$ it follows that $P_4(z) > 0$ for $z \in (0, z_1) \subset (z_2, z_1)$. Finally, we conclude that

$$P_8(x) > 0 \text{ for } x \in (0, \sqrt{z_1}) = (0, 2.122...) \Longrightarrow P_{16}(x) > 0 \text{ for } x \in (0, 2.122...) \\ \Longrightarrow f(x) > 0 \text{ for } x \in \left(0, \frac{\pi}{2}\right) \subset (0, 2.122...). \tag{63}$$

Let us notice that the least positive real root of the downward approximation of the function $f(x)$, i.e. of the polynomial $P_{16}(x)$, is $x^* = \sqrt{z_1} = 2.122175... > \pi/2$. Elementary calculus gives that the constant $\frac{2}{45}$ is the best possible. □

3.2. A proof of an inequality from the paper [35]

In the paper [34] Z.-J. Sun and L. Zhu have posed an open problem, to prove the statement:

THEOREM 3.2. *Let* $0 < x < \pi/2$. *Then*

$$\frac{(2\pi^4/3)x^3 + (8\pi^4/15 - 16\pi^2/3)x^5}{(\pi^2 - 4x^2)^2} < x\sec^2 x - \tan x \\ < \frac{(2\pi^4/3)x^3 + (256/\pi^2 - 8\pi^2/3)x^5}{(\pi^2 - 4x^2)^2}, \tag{64}$$

hold, where $(8\pi^4/15 - 16\pi^2/3)$ *and* $(256/\pi^2 - 8\pi^2/3)$ *are the best constants in* (64).

Now we present a proof of the previous statement.

Proof.

I We prove the inequality:

$$\frac{(2\pi^4/3)x^3 + (8\pi^4/15 - 16\pi^2/3)x^5}{(\pi^2 - 4x^2)^2} < x\sec^2 x - \tan x \tag{65}$$

for $x \in (0, \pi/2)$. The requested inequality is equivalent to the inequality $f(x) > 0$ for $x \in (0, \pi/2)$, where

$$f(x) = x(\pi^2 - 4x^2)^2 - (\pi^2 - 4x^2)^2 \cos x \sin x - \left((2\pi^4/3)x^3 + (8\pi^4/15 - 16\pi^2/3)x^5\right)\cos^2 x \tag{66}$$

is a concrete mixed trigonometric polynomial. Let us notice that $x = 0$ is zero of the seventh order and $x = \pi/2$ is zero of the second order of the function $f(x)$. Let us consider two cases:

1) If $x \in (0, 1.136)$:

According to the Theorem 1.5. the function $f(x)$ can be written in the following way:

$$f(x) = x(\pi^2 - 4x^2)^2 - \frac{(\pi^2 - 4x^2)^2}{2}\sin 2x - \left((\pi^4/3)x^3 + (4\pi^4/15 - 8\pi^2/3)x^5\right) \\ - \Big(\underbrace{(\pi^4/3)x^3 + (4\pi^4/15 - 8\pi^2/3)x^5}_{(>0)}\Big)\cos 2x. \tag{67}$$

Then, according to the Lemmas 1.1. and 1.2. and the description of the method, the following inequalities are true: $\sin y < \overline{T}_k^{\sin,0}(y)$ $(k=9)$ and $\cos y < \overline{T}_k^{\cos,0}(y)$ $(k=8)$, for $y \in \big(0, \sqrt{(k+3)(k+4)}\big)$.

For $x \in (0, 1.136)$ it is valid:

$$\begin{aligned} f(x) > x(\pi^2-4x^2)^2 - \frac{(\pi^2-4x^2)^2}{2}\overline{T}_9^{\sin,0}(2x) - \big((\pi^4/3)x^3 + (4\pi^4/15 - 8\pi^2/3)x^5\big) \\ -\big(\underbrace{(\pi^4/3)x^3 + (4\pi^4/15-8\pi^2/3)x^5}_{(>0)}\big)\overline{T}_8^{\cos,0}(2x) = P_{13}(x), \end{aligned} \tag{68}$$

where $P_{13}(x)$ is the polynomial

$$\begin{aligned} P_{13}(x) &= \frac{2x^7}{14175}\big((-12\pi^4+120\pi^2-80)x^6 - (-153\pi^4+1640\pi^2-1440)x^4 \\ &\quad -(1055\pi^4-11880\pi^2+15120)x^2 + 2295\pi^4 - 30240\pi^2 + 75600\big) \\ &= \frac{2x^7}{14175}P_6(x). \end{aligned} \tag{69}$$

Then we determine the sign of the polynomial $P_6(x)$ for $x \in (0, 1.136)$. By introducing the substitute $z = x^2$, we get the third degree polynomial:

$$\begin{aligned} P_3(z) &= (-12\pi^4+120\pi^2-80)z^3 - (-153\pi^4+1640\pi^2-1440)z^2 \\ &\quad -(1055\pi^4-11880\pi^2+15120)z + 2295\pi^4 - 30240\pi^2 + 75600. \end{aligned} \tag{70}$$

A real numerical factorization of the polynomial $P_3(z)$, has been determined via Matlab software, and given with

$$P_3(z) = \alpha(z-z_1)(z^2+pz+q), \tag{71}$$

where $\alpha = -64.556...$, $z_1 = 1.290721...$, $p = -1.148...$, $q = 8.365...$; whereby the inequality $p^2 - 4q < 0$ is true. The polynomial $P_3(z)$ has exactly one simple real root with a symbolic radical representation and the corresponding numerical value z_1. Let us notice that $\sqrt{z_1} = 1.136099... > 1.136$. Since $P_3(0) > 0$ it follows that $P_3(z) > 0$ for $z \in (0, 1.136)$. Finally, we conclude that

$$\begin{aligned} P_6(x) > 0 \text{ for } x \in (0, 1.136) &\Longrightarrow P_{13}(x) > 0 \text{ for } x \in (0, 1.136) \\ &\Longrightarrow f(x) > 0 \text{ for } x \in (0, 1.136). \end{aligned} \tag{72}$$

Let us notice that the least positive real root of the downward approximation of the function $f(x)$, i.e. of the polynomial $P_{13}(x)$, is $x^* = \sqrt{z_1} = 1.136099...$.

2) If $x \in [1.136, \pi/2)$, let us define the function

$$\begin{aligned} \varphi(x) = f\big(\pi/2 - x\big) &= -16x^5 + 40\pi x^4 - 32\pi^2x^3 + 8\pi^3x^2 \\ &\quad -(16x^4 - 32\pi x^3 + 16\pi^2x^2)\sin x\cos x \\ &\quad -(\pi^2/60)(\pi-2x)^3\big((4\pi^2-40)x^2 + (-4\pi^3+40\pi)x + \pi^4 - 5\pi^2\big)\sin^2 x. \end{aligned} \tag{73}$$

Now we prove that $f(x) > 0$ for $x \in [1.136, \pi/2)$, which is equivalent to $\varphi(x) > 0$ for $x \in (0, c]$, where $c = \pi/2 - 1.136 = \pi/2 - 142/125$ ($c = 0.434\ldots$). The function $\varphi(x)$ is also a concrete mixed trigonometric polynomial. According to the Theorem 1.5. the function $\varphi(x)$ can be written in the following way:

$$\begin{aligned}\varphi(x) = & -16x^5 + 40\pi x^4 - 32\pi^2 x^3 + 8\pi^3 x^2 - \big(\underbrace{8x^4 - 16\pi x^3 + 8\pi^2 x^2}_{(>0)}\big)\sin 2x \\ & -(\pi^2/60)(\pi - 2x)^3\big((2\pi^2 - 20)x^2 + (-2\pi^3 + 20\pi)x + \pi^4/2 - 5\pi^2/2\big) \\ & +(\pi^2/60)(\pi - 2x)^3\big(\underbrace{(2\pi^2 - 20)x^2 + (-2\pi^3 + 20\pi)x + \pi^4/2 - 5\pi^2/2}_{(>0)}\big)\cos 2x.\end{aligned} \tag{74}$$

Then, according to the Lemmas 1.1. and 1.2. and the description of the method, the following inequalities are true: $\sin y < \overline{T}_k^{\sin,0}(y)$ $(k = 5)$ and $\cos y > \underline{T}_k^{\cos,0}(y)$ $(k = 6)$, for $y \in \big(0, \sqrt{(k+3)(k+4)}\big)$.

For $x \in (0, c]$ it is valid:

$$\begin{aligned}\varphi(x) > & -16x^5 + 40\pi x^4 - 32\pi^2 x^3 + 8\pi^3 x^2 - \big(\underbrace{8x^4 - 16\pi x^3 + 8\pi^2 x^2}_{(>0)}\big)\overline{T}_5^{\sin,0}(2x) \\ & -(\pi^2/60)(\pi - 2x)^3\big((2\pi^2 - 20)x^2 + (-2\pi^3 + 20\pi)x + \pi^4/2 - 5\pi^2/2\big) \\ & +(\pi^2/60)(\pi - 2x)^3\big(\underbrace{(2\pi^2 - 20)x^2 + (-2\pi^3 + 20\pi)x + \pi^4/2 - 5\pi^2/2}_{(>0)}\big)\underline{T}_6^{\cos,0}(2x) \\ = & \; Q_{11}(x),\end{aligned} \tag{75}$$

where $Q_{11}(x)$ is the polynomial

$$\begin{aligned}Q_{11}(x) = & \frac{x^2}{2700}\big((64\pi^4 - 640\pi^2)x^9 + (-160\pi^5 + 1600\pi^3)x^8 \\ & +(160\pi^6 - 2000\pi^4 + 4800\pi^2 - 5760)x^7 \\ & +(-80\pi^7 + 1880\pi^5 - 12000\pi^3 + 11520\pi)x^6 \\ & +(20\pi^8 - 1340\pi^6 + 12840\pi^4 - 20160\pi^2 + 28800)x^5 \\ & +(-2\pi^9 + 610\pi^7 - 8700\pi^5 + 36000\pi^3 - 57600\pi)x^4 \\ & +(-150\pi^8 + 4650\pi^6 - 34200\pi^4 + 28800\pi^2 - 86400)x^3 \\ & +(15\pi^9 - 1875\pi^7 + 15300\pi^5 + 194400\pi)x^2 \\ & +(450\pi^8 - 3150\pi^6 - 129600\pi^2)x - 45\pi^9 + 225\pi^7 + 21600\pi^3\big) \\ = & \frac{x^2}{2700}Q_9(x).\end{aligned} \tag{76}$$

Then we determine the sign of the polynomial $Q_9(x)$ for $x \in (0, c]$. Let us look at the fifth derivative of the polynomial $Q_9(x)$, as the fourth degree polynomial, in the following form

$$
\begin{aligned}
Q_9^{(5)}(x) = {} & (967680\pi^4 - 9676800\pi^2)x^4 + (-1075200\pi^5 + 10752000\pi^3)x^3 \\
& +(403200\pi^6 - 5040000\pi^4 + 12096000\pi^2 - 14515200)x^2 \\
& +(-57600\pi^7 + 1353600\pi^5 - 8640000\pi^3 + 8294400\pi)x \\
& +2400\pi^8 - 160800\pi^6 + 1540800\pi^4 - 2419200\pi^2 + 3456000.
\end{aligned} \tag{77}
$$

A real numerical factorization of the polynomial $Q_9^{(5)}(x)$ has been determined via Matlab software, and given with

$$
Q_9^{(5)}(x) = \beta(x - x_1)(x - x_2)(x^2 + px + q), \tag{78}
$$

where $\beta = -1245358.656...$, $x_1 = 0.894...$, $x_2 = 3.702...$, $p = 1.106...$, $q = 0.521...$, whereby the inequality $p^2 - 4q < 0$ is true. The polynomial $Q_9^{(5)}(x)$ has exactly two simple real roots with a symbolic radical representation and the corresponding numerical values x_1 and x_2. Therefore, the polynomial $Q_9^{(5)}(x)$ has no real roots for $x \in (0, c]$. Since $Q_9^{(5)}(0) < 0$ it follows that $Q_9^{(5)}(x) < 0$ for $x \in (0, c]$. Furthermore, the polynomial $Q_9^{(4)}(x)$ is a monotonically decreasing function for $x \in (0, c]$ and $Q_9^{(4)}(c) > 0$, so it follows that $Q_9^{(4)}(x) > 0$ for $x \in (0, c]$. Then, since the polynomial $Q_9'''(x)$ is a monotonically increasing function for $x \in (0, c]$ and $Q_9'''(c) < 0$ it follows that $Q_9'''(x) < 0$ for $x \in (0, c]$. This implies that the polynomial $Q_9''(x)$ is a monotonically decreasing function for $x \in (0, c]$ and since $Q_9''(c) > 0$ it follows that $Q_9''(x) > 0$ for $x \in (0, c]$. Hence, the polynomial $Q_9'(x)$ is a monotonically increasing function for $x \in (0, c]$ and $Q_9'(c) < 0$, so it follows that $Q_9'(x) < 0$ for $x \in (0, c]$. Finally, since the polynomial $Q_9(x)$ is a monotonically decreasing function for $x \in (0, c]$ and $Q_9(c) > 0$, we conclude that

$$
\begin{aligned}
Q_9(x) > 0 \text{ for } x \in (0, c] & \Longrightarrow Q_{11}(x) > 0 \text{ for } x \in (0, c] \\
& \Longrightarrow \varphi(x) > 0 \text{ for } x \in (0, c] \\
& \Longrightarrow f(x) > 0 \text{ for } x \in [1.136, \pi/2).
\end{aligned} \tag{79}
$$

Let us notice that the least positive real root of the downward approximation of the function $\varphi(x)$, i.e. of the polynomial $Q_{11}(x)$, is $x^* = 0.630862\ldots > c = 0.434\ldots$. Elementary calculus gives that the constant $(8\pi^4/15 - 16\pi^2/3)$ is the best possible. The proof of the first inequality is completed.

II We prove the inequality:

$$
x\sec^2 x - \tan x < \frac{(2\pi^4/3)x^3 + (256/\pi^2 - 8\pi^2/3)x^5}{(\pi^2 - 4x^2)^2} \tag{80}
$$

for $x \in (0, \pi/2)$. The requested inequality is equivalent to the inequality $f(x) > 0$

for $x \in (0, \pi/2)$, where

$$f(x) = -x(\pi^2-4x^2)^2+(\pi^2-4x^2)^2\cos x\sin x+\big((2\pi^4/3)x^3+(256/\pi^2-8\pi^2/3)x^5\big)\cos^2 x \tag{81}$$

is a concrete mixed trigonometric polynomial. Let us notice that $x=0$ is zero of the fifth order and $x=\pi/2$ is zero of the third order of the function $f(x)$.

Let us consider two cases:

1) If $x \in (0, 0.858)$:

According to the Theorem 1.5. the function $f(x)$ can be written in the following way:

$$\begin{aligned} f(x) &= -x(\pi^2-4x^2)^2 + \frac{(\pi^2-4x^2)^2}{2}\sin 2x + (\pi^4/3)x^3 + (128/\pi^2-4\pi^2/3)x^5 \\ &\quad + \big(\underbrace{(\pi^4/3)x^3+(128/\pi^2-4\pi^2/3)x^5}_{(>0)}\big)\cos 2x. \end{aligned} \tag{82}$$

Then, according to the Lemmas 1.1. and 1.2. and the description of the method, the inequalities are true: $\sin y > \underline{T}_k^{\sin,0}(y)$ $(k=7)$ and $\cos y > \underline{T}_k^{\cos,0}(y)$ $(k=6)$, for $y \in \big(0, \sqrt{(k+3)(k+4)}\big)$.

For $x \in (0, 0.858)$ it is valid:

$$\begin{aligned} f(x) &> -x(\pi^2-4x^2)^2 + \frac{(\pi^2-4x^2)^2}{2}\underline{T}_7^{\sin,0}(2x) + (\pi^4/3)x^3 + (128/\pi^2-4\pi^2/3)x^5 \\ &\quad + \big(\underbrace{(\pi^4/3)x^3+(128/\pi^2-4\pi^2/3)x^5}_{(>0)}\big)\underline{T}_6^{\cos,0}(2x) = P_{11}(x), \end{aligned} \tag{83}$$

where $P_{11}(x)$ is the polynomial

$$\begin{aligned} P_{11}(x) &= \frac{2x^5}{945\pi^2}\big((56\pi^4-96\pi^2-5376)x^6+(-14\pi^6-372\pi^4+1008\pi^2+40320)x^4 \\ &\quad +(99\pi^6+756\pi^4-5040\pi^2-120960)x^2-252\pi^6+1260\pi^4+120960\big) \\ &= \frac{2x^5}{945\pi^2}P_6(x). \end{aligned} \tag{84}$$

Then we determine the sign of the polynomial $P_6(x)$ for $x \in (0, 0.858)$. By introducing the substitute $z = x^2$, we get the third degree polynomial:

$$\begin{aligned} P_3(z) &= (56\pi^4-96\pi^2-5376)z^3+(-14\pi^6-372\pi^4+1008\pi^2+40320)z^2 \\ &\quad +(99\pi^6+756\pi^4-5040\pi^2-120960)z-252\pi^6+1260\pi^4+120960. \end{aligned} \tag{85}$$

A real numerical factorization of the polynomial $P_3(z)$, has been determined via Matlab software, and given with

$$P_3(z) = \alpha(z-z_1)(z^2+pz+q), \tag{86}$$

where $\alpha = -868.572...$, $z_1 = 0.737147...$, $p = 0.077...$, $q = 2.226...$; whereby the inequality $p^2-4q<0$ is true. The polynomial $P_3(z)$ has exactly one simple real root with a symbolic radical representation and the corresponding numerical value z_1. Let us notice that $\sqrt{z_1} = 0.858573... > 0.858$. Since $P_3(0) > 0$ it follows that $P_3(z) > 0$ for $z \in (0, 0.858)$. Finally, we conclude that

$$\begin{aligned} P_6(x) > 0 \text{ for } x \in (0, 0.858) &\Longrightarrow P_{11}(x) > 0 \text{ for } x \in (0, 0.858) \\ &\Longrightarrow f(x) > 0 \text{ for } x \in (0, 0.858). \end{aligned} \tag{87}$$

Let us notice that the least positive real root of the downward approximation of the function $f(x)$, i.e. of the polynomial $P_{11}(x)$, is $x^* = \sqrt{z_1} = 0.858573...$.

2) If $x \in [0.858, \pi/2)$, let us define the function

$$\begin{aligned} \varphi(x) = f\big(\pi/2 - x\big) &= 16x^5 - 40\pi x^4 + 32\pi^2 x^3 - 8\pi^3 x^2 \\ &+ (16x^4 - 32\pi x^3 + 16\pi^2 x^2)\sin x \cos x \\ &+ (1/(3\pi^2))(\pi - 2x)^3\big((96 - \pi^4)x^2 + (\pi^5 - 96\pi)x + 24\pi^2\big)\sin^2 x. \end{aligned} \tag{88}$$

Now we prove that $f(x) > 0$ for $x \in [0.858, \pi/2)$, which is equivalent to $\varphi(x) > 0$ for $x \in (0, c]$, where $c = \pi/2 - 0.858 = \pi/2 - 429/500$ ($c = 0.712\ldots$). The function $\varphi(x)$ is also a concrete mixed trigonometric polynomial. According to the Theorem 1.5. the function $\varphi(x)$ can be written in the following way:

$$\begin{aligned} \varphi(x) = {} & 16x^5 - 40\pi x^4 + 32\pi^2 x^3 - 8\pi^3 x^2 + \big(\underbrace{8x^4 - 16\pi x^3 + 8\pi^2 x^2}_{(>0)}\big)\sin 2x \\ & + (1/(6\pi^2))(\pi - 2x)^3\big((96 - \pi^4)x^2 + (\pi^5 - 96\pi)x + 24\pi^2\big) \\ & - (1/(6\pi^2))(\pi - 2x)^3\big(\underbrace{(96 - \pi^4)x^2 + (\pi^5 - 96\pi)x + 24\pi^2}_{(>0)}\big)\cos 2x. \end{aligned} \tag{89}$$

Then, according to the Lemmas 1.1. and 1.2. and the description of the method, the inequalities are true: $\sin y > \underline{T}_k^{\sin,0}(y)$ $(k = 7)$ and $\cos y < \overline{T}_k^{\cos,0}(y)$ $(k = 8)$, for $y \in \big(0, \sqrt{(k+3)(k+4)}\big)$.

For $x \in (0, c]$ it is valid:

$$\begin{aligned} \varphi(x) > {} & 16x^5 - 40\pi x^4 + 32\pi^2 x^3 - 8\pi^3 x^2 + \big(\underbrace{8x^4 - 16\pi x^3 + 8\pi^2 x^2}_{(>0)}\big)\underline{T}_7^{\sin,0}(2x) \\ & + (1/(6\pi^2))(\pi - 2x)^3\big((96 - \pi^4)x^2 + (\pi^5 - 96\pi)x + 24\pi^2\big) \\ & - (1/(6\pi^2))(\pi - 2x)^3\big(\underbrace{(96 - \pi^4)x^2 + (\pi^5 - 96\pi)x + 24\pi^2}_{(>0)}\big)\overline{T}_8^{\cos,0}(2x) \\ = {} & Q_{13}(x), \end{aligned} \tag{90}$$

where $Q_{13}(x)$ is the polynomial

$$\begin{aligned} Q_{13}(x) = & \frac{x^3}{945\pi^2}\big((-8\pi^4+768)x^{10}+(20\pi^5-1920\pi)x^9 \\ & +(-18\pi^6+112\pi^4+1728\pi^2-10752)x^8 \\ & +(7\pi^7-280\pi^5-576\pi^3+26880\pi)x^7 \\ & +(-\pi^8+252\pi^6-792\pi^4-24864\pi^2+80640)x^6 \\ & +(-98\pi^7+2076\pi^5+9408\pi^3-201600\pi)x^5 \\ & +(14\pi^8-1890\pi^6+1176\pi^4+191520\pi^2-241920)x^4 \\ & +(735\pi^7-5964\pi^5-80640\pi^3+604800\pi)x^3 \\ & +(-105\pi^8+5670\pi^6+15120\pi^4-574560\pi^2)x^2 \\ & +(-2205\pi^7-2520\pi^5+234360\pi^3)x+315\pi^8-30240\pi^4\big) \\ = & \frac{x^3}{945\pi^2}Q_{10}(x). \end{aligned} \tag{91}$$

Then we determine the sign of the polynomial $Q_{10}(x)$ for $x \in (0,c]$. Let us look at the sixth derivative of the polynomial $Q_{10}(x)$, as the fourth degree polynomial, in the following form

$$\begin{aligned} Q_{10}^{(6)}(x) = & (-1209600\pi^4+116121600)x^4+(1209600\pi^5-116121600\pi)x^3 \\ & +(-362880\pi^6+2257920\pi^4+34836480\pi^2-216760320)x^2 \\ & +(35280\pi^7-1411200\pi^5-2903040\pi^3+135475200\pi)x \\ & -720\pi^8+181440\pi^6-570240\pi^4-17902080\pi^2+58060800. \end{aligned} \tag{92}$$

A real numerical factorization of the polynomial $Q_{10}^{(6)}(x)$ has been determined via Matlab software, and given with

$$Q_{10}^{(6)}(x) = \beta(x-x_1)(x-x_2)(x^2+px+q), \tag{93}$$

where $\beta = -1704436.514...$, $x_1 = 0.610...$, $x_2 = 3.262...$, $p = 0.731...$, $q = 1.935...$, whereby the inequality $p^2-4q<0$ is true. The polynomial $Q_{10}^{(6)}(x)$ has exactly two simple real roots with a symbolic radical representation and the corresponding numerical values x_1 and x_2. Since $Q_{10}^{(6)}(0)<0$, it follows that $Q_{10}^{(6)}(x)<0$ for $x<x_1$ and since $Q_{10}^{(6)}(c)>0$, hence it follows that $Q_{10}^{(6)}(x)>0$ for $x \in (x_1,x_2)$. Therefore, $Q_{10}^{(5)}(x)$ is a monotonically decreasing function for $x<x_1$ and a monotonically increasing function for $x \in (x_1,x_2)$, hence $Q_{10}^{(5)}(x)$ reaches the minimum at the point $x_1 = 0.610...$ in the interval $(0,c]$. Then, since $Q_{10}^{(5)}(0)<0$ and $Q_{10}^{(5)}(c)<0$, it follows that $Q_{10}^{(5)}(x)<0$ for $x \in (0,c]$. Furthermore, since the polynomial $Q_{10}^{(4)}(x)$ is a monotonically decreasing function for $x \in (0,c]$ and $Q_{10}^{(4)}(c)>0$ it follows that $Q_{10}^{(4)}(x)>0$ for $x \in (0,c]$. This implies that the polynomial $Q_{10}'''(x)$ is a monotonically increasing function for $x \in (0,c]$ and $Q_{10}'''(c) < 0$, so it follows that $Q_{10}'''(x) < 0$ for $x \in (0,c]$. Hence, the polynomial $Q_{10}''(x)$ is a monotonically decreasing function for $x \in (0,c]$ and since $Q_{10}''(c) > 0$ it follows that $Q_{10}''(x) > 0$ for $x \in (0,c]$. Then, since the polynomial $Q_{10}'(x)$ is a monotonically increasing function for $x \in (0,c]$ and $Q_{10}'(c) < 0$ it follows that $Q_{10}'(x) < 0$ for $x \in (0,c]$. Finally, since the polynomial $Q_{10}(x)$ is a monotonically decreasing function for $x \in (0,c]$ and $Q_{10}(c) > 0$, we conclude that

$$\begin{aligned} Q_{10}(x) > 0 \text{ for } x \in (0,c] &\Longrightarrow Q_{13}(x) > 0 \text{ for } x \in (0,c] \\ &\Longrightarrow \varphi(x) > 0 \text{ for } x \in (0,c] \\ &\Longrightarrow f(x) > 0 \text{ for } x \in [0.858, \pi/2). \end{aligned} \tag{94}$$

Let us notice that the least positive real root of the downward approximation of the function $\varphi(x)$, i.e. of the polynomial $Q_{13}(x)$, is $x^* = 0.910490\ldots > c = 0.712\ldots$. Elementary calculus gives that the constant $(256/\pi^2 - 8\pi^2/3)$ is the best possible. The proof of the second inequality is completed. □

4. Conclusion

The previous method can be applied to numerous trigonometric inequalities which correspond to univariate mixed trigonometric polynomial functions. By using this method new results can be obtained and the existing ones can be improved from the articles [1]–[2], [4]–[9], [13]–[19], [21], [26]–[45] and the books [24], [25]. Concrete results of the presented method for proving some inequalities, have been obtained in this article through the applications, as well as in the articles [3] and [23].

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