SOME NEW RESULTS OF TWO OPEN PROBLEMS RELATED TO INTEGRAL INEQUALITIES

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Abstract. In this paper, we have solved two open problems, and as consequence some interesting integral inequalities are obtained.

1. Introduction

More recently, Liu et al. (see [1]) obtained the following theorem.

**Theorem 1.1.** Let \( f(x) \geq 0 \) be a continuous function on \([a, b]\) satisfying

\[
\int_a^b f_{\min\{1, \beta\}}(t) \, dt \geq \int_a^b (t - a)^{\min\{1, \beta\}} \, dt, \quad \forall x \in [a, b]
\]

Then the inequality

\[
\int_a^b f^{\alpha + \beta}(x) \, dx \geq \int_a^b (x - a)^{\alpha} f^\beta (x) \, dx
\]

holds for every positive real number \( \alpha > 0 \) and \( \beta > 0 \).

**Theorem 1.2.** Let \( f(x), g(x), h(x) > 0 \) be continuous functions on \([a, b]\) with \( f(x) \leq h(x) \) for all \( x \) and such that \( \frac{f(x)}{h(x)} \) is decreasing and \( f(x), g(x) \) are increasing. Assume that \( \phi(x) \) is a convex function with \( \phi(0) = 0 \).

Then the inequality

\[
\frac{\int_a^b f(x) \, dx}{\int_a^b h(x) \, dx} \geq \frac{\int_a^b \phi(f(x))g(x) \, dx}{\int_a^b \phi(h(x))g(x) \, dx}
\]

holds.

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Liu et al. (see [2]) presented the following two open problems.

Open Problem 1. Under what conditions does the inequality

\[
\int_a^b f^{\alpha+\beta}(x)dx \geq \left( \int_a^b (x-a)^\alpha f^\beta(x)dx \right)^\lambda
\]

(1.4)

hold for \(\alpha, \beta\) and \(\lambda\)?

Open Problem 2. Assume that \(\phi(x)\) is a convex function with \(\phi(0) = 0\). Under what conditions does the inequality

\[
\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \geq \left( \frac{\int_a^b \phi(f(x))g(x)dx}{\int_a^b \phi(h(x))g(x)dx} \right)^\delta
\]

(1.5)

hold for \(\delta\) and \(\lambda\)?

2. Main results

Theorem 2.1. Let \(f(x) \geq 0\) be a continuous function on \([a,b]\) satisfying

\[
\int_x^b (t-a)^{\min\{1, \beta\}}dt \leq \int_x^b f^{\min\{1, \beta\}}(t)dt, \quad \forall x \in [a,b]
\]

(2.1)

Then the inequality

\[
\int_a^b f^{\alpha+\beta}(x)dx \geq \left( \int_a^b (x-a)^\alpha f^\beta(x)dx \right)^\lambda, \quad \forall \lambda \geq 1
\]

(2.2)

holds under each of the following conditions:

1. For all \(\beta > 1\) and \(\alpha > 0\) such that

\[
\frac{(b-a)^{\alpha+2}}{\alpha+2} \leq 1
\]

2. For \(\beta \in (0,1]\) and \(\alpha > 0\) such that

\[
\frac{(b-a)^{\alpha+\beta+1}}{\alpha+\beta+1} \leq 1
\]

Proof. If \(\lambda = 1\) then (2.2) holds for every positive real number \(\alpha > 0\) and \(\beta > 0\) by theorem 1.1. Let \(\lambda > 1\).

Then

\[
\left( \int_a^b (x-a)^\alpha f^\beta(x)dx \right)^\lambda = \left( \int_a^b (x-a)^\alpha f^\beta(x)dx \right) \cdot \left( \int_a^b (x-a)^\alpha f^\beta(x)dx \right)^{\lambda-1}
\]
By using integration by parts, we obtain the following relation

\[ \int_a^b (x-a)^\alpha f^\beta(x)\,dx = \alpha \int_a^b (x-a)^{\alpha-1} \left( \int_x^b f^\beta(t)\,dt \right) \,dx \]  
(2.4)

But, by the hypothesis of theorem 2.1

\[ \int_x^b (t-a)^{\min\{1,\beta\}}\,dt \leq \int_x^b f^{\min\{1,\beta\}}(t)\,dt, \quad \forall x \in [a,b] \]

We have the following two cases:

1. For all \( \beta > 1 \) and \( \alpha > 0 \) such that

\[ \frac{(b-a)^{\alpha+2}}{\alpha+2} \leq 1 \]

by simple calculations inequality (2.3) follows.

2. For \( \beta \in (0,1] \) and \( \alpha > 0 \) such that

\[ \frac{(b-a)^{\alpha+\beta+1}}{\alpha+\beta+1} \leq 1 \]

by simple calculations inequality (2.3) holds. \( \square \)

**Theorem 2.2.** Let \( f(x), g(x), h(x) > 0 \) be continuous functions on \( [a,b] \) with \( f(x) \leq h(x) \) for all \( x \) and such that \( \frac{f(x)}{h(x)} \) is decreasing and \( f(x), g(x) \) are increasing.

Assume that \( \varphi(x) \) is positive and convex function with \( \varphi(0) = 0 \).

Then the inequality

\[ \frac{\int_a^b f(x)\,dx}{\int_a^b h(x)\,dx} \geq \left( \frac{\int_a^b \varphi(f(x))g(x)\,dx}{\int_a^b \varphi(h(x))g(x)\,dx} \right)^\delta \]  
(2.5)

holds under each of the following conditions:

1. \( \lambda = \delta = 0 \) and \( f(x) = h(x) \), for all \( x \in [a,b] \);
2. \( \lambda = \delta \in [1, +\infty) \), for all \( x \in [a,b] \);
3. \( \varphi(f(a)) \geq \frac{1}{(b-a)g(a)} \) for \( 1 \leq \delta < \lambda \);
4. \( \varphi(f(b)) \leq \frac{1}{(b-a)g(b)} \) for \( 1 \leq \lambda < \delta \).

**Proof.**

1. If \( \lambda = \delta = 0 \) and \( f(x) = h(x) \), for all \( x \in [a, b] \) then inequality (2.5) turns into an equality.

2. If \( \lambda = \delta = 1 \) inequality (2.5) coincides with theorem 1.2.

Now let \( \lambda = \delta > 1 \) and denote by \( d = \frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \). Since \( 0 < f(x) \leq h(x) \), for all \( x \in [a, b] \) then \( d \in [0, 1] \). By theorem 1.2 and the fact that \( \varphi(x) \) is positive and convex function with \( \varphi(0) = 0 \), we have the following inequalities

\[
\left( \frac{\int_a^b \varphi(f(x))g(x)dx}{\int_a^b \varphi(h(x))g(x)dx} \right)^\delta \leq \left( \frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \right)^\delta \leq \frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \tag{2.6}
\]

since \( d \in [0, 1] \), for all \( \delta > 1 \). So inequality (2.5) follows.

3. For \( 1 \leq \delta < \lambda \) there exists a real positive number \( r \) such that \( \lambda = \delta + r \). Using case (2) for \( \lambda = \delta \in [1, +\infty) \) we have

\[
\left( \frac{\int_a^b \varphi(f(x))g(x)dx}{\int_a^b \varphi(h(x))g(x)dx} \right)^\lambda = \left( \frac{\int_a^b \varphi(f(x))g(x)dx}{\int_a^b \varphi(h(x))g(x)dx} \right)^\delta \cdot \frac{1}{\left( \frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \right)^r} \leq \left( \frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \right)^\delta \cdot \frac{1}{\left( \frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \right)^r} \leq \frac{\int_a^b f(x)dx}{\int_a^b h(x)dx}
\]

The last inequality above follows by the fact that \( \left( \frac{\int_a^b \varphi(h(x))g(x)dx}{\int_a^b \varphi(h(x))g(x)dx} \right)^r \geq 1 \) for \( r > 0 \), since we have assumed that \( \varphi(f(a)) \geq \frac{1}{(b-a)g(a)} \). So inequality (2.5) holds.

4. For \( 1 \leq \lambda < \delta \) there exists a real positive number \( r_1 \) such that \( \delta = \lambda + r_1 \). Using case (2) for \( \lambda = \delta \in [1, +\infty) \) we have

\[
\left( \frac{\int_a^b \varphi(f(x))g(x)dx}{\int_a^b \varphi(h(x))g(x)dx} \right)^\lambda = \left( \frac{\int_a^b \varphi(f(x))g(x)dx}{\int_a^b \varphi(h(x))g(x)dx} \right)^\delta \cdot \left( \frac{\int_a^b \varphi(f(x))g(x)dx}{\int_a^b \varphi(h(x))g(x)dx} \right)^{r_1} \leq \left( \frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \right)^\delta \cdot \left( \frac{\int_a^b \varphi(f(x))g(x)dx}{\int_a^b \varphi(h(x))g(x)dx} \right)^{r_1} \leq \frac{\int_a^b f(x)dx}{\int_a^b h(x)dx}
\]
In the last inequality we have used the fact that \( \left( \int_a^b \varphi(f(x))g(x)dx \right)^{r_1} \leq 1 \) for \( r_1 > 0 \), since we have assumed that \( \varphi(f(b)) \leq \frac{1}{(b-a)g(b)} \). So inequality (2.5) follows. □

3. Applications

**Corollary 3.1.** Let \( f(x) \geq 0 \) be a continuous function on \([0, 1]\) satisfying
\[
\int_0^1 t^\min\{1, \beta\} dt \leq \int_0^1 f^\min\{1, \beta\}(t) dt, \quad \forall x \in [0, 1]
\]
Then the inequality
\[
\int_0^1 f^{\alpha+\beta}(x)dx \geq \left( \int_0^1 x^\alpha f^\beta(x)dx \right)^\lambda, \quad \forall \lambda > 1
\]
holds for \( \alpha, \beta > 0 \).

**Corollary 3.2.** Let \( f(x) \geq 0 \) be a continuous function on \([a, b]\) satisfying
\[
\int_a^b (t-a)^\min\{1, \alpha\} dt \leq \int_a^b f^\min\{1, \alpha\}(t) dt, \quad \forall x \in [a, b]
\]
Then the inequality
\[
\int_a^b f^{2\alpha}(x)dx \geq \left( \int_a^b ((x-a) \cdot f(x))^\alpha dx \right)^\lambda, \quad \forall \lambda > 1
\]
holds under each of the following conditions:

1. For \( \alpha > 1 \) such that
\[
\frac{(b-a)^{\alpha+2}}{\alpha+2} \leq 1
\]
2. For \( \alpha \in (0, 1] \) such that
\[
\frac{(b-a)^{2\alpha+1}}{2\alpha+1} \leq 1
\]

**Proof.** Let \( \alpha = \beta \) and applying theorem 2.1. □

**Corollary 3.3.** Let \( f(x), g(x), h(x) > 0 \) be continuous functions on \([a, b]\) with \( f(x) \leq h(x) \) for all \( x \) and such that \( \frac{f(x)}{h(x)} \) is decreasing and \( f(x), g(x) \) are increasing. Assume that \( \varphi(x) \) is positive and convex function with \( \varphi(0) = 0 \). Then the inequality
\[
\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \geq \left( \frac{\int_a^b \varphi(f(x))g^p(x)dx}{\int_a^b \varphi(h(x))g^p(x)dx} \right)^\delta, \quad \forall p \geq 0
\]
holds under each of the following conditions:
1. \( \lambda = \delta = 0 \) and \( f(x) = h(x) \), for all \( x \in [a, b] \);

2. \( \lambda = \delta \in [1, +\infty) \), for all \( x \in [a, b] \);

3. \( \varphi(f(a)) \geq \frac{1}{(b-a)g^p(a)} \) for \( 1 \leq \delta < \lambda \);

4. \( \varphi(f(b)) \leq \frac{1}{(b-a)g^p(b)} \) for \( 1 \leq \lambda < \delta \).

**Proof.** Let \( g_p(x) = g^p(x) \), for all \( x \in [a, b] \) and for all \( p \geq 0 \). Since \( g(x) \) is increasing function and \( g(x) > 0 \), then \( g_p(x) \) are increasing functions for all \( p \geq 0 \). By applying theorem 2.2, inequality (3.5) follows. \( \Box \)

**Corollary 3.4.** Let \( f(x), g(x), h(x) > 0 \) be continuous functions on \([a, b]\) with \( f(x) \leq h(x) \) for all \( x \) and such that \( \frac{f(x)}{h(x)} \) is decreasing and \( f(x), g(x) \) are increasing. Then the inequality

\[
\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \geq \left( \frac{\int_a^b f^k(x)g^p(x)dx}{\int_a^b h^k(x)g^p(x)dx} \right)^{\delta}, \quad \forall k \geq 1 \text{ and } \forall p \geq 0
\]  

(3.6)

holds under each of the following conditions:

1. \( \lambda = \delta = 0 \) and \( f(x) = h(x) \), for all \( x \in [a, b] \);

2. \( \lambda = \delta \in [1, +\infty) \), for all \( x \in [a, b] \);

3. \( f^k(a) \geq \frac{1}{(b-a)g^p(a)} \) for \( 1 \leq \delta < \lambda \);

4. \( f^k(b) \leq \frac{1}{(b-a)g^p(b)} \) for \( 1 \leq \lambda < \delta \).

**Proof.** Let \( \varphi(x) = x^k \) where \( k \geq 1 \). \( \varphi \) is a convex function and \( \varphi(0) = 0 \). By corollary 3.3, inequality (3.6) follows. \( \Box \)

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REFERENCES


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