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ON D'AURIZIO'S TRIGONOMETRIC INEQUALITY

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Abstract. We offer new proof of the recent sharp trigonometric inequality $\cos x/\cos(x/2) \geqslant 1-4x^2/\pi^2$ for $x \in (0,\pi/2)$, discovered by Jacopo D'aurizio [1]. The converse inequality, as well as sharp analogous inequalities are pointed out, too.

1. Introduction

By studying refinements of the famous Shafer-Fink inequality for the arctangent function, J. D'aurizio [1] recently proved the following new trigonometric inequality (see the Proof of Theorem 4 of [1])

$$\frac{\cos x}{\cos\frac{x}{2}} \geqslant 1-\frac{4x^2}{\pi^2}, \quad x \in \left[0,\frac{\pi}{2}\right] \tag{1}$$

D'aurizio's proof is based on infinite product expansions, as well as inequalities on series and $\zeta(2n)$, where ζ is the Riemann's zeta function.

Our aim will be to offer a new proof of (1), based on trigonometric inequalities, using an auxiliary function. The method will provide also the following converse to (1):

$$\frac{\cos x}{\cos\frac{x}{2}} \leqslant 1-\frac{3}{8}x^2 \tag{2}$$

2. Main results

In fact, (1) and (2) are consequences to the following:

THEOREM 1.

$$\frac{3}{8} < \frac{1-\dfrac{\cos x}{\cos\frac{x}{2}}}{x^2} < \frac{4}{\pi^2}, \tag{3}$$

for any $x \in (0,\pi/2)$

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Proof. Let us consider the application

$$f(x) = \frac{1 - \dfrac{\cos x}{\cos \frac{x}{2}}}{x^2}, \quad x \in (0, \pi/2) \tag{4}$$

By letting $x = 2t$, for $t \in (0, \pi/4)$, and by using $\cos 2t = 2\cos^2 t - 1$, we get

$$4f(x) = g(t) = \frac{\cos t - 2\cos^2 t + 1}{t^2 \cos t}, \quad t \in (0, \pi/4) \tag{5}$$

A simple computation shows that one has

$$t^3 \cos^2 t \cdot g'(t) = 2t \sin t \cdot \cos^2 t + 4\cos^3 t - 2\cos^2 t - 2\cos t = h(t)$$

We will show that $h(t) > 0$ for any $t \in (0, \pi/4)$, or equivalently:

$$(t \sin t)(2\cos^2 t + 1) > 2\cos t \cdot (1 - \cos t) \cdot (2\cos t + 1) \tag{6}$$

As $\sin t = 2\sin \frac{t}{2} \cos \frac{t}{2}$, and $1 - \cos t = 2\sin^2 \frac{t}{2}$, relation (6) may be written as

$$\frac{\sin(t/2)}{(t/2)} < (\cos(t/2)) \cdot A(t), \tag{7}$$

where $A(t) = \dfrac{2\cos^2 t + 1}{2\cos^2 t + \cos t}$.

First we remark that, it is immediate that

$$A(t) > \frac{3}{2 + \cos t}, \tag{8}$$

as this becomes equivalent with $\cos^2 t - 2\cos t + 1 > 0$, or $(\cos t - 1)^2 > 0$. On the other hand, by the famous Cusa–Huygens inequality (see e.g. [2], [3])

$$\frac{\sin z}{z} < \frac{\cos z + 2}{3}, \quad z \in (0, \pi/2) \tag{9}$$

the validity of (7) will be a consequence of the relation

$$\frac{\cos \frac{t}{2} + 2}{3} < \left(\cos \frac{t}{2}\right) \cdot \frac{3}{2 + \cos t} \tag{10}$$

By letting $\cos \frac{t}{2} = u$, inequality (10) becomes, after certain elementary transformations:

$$P(u) = u^3 + 2u^2 - 4u + 1 < 0 \tag{11}$$

As $P(u)=(u-1)(u^2+3u-1)$ and $u-1<0$, it is sufficient to remark that $u^2+3u-1>0$. This holds clearly true, as $u=\cos\frac{t}{2}>\frac{1}{\sqrt{2}}$ for $0<\frac{t}{2}<\frac{\pi}{8}<\frac{\pi}{4}$, and so $u^2+3u>\frac{3\sqrt{2}+1}{2}>1$.

Therefore, relation (7) follows, and this means that (6) is also true, implying $h(t)>0$. Thus, the function $g(t)$ will be strictly increasing, and as $x=2t$, clearly the function $f(x)$ will have this property, too.

As $\lim_{x\to\pi/2} f(x)=\frac{4}{\pi^2}$ and $\lim_{x\to 0} f(x)=\frac{3}{8}$, Theorem 1 follows. □

REMARK 1. The proof shows that, there is equality in (1) only for $x=0$ and $x=\pi/2$; and in (2) only for $x=0$. Clearly, both inequalities are best possible.

The following analogue for the case of sin functions holds true:

THEOREM 2.

$$\frac{4}{\pi^2}(2-\sqrt{2})<\frac{2-\dfrac{\sin x}{\sin\frac{x}{2}}}{x^2}<\frac{1}{4}, \tag{12}$$

for any $x\in(0,\pi/2)$.

Proof. As $\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}$, we can write

$$\frac{2-\dfrac{\sin x}{\sin\frac{x}{2}}}{x^2}=2\left(\frac{1-\cos\frac{x}{2}}{x^2}\right)=\frac{1}{2}\cdot q(t), \tag{13}$$

where

$$q(t)=\frac{1-\cos t}{t^2}, \tag{14}$$

and $t=\frac{x}{2}$. In our recent paper [3] it is proved that $q(p)$ of (14) is a strictly decreasing (and concave) function of p, for any $p\in(0,\pi/2)$. As $x\in(0,\pi/2)$, we get $\lim_{t\to 0} q(t)>q(t)>\lim_{t\to\pi/4} q(t)$, giving the double inequality

$$\frac{16}{\pi^2}\cdot\left(1-\frac{\sqrt{2}}{2}\right)<q(t)<\frac{1}{2}, \tag{15}$$

which by (13) immediately implies (12). □

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