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NEW INEQUALITIES FOR G–FRAMES IN HILBERT C^*–MODULES

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Abstract. In this paper, we establish several new inequalities for g-frames in Hilbert C^*-modules which are different in structure from those previously obtained by Balan et al. for Hilbert space frames. We also present some equalities and inequalities for g-frames in Hilbert C^*-modules with Moore–Penrose inverses and show that they are more general and cover some results in [Xiao, XC, Zeng, XM: Some properties of g-frames in Hilbert C^*-modules. J. Math. Anal. Appl. 363 (2), 399–408 (2010)].

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [1] to study some deep problems in nonharmonic Fourier series. The importance of frames was not realized until 1986 when Daubechies et al. [2] found a fundamental new application, to wavelet and window Fourier transform. Since then, frames have become the focus of active research, both in theory and in applications, such as the characterization of function spaces, digital signal processing, scientific computations etc. The interested reader can consult [3, 4] for an introduction to frame theory and its applications. Some authors devoted their efforts to the extension of frame theory in Hilbert spaces and thereby leads to various generalizations of frame concept, in which g-frames, proposed by Sun in [5], include many other generalizations of frames, e.g., frames of subspaces [6], oblique frames [7], pseudo-frames [8], outer frames [9], etc.

Frames and g-frames for Hilbert spaces have natural analogues for Hilbert C^*-modules [10, 11]. Although Hilbert C^*-modules are generalizations of Hilbert spaces, there are many essential differences between them. For instance, a closed topologically complemented submodule may not be orthogonally complemented, meanwhile the fundamental Riesz representation theorem concerning the bounded linear functionals in Hilbert spaces may also be not true in Hilbert C^*-modules. This suggests that the generalization of frame theory from Hilbert spaces to Hilbert C^*-modules is not a trivial task. G-frames in Hilbert C^*-modules have been studied intensively, for more details see [12–15].

We need recall some definitions and basic properties of g-frames in Hilbert C^*-modules.

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Throughout this paper, the symbols $\mathbb{J}$ and $\mathscr{A}$ refer, respectively, to a finite or countable index set and a unital C^*-algebra. $\mathscr{H}$, $\mathscr{K}$ and $\mathscr{K}_j$'s are Hilbert C^*-modules over $\mathscr{A}$, and set $\langle x,x\rangle = |x|^2$ for every $x \in \mathscr{H}$. We denote by $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H},\mathscr{K})$ the set of all adjointable operators from $\mathscr{H}$ to $\mathscr{K}$, and $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H},\mathscr{H})$ is abbreviated to $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$.

A sequence $\{\Lambda_j \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H},\mathscr{K}_j)\}_{j\in\mathbb{J}}$ is called a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$, if there exist two constants $C,D > 0$ such that

$$C\langle f,f\rangle \leqslant \sum_{j\in\mathbb{J}} \langle \Lambda_j f, \Lambda_j f\rangle \leqslant D\langle f,f\rangle \tag{1.1}$$

for all $f \in \mathscr{H}$. We call C and D the lower and upper g-frame bounds, respectively. The g-frame $\{\Lambda_j\}_{j\in\mathbb{J}}$ is said to be λ-tight if $C = D = \lambda$, and said to be Parseval if $C = D = 1$. The sequence $\{\Lambda_j\}_{j\in\mathbb{J}}$ is called a g-Bessel sequence with g-Bessel bound D if we only require the right hand inequality of (1.1).

Let $\{\Lambda_j \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H},\mathscr{K}_j)\}_{j\in\mathbb{J}}$ be a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$, then the g-frame operator $S : \mathscr{H} \to \mathscr{H}$ defined by

$$Sf = \sum_{j\in\mathbb{J}} \Lambda_j^* \Lambda_j f \tag{1.2}$$

is a positive, self-adjoint and invertible operator. Denote $\widetilde{\Lambda}_j = \Lambda_j S^{-1}$ for each $j \in \mathbb{J}$, then it is easy to check that $\{\widetilde{\Lambda}_j\}_{j\in\mathbb{J}}$ is still a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$, which is called the canonical dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$. For any $\mathbb{K} \subset \mathbb{J}$, let $\mathbb{K}^c = \mathbb{J}\backslash\mathbb{K}$, and define adjointable operators $S_{\mathbb{K}}, S_{\mathbb{K}^c} : \mathscr{H} \to \mathscr{H}$ as follows:

$$S_{\mathbb{K}} f = \sum_{j\in\mathbb{K}} \Lambda_j^* \Lambda_j f, \quad S_{\mathbb{K}^c} f = \sum_{j\in\mathbb{K}^c} \Lambda_j^* \Lambda_j f. \tag{1.3}$$

Balan et al. [16] discovered a new identity for Parseval frames in Hilbert spaces when working on algorithms for computing the reconstruction. Moreover, in [16] the following inequality was obtained:

THEOREM 1.1. *Let $\{f_j\}_{j\in\mathbb{J}}$ be a Parseval frame for a Hilbert space $\mathscr{M}$, then for every $\mathbb{K} \subset \mathbb{J}$ and every $f \in \mathscr{M}$, we have*

$$\sum_{j\in\mathbb{K}} |\langle f, f_j\rangle|^2 + \left\| \sum_{j\in\mathbb{K}^c} \langle f, f_j\rangle f_j \right\|^2 \geqslant \frac{3}{4}\|f\|^2. \tag{1.4}$$

Later on, Găvruţa in [17] generalized inequality (1.4) to general frames:

THEOREM 1.2. *Let $\{f_j\}_{j\in\mathbb{J}}$ be a frame for a Hilbert space $\mathscr{M}$ with canonical dual frame $\{\tilde{f}_j\}_{j\in\mathbb{J}}$. Then for all $\mathbb{K} \subset \mathbb{J}$ and all $f \in \mathscr{M}$, we have*

$$\sum_{j\in\mathbb{K}} |\langle f, f_j\rangle|^2 + \sum_{j\in\mathbb{J}} |\langle S_{\mathbb{K}^c} f, \tilde{f}_j\rangle|^2 \geqslant \frac{3}{4} \sum_{j\in\mathbb{J}} |\langle f, f_j\rangle|^2. \tag{1.5}$$

Recently, Xiao and Zeng in [12] showed that g-frames in Hilbert C^*-modules also have their equalities and inequalities. In Section 2 of this work we first establish several new inequalities for g-frames in Hilbert C^*-modules which differ in structure from those in Theorems 1.1 and 1.2. We then give some equalities and inequalities for g-frames in Hilbert C^*-modules with Moore–Penrose inverses and we show that Theorems 4.1 and 4.2 in [12] are special cases of our results.

2. The main results and their proofs

Let $\{\Lambda_j \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H},\mathscr{K}_j)\}_{j\in\mathbb{J}}$, $\{\Gamma_j \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H},\mathscr{K}_j)\}_{j\in\mathbb{J}}$ and $\{\Theta_j \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H},\mathscr{K}_j)\}_{j\in\mathbb{J}}$ be g-Bessel sequences for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. In [12], the authors defined an adjointable operator as follows:

$$L:\mathscr{H}\to\mathscr{H},\quad Lf=\sum_{j\in\mathbb{J}}\Gamma_j^*\Lambda_j f,\quad \forall f\in\mathscr{H}. \tag{2.1}$$

It is easily seen that the operator $Q:\mathscr{H}\to\mathscr{H}$ defined by

$$Qf=\sum_{j\in\mathbb{J}}(\Gamma_j-\Theta_j)^*\Lambda_j f,\quad \forall f\in\mathscr{H}, \tag{2.2}$$

is well defined and adjointable.

In order to derive our main results, we need the following lemmas.

LEMMA 2.1. (see [18]) *Suppose that $T\in\mathrm{End}^*_{\mathscr{A}}(\mathscr{H},\mathscr{K})$ has closed range, then there exits a unique operator $T^\dagger\in\mathrm{End}^*_{\mathscr{A}}(\mathscr{K},\mathscr{H})$, called the Moore–Penrose inverse of T, satisfying*

$$TT^\dagger T=T,\quad T^\dagger TT^\dagger=T^\dagger,\quad (TT^\dagger)^*=TT^\dagger,\quad (T^\dagger T)^*=T^\dagger T. \tag{2.3}$$

The notation $T^\dagger$ is reserved to denote the Moore–Penrose inverse of T (if it exists).

LEMMA 2.2. *Suppose that $T\in\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ is a self-adjoint operator. Let $a,b,c\in\mathbb{R}$ and $U=aT^2+bT+c\mathrm{Id}_{\mathscr{H}}$. We have the following statements.*

(1) If $a>0$, then

$$\langle Uf,f\rangle \geqslant \frac{4ac-b^2}{4a}\langle f,f\rangle,\quad \forall f\in\mathscr{H}.$$

(2) If $a<0$, then

$$\langle Uf,f\rangle \leqslant \frac{4ac-b^2}{4a}\langle f,f\rangle,\quad \forall f\in\mathscr{H}.$$

Proof. (1) A direct calculation shows that

$$U=a\left(T+\frac{b}{2a}\mathrm{Id}_{\mathscr{H}}\right)^2+\frac{4ac-b^2}{4a}\mathrm{Id}_{\mathscr{H}}.$$

Since $(T+\frac{b}{2a}\mathrm{Id}_{\mathscr{H}})$ is self-adjoint, its square is positive. Hence

$$\langle Uf,f\rangle \geqslant \frac{4ac-b^2}{4a}\langle f,f\rangle, \quad \forall f\in\mathscr{H}.$$

(2) It is a direct consequence of (1). □

LEMMA 2.3. *Suppose that $L,P,Q\in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$, that $P+Q=L$, and that L has closed range. Then*

$$L^*L^\dagger P+Q^*L^\dagger Q=Q^*L^\dagger L+P^*L^\dagger P. \tag{2.4}$$

Moreover, if L and $LL^\dagger Q$ are positive and self-adjoint, then for all $f\in\mathscr{H}$,

$$\langle (L^*L^\dagger P+Q^*L^\dagger Q)f,f\rangle \geqslant \frac{3}{4}\langle Lf,f\rangle. \tag{2.5}$$

Proof. The formula (2.4) follows from the following:

$$L^*L^\dagger P-P^*L^\dagger P=(L^*-P^*)L^\dagger P=Q^*L^\dagger(L-Q)=Q^*L^\dagger L-Q^*L^\dagger Q.$$

Next we show that inequality (2.5) holds. Set $T=L^{\frac{1}{2}}L^\dagger Q$, then $T^*=Q^*L^\dagger L^{\frac{1}{2}}$. Now for all $f\in\mathscr{H}$, we compute

$$\begin{aligned}\langle (L^*L^\dagger P+Q^*L^\dagger Q)f,f\rangle &= \langle (LL^\dagger(L-Q)+Q^*L^\dagger Q)f,f\rangle\\ &=\langle Lf,f\rangle-\frac{1}{2}\langle L^{\frac{1}{2}}f,Tf\rangle-\frac{1}{2}\langle Tf,L^{\frac{1}{2}}f\rangle+\langle Tf,Tf\rangle\\ &=\frac{3}{4}\langle L^{\frac{1}{2}}f,L^{\frac{1}{2}}f\rangle+\left\langle \frac{1}{2}L^{\frac{1}{2}}f-Tf,\frac{1}{2}L^{\frac{1}{2}}f-Tf\right\rangle \geqslant \frac{3}{4}\langle Lf,f\rangle.\end{aligned}$$

□

THEOREM 2.4. *Let $\{\Lambda_j\in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H},\mathscr{K}_j)\}_{j\in\mathbb{J}}$ be a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ with canonical dual g-frame $\{\widetilde{\Lambda}_j\}_{j\in\mathbb{J}}$. Then for any $\mathbb{K}\subset\mathbb{J}$ and any $f\in\mathscr{H}$, we have*

$$0\leqslant \sum_{j\in\mathbb{K}}|\Lambda_j f|^2-\sum_{j\in\mathbb{J}}|\widetilde{\Lambda}_j S_{\mathbb{K}}f|^2\leqslant \frac{1}{4}\sum_{j\in\mathbb{J}}|\Lambda_j f|^2. \tag{2.6}$$

$$\frac{1}{2}\sum_{j\in\mathbb{J}}|\Lambda_j f|^2\leqslant \sum_{j\in\mathbb{J}}|\widetilde{\Lambda}_j S_K f|^2+\sum_{j\in\mathbb{J}}|\widetilde{\Lambda}_j S_{\mathbb{K}^c}f|^2\leqslant \sum_{j\in\mathbb{J}}|\Lambda_j f|^2. \tag{2.7}$$

Proof. Denote S the g-frame operator of $\{\Lambda_j\}_{j\in\mathbb{J}}$. It is easy to check that $\{\Gamma_j=\Lambda_j S^{-\frac{1}{2}}\}_{j\in\mathbb{J}}$ is a Parseval g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. Let $\widetilde{S}_{\mathbb{K}}f=\sum_{j\in\mathbb{K}}\Gamma_j^*\Gamma_j f$ and $\widetilde{S}_{\mathbb{K}^c}f=\sum_{j\in\mathbb{K}^c}\Gamma_j^*\Gamma_j f$ for each $f\in\mathscr{H}$, then

$$\widetilde{S}_{\mathbb{K}}f+\widetilde{S}_{\mathbb{K}^c}f=\sum_{j\in\mathbb{K}}\Gamma_j^*\Gamma_j f+\sum_{j\in\mathbb{K}^c}\Gamma_j^*\Gamma_j f=\sum_{j\in\mathbb{J}}\Gamma_j^*\Gamma_j f=f,$$

equivalently, $\widetilde{S}_{\mathbb{K}}+\widetilde{S}_{\mathbb{K}^c}=\mathrm{Id}_{\mathscr{H}}$. Thus,

$$0 \leqslant \widetilde{S}_{\mathbb{K}}\widetilde{S}_{\mathbb{K}^c}=\widetilde{S}_{\mathbb{K}}(\mathrm{Id}_{\mathscr{H}}-\widetilde{S}_{\mathbb{K}})=\widetilde{S}_{\mathbb{K}}-\widetilde{S}_{\mathbb{K}}^2. \tag{2.8}$$

From Lemma 2.2 we see that

$$\widetilde{S}_{\mathbb{K}}-\widetilde{S}_{\mathbb{K}}^2 \leqslant \frac{1}{4}\mathrm{Id}_{\mathscr{H}}. \tag{2.9}$$

Noting that

$$\widetilde{S}_{\mathbb{K}}f=\sum_{j\in\mathbb{K}}(\Lambda_j S^{-\frac{1}{2}})^*(\Lambda_j S^{-\frac{1}{2}})f=S^{-\frac{1}{2}}\sum_{j\in\mathbb{K}}\Lambda_j^*\Lambda_j S^{-\frac{1}{2}}f=S^{-\frac{1}{2}}S_{\mathbb{K}}S^{-\frac{1}{2}}f, \quad \forall f\in\mathscr{H},$$

we have $\widetilde{S}_{\mathbb{K}}=S^{-\frac{1}{2}}S_{\mathbb{K}}S^{-\frac{1}{2}}$. Combination of (2.8) and (2.9) we get

$$0 \leqslant S^{-\frac{1}{2}}(S_{\mathbb{K}}-S_{\mathbb{K}}S^{-1}S_{\mathbb{K}})S^{-\frac{1}{2}} \leqslant \frac{1}{4}\mathrm{Id}_{\mathscr{H}}, \tag{2.10}$$

implying that

$$0 \leqslant S_{\mathbb{K}}-S_{\mathbb{K}}S^{-1}S_{\mathbb{K}} \leqslant \frac{1}{4}S. \tag{2.11}$$

Since

$$\begin{aligned}
\langle S_{\mathbb{K}}f,f\rangle-\langle S_{\mathbb{K}}S^{-1}S_{\mathbb{K}}f,f\rangle &= \langle S_{\mathbb{K}}f,f\rangle-\langle S^{-1}S_{\mathbb{K}}f,S_{\mathbb{K}}f\rangle\\
&= \langle S_{\mathbb{K}}f,f\rangle-\langle SS^{-1}S_{\mathbb{K}}f,\,S^{-1}S_{\mathbb{K}}f\rangle\\
&= \sum_{j\in\mathbb{K}}\langle\Lambda_j f,\Lambda_j f\rangle-\sum_{j\in\mathbb{J}}\langle\Lambda_j^*\Lambda_j S^{-1}S_{\mathbb{K}}f,S^{-1}S_{\mathbb{K}}f\rangle\\
&= \sum_{j\in\mathbb{K}}|\Lambda_j f|^2-\sum_{j\in\mathbb{J}}|\widetilde{\Lambda}_j S_{\mathbb{K}}f|^2
\end{aligned}$$

for each $f\in\mathscr{H}$, it follows from (2.11) that

$$0 \leqslant \sum_{j\in\mathbb{K}}|\Lambda_j f|^2-\sum_{j\in\mathbb{J}}|\widetilde{\Lambda}_j S_{\mathbb{K}}f|^2 \leqslant \frac{1}{4}\sum_{j\in\mathbb{J}}|\Lambda_j f|^2. \tag{2.12}$$

It remains to show that (2.7) holds. Since

$$\widetilde{S}_{\mathbb{K}}^2+\widetilde{S}_{\mathbb{K}^c}^2=\widetilde{S}_{\mathbb{K}}^2+(\mathrm{Id}_{\mathscr{H}}-\widetilde{S}_{\mathbb{K}})^2=2\widetilde{S}_{\mathbb{K}}^2-2\widetilde{S}_{\mathbb{K}}+\mathrm{Id}_{\mathscr{H}}, \tag{2.13}$$

again by Lemma 2.2 we have

$$\widetilde{S}_{\mathbb{K}}^2+\widetilde{S}_{\mathbb{K}^c}^2 \geqslant \frac{1}{2}\mathrm{Id}_{\mathscr{H}}. \tag{2.14}$$

Since $\widetilde{S}_{\mathbb{K}}-\widetilde{S}_{\mathbb{K}}^2\geqslant 0$, it follows from (2.13) that

$$\widetilde{S}_{\mathbb{K}}^2+\widetilde{S}_{\mathbb{K}^c}^2=\mathrm{Id}_{\mathscr{H}}-2(\widetilde{S}_{\mathbb{K}}-\widetilde{S}_{\mathbb{K}}^2) \leqslant \mathrm{Id}_{\mathscr{H}}. \tag{2.15}$$

Applying the fact $\widetilde{S}_{\mathbb{K}} = S^{-\frac{1}{2}} S_{\mathbb{K}} S^{-\frac{1}{2}}$ and $\widetilde{S}_{\mathbb{K}^c} = S^{-\frac{1}{2}} S_{\mathbb{K}^c} S^{-\frac{1}{2}}$ to (2.14) and (2.15) yields

$$\frac{1}{2}\mathrm{Id}_{\mathscr{H}} \leqslant S^{-\frac{1}{2}}(S_{\mathbb{K}} S^{-1} S_{\mathbb{K}} + S_{\mathbb{K}^c} S^{-1} S_{\mathbb{K}^c}) S^{-\frac{1}{2}} \leqslant \mathrm{Id}_{\mathscr{H}}.$$

Therefore,

$$\frac{1}{2}S \leqslant S_{\mathbb{K}} S^{-1} S_{\mathbb{K}} + S_{\mathbb{K}^c} S^{-1} S_{\mathbb{K}^c} \leqslant S. \tag{2.16}$$

Finally, for each $f \in \mathscr{H}$ we have

$$\begin{aligned}
&\langle S_{\mathbb{K}} S^{-1} S_{\mathbb{K}} f, f\rangle + \langle S_{\mathbb{K}^c} S^{-1} S_{\mathbb{K}^c} f, f\rangle \\
&\quad = \langle S S^{-1} S_{\mathbb{K}} f, S^{-1} S_{\mathbb{K}} f\rangle + \langle S S^{-1} S_{\mathbb{K}^c} f, S^{-1} S_{\mathbb{K}^c} f\rangle \\
&\quad = \sum_{j\in\mathbb{J}} \langle \Lambda_j^* \Lambda_j S^{-1} S_{\mathbb{K}} f, S^{-1} S_{\mathbb{K}} f\rangle + \sum_{j\in\mathbb{J}} \langle \Lambda_j^* \Lambda_j S^{-1} S_{\mathbb{K}^c} f, S^{-1} S_{\mathbb{K}^c} f\rangle \\
&\quad = \sum_{j\in\mathbb{J}} |\widetilde{\Lambda}_j S_{\mathbb{K}} f|^2 + \sum_{j\in\mathbb{J}} |\widetilde{\Lambda}_j S_{\mathbb{K}^c} f|^2.
\end{aligned} \tag{2.17}$$

This along with (2.16) leads to (2.7). □

If $\{\Lambda_j \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K}_j)\}_{j\in\mathbb{J}}$ is a Parseval g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$, then $S = \mathrm{Id}_{\mathscr{H}}$. For any $\mathbb{K} \subset \mathbb{J}$ and any $f \in \mathscr{H}$, we have

$$\begin{aligned}
\sum_{j\in\mathbb{J}} |\widetilde{\Lambda}_j S_{\mathbb{K}} f|^2 &= \sum_{j\in\mathbb{J}} \langle \widetilde{\Lambda}_j S_{\mathbb{K}} f, \widetilde{\Lambda}_j S_{\mathbb{K}} f\rangle = \sum_{j\in\mathbb{J}} \langle \Lambda_j S_{\mathbb{K}} f, \Lambda_j S_{\mathbb{K}} f\rangle \qquad (2.18)\\
&= \sum_{j\in\mathbb{J}} \langle \Lambda_j^* \Lambda_j S_{\mathbb{K}} f, S_{\mathbb{K}} f\rangle = \langle S_{\mathbb{K}} f, S_{\mathbb{K}} f\rangle = \left|\sum_{j\in\mathbb{K}} \Lambda_j^* \Lambda_j f\right|^2.
\end{aligned}$$

Similarly, we have

$$\sum_{j\in\mathbb{J}} |\widetilde{\Lambda}_j S_{\mathbb{K}^c} f|^2 = \left|\sum_{j\in\mathbb{K}^c} \Lambda_j^* \Lambda_j f\right|^2. \tag{2.19}$$

Thus, Theorem 2.4 leads to an immediate consequence as follows.

COROLLARY 2.5. *Let $\{\Lambda_j \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K}_j)\}_{j\in\mathbb{J}}$ be a Parseval g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. Then for any $\mathbb{K} \subset \mathbb{J}$ and any $f \in \mathscr{H}$, we have*

$$0 \leqslant \sum_{j\in\mathbb{K}} |\Lambda_j f|^2 - \left|\sum_{j\in\mathbb{K}} \Lambda_j^* \Lambda_j f\right|^2 \leqslant \frac{1}{4}\langle f, f\rangle. \tag{2.20}$$

$$\frac{1}{2}\langle f, f\rangle \leqslant \left|\sum_{j\in\mathbb{K}} \Lambda_j^* \Lambda_j f\right|^2 + \left|\sum_{j\in\mathbb{K}^c} \Lambda_j^* \Lambda_j f\right|^2 \leqslant \langle f, f\rangle. \tag{2.21}$$

Since each λ-tight g-frame in Hilbert C^*-modules can be turned into a Parseval g-frame by a change of scale, by Corollary 2.5 we immediately obtain the following result.

COROLLARY 2.6. *Let $\{\Lambda_j \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H},\mathscr{K}_j)\}_{j\in\mathbb{J}}$ be a λ-tight g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. Then for any $\mathbb{K}\subset\mathbb{J}$ and any $f\in\mathscr{H}$, we have*

$$0 \leqslant \lambda \sum_{j\in\mathbb{K}} |\Lambda_j f|^2 - \Big|\sum_{j\in\mathbb{K}} \Lambda_j^*\Lambda_j f\Big|^2 \leqslant \frac{\lambda^2}{4}\langle f,f\rangle. \tag{2.22}$$

$$\frac{\lambda^2}{2}\langle f,f\rangle \leqslant \Big|\sum_{j\in\mathbb{K}} \Lambda_j^*\Lambda_j f\Big|^2 + \Big|\sum_{j\in\mathbb{K}^c} \Lambda_j^*\Lambda_j f\Big|^2 \leqslant \lambda^2\langle f,f\rangle. \tag{2.23}$$

We next give equalities and inequalities for g-frames in Hilbert C^*-modules with the Moore–Penrose inverses of adjointable operators.

THEOREM 2.7. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. Let $\{\Gamma_j\}_{j\in\mathbb{J}}$ and $\{\Theta_j\}_{j\in\mathbb{J}}$ be g-Bessel sequences for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. Suppose that the operator L defined by (2.1) has closed range. Then for all $f\in\mathscr{H}$ we have*

$$\begin{aligned}\sum_{j\in\mathbb{J}}\langle\Lambda_j f,\Theta_j(L^\dagger)^*Lf\rangle + \Big\langle \sum_{j\in\mathbb{J}} L^\dagger(\Gamma_j-\Theta_j)^*\Lambda_j f, \sum_{j\in\mathbb{J}}(\Gamma_j-\Theta_j)^*\Lambda_j f\Big\rangle \\ = \sum_{j\in\mathbb{J}}\langle(\Gamma_j-\Theta_j)L^\dagger Lf,\Lambda_j f\rangle + \Big\langle \sum_{j\in\mathbb{J}} L^\dagger\Theta_j^*\Lambda_j f, \sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f\Big\rangle.\end{aligned}$$

Moreover, if L and $LL^\dagger Q$ are positive and self-adjoint, then for all $f\in\mathscr{H}$,

$$\sum_{j\in\mathbb{J}}\langle\Lambda_j f,\Theta_j(L^\dagger)^*Lf\rangle + \Big\langle \sum_{j\in\mathbb{J}} L^\dagger(\Gamma_j-\Theta_j)^*\Lambda_j f, \sum_{j\in\mathbb{J}}(\Gamma_j-\Theta_j)^*\Lambda_j f\Big\rangle \geqslant \frac{3}{4}\langle Lf,f\rangle,$$

where the operator Q is defined by (2.2).

Proof. Let $Pf=\sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f$ for $f\in\mathscr{H}$ and Q be defined by (2.2). Clearly

$$Pf+Qf=\sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f+\sum_{j\in\mathbb{J}}(\Gamma_j-\Theta_j)^*\Lambda_j f=\sum_{j\in\mathbb{J}}\Gamma_j^*\Lambda_j f=Lf.$$

By Lemma 2.3, we obtain

$$\begin{aligned}&\sum_{j\in\mathbb{J}}\langle\Lambda_j f,\Theta_j(L^\dagger)^*Lf\rangle + \Big\langle \sum_{j\in\mathbb{J}} L^\dagger(\Gamma_j-\Theta_j)^*\Lambda_j f, \sum_{j\in\mathbb{J}}(\Gamma_j-\Theta_j)^*\Lambda_j f\Big\rangle \\ &= \langle Pf,(L^\dagger)^*Lf\rangle + \langle L^\dagger Qf,Qf\rangle = \langle L^*L^\dagger Pf,f\rangle + \langle Q^*L^\dagger Qf,f\rangle \\ &= \langle (L^*L^\dagger P+Q^*L^\dagger Q)f,f\rangle = \langle (Q^*L^\dagger L+P^*L^\dagger P)f,f\rangle \\ &= \sum_{j\in\mathbb{J}}\langle(\Gamma_j-\Theta_j)L^\dagger Lf,\Lambda_j f\rangle + \Big\langle \sum_{j\in\mathbb{J}} L^\dagger\Theta_j^*\Lambda_j f, \sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f\Big\rangle.\end{aligned}$$

By Lemma 2.3 again, we have

$$\begin{aligned}&\sum_{j\in\mathbb{J}}\langle\Lambda_j f,\Theta_j(L^\dagger)^*Lf\rangle + \Big\langle \sum_{j\in\mathbb{J}} L^\dagger(\Gamma_j-\Theta_j)^*\Lambda_j f, \sum_{j\in\mathbb{J}}(\Gamma_j-\Theta_j)^*\Lambda_j f\Big\rangle \\ &= \langle Pf,(L^\dagger)^*Lf\rangle + \langle L^\dagger Qf,Qf\rangle = \langle (L^*L^\dagger P+Q^*L^\dagger Q)f,f\rangle \geqslant \frac{3}{4}\langle Lf,f\rangle. \qquad \square\end{aligned}$$

COROLLARY 2.8. *Let $\{\Lambda_j \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H},\mathscr{K}_j)\}_{j\in\mathbb{J}}$ be a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$, and $\{\widetilde{\Lambda}_j\}_{j\in\mathbb{J}}$ be the canonical dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$, then for any $\mathbb{K}\subset\mathbb{J}$ and any $f\in\mathscr{H}$, we have*

$$\sum_{j\in\mathbb{K}}|\Lambda_j f|^2+\sum_{j\in\mathbb{J}}|\widetilde{\Lambda}_j S_{\mathbb{K}^c}f|^2=\sum_{j\in\mathbb{K}^c}|\Lambda_j f|^2+\sum_{j\in\mathbb{J}}|\widetilde{\Lambda}_j S_{\mathbb{K}}f|^2\geqslant\frac{3}{4}\sum_{j\in\mathbb{J}}|\Lambda_j f|^2.$$

Proof. For each $j\in\mathbb{J}$, set $\Gamma_j=\Lambda_j$. Let S be the g-frame operator of $\{\Lambda_j\}_{j\in\mathbb{J}}$, then $L=S$ and $L^\dagger=S^{-1}$. For each $j\in\mathbb{J}$, let

$$\Theta_j=\begin{cases}\Lambda_j & \text{if } j\in\mathbb{K},\\ 0 & \text{if } j\in\mathbb{K}^c.\end{cases}$$

Then, clearly, $\{\Theta_j\}_{j\in\mathbb{J}}$ is a g-Bessel sequence for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$, and $(\Gamma_j-\Theta_j)$ has a form as follows:

$$\Gamma_j-\Theta_j=\begin{cases}0 & \text{if } j\in\mathbb{K},\\ \Lambda_j & \text{if } j\in\mathbb{K}^c.\end{cases}$$

For each $f\in\mathscr{H}$ we have

$$\begin{aligned}&\sum_{j\in\mathbb{J}}\langle\Lambda_j f,\Theta_j(L^\dagger)^*Lf\rangle+\left\langle\sum_{j\in\mathbb{J}}L^\dagger(\Gamma_j-\Theta_j)^*\Lambda_j f,\sum_{j\in\mathbb{J}}(\Gamma_j-\Theta_j)^*\Lambda_j f\right\rangle\\&\quad=\sum_{j\in\mathbb{J}}\langle\Lambda_j f,\Theta_j f\rangle+\langle SS^{-1}S_{\mathbb{K}^c}f,S^{-1}S_{\mathbb{K}^c}f\rangle\\&\quad=\sum_{j\in\mathbb{K}}|\Lambda_j f|^2+\sum_{j\in\mathbb{J}}|\widetilde{\Lambda}_j S_{\mathbb{K}^c}f|^2.\end{aligned}$$

Similarly, we have

$$\sum_{j\in\mathbb{J}}\langle(\Gamma_j-\Theta_j)L^\dagger Lf,\Lambda_j f\rangle+\left\langle\sum_{j\in\mathbb{J}}L^\dagger\Theta_j^*\Lambda_j f,\sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f\right\rangle=\sum_{j\in\mathbb{K}^c}|\Lambda_j f|^2+\sum_{j\in\mathbb{J}}|\widetilde{\Lambda}_j S_{\mathbb{K}}f|^2.$$

Hence, by Theorem 2.7, we obtain

$$\begin{aligned}\sum_{j\in\mathbb{K}}|\Lambda_j f|^2+\sum_{j\in\mathbb{J}}|\widetilde{\Lambda}_j S_{\mathbb{K}^c}f|^2&=\sum_{j\in\mathbb{K}^c}|\Lambda_j f|^2+\sum_{j\in\mathbb{J}}|\widetilde{\Lambda}_j S_{\mathbb{K}}f|^2\\&\geqslant\frac{3}{4}\langle Sf,f\rangle=\frac{3}{4}\sum_{j\in\mathbb{J}}|\Lambda_j f|^2.\quad\square\end{aligned}$$

Let $\{\Lambda_j\in\mathrm{End}^*_{\mathscr{A}}(\mathscr{H},\mathscr{K}_j)\}_{j\in\mathbb{J}}$ be a Parseval g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$, then $S=L=L^\dagger=\mathrm{Id}_{\mathscr{H}}$. Hence, combination of formulas (2.18), (2.19) and Corollary 2.8 yields:

COROLLARY 2.9. *Let $\{\Lambda_j\in\mathrm{End}^*_{\mathscr{A}}(\mathscr{H},\mathscr{K}_j)\}_{j\in\mathbb{J}}$ be a Parseval g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$, then for any $\mathbb{K}\subset\mathbb{J}$ and any $f\in\mathscr{H}$, we have*

$$\sum_{j\in\mathbb{K}}|\Lambda_j f|^2+\left|\sum_{j\in\mathbb{K}^c}\Lambda_j^*\Lambda_j f\right|^2=\sum_{j\in\mathbb{K}^c}|\Lambda_j f|^2+\left|\sum_{j\in\mathbb{K}}\Lambda_j^*\Lambda_j f\right|^2\geqslant\frac{3}{4}\langle f,f\rangle.$$

REMARK 2.10. Corollaries 2.8 and 2.9 are respectively Theorems 4.1 and 4.2 in [12].

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