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A REMARK ON "INEQUALITIES FOR THE FROBENIUS NORM"

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Abstract. In a recent paper, a refinement of Heinz inequality was shown and compared with an inequality obtained by Kittaneh and Manasrah. This paper shows that the refinement is a trivial result and that the comparison is not proper.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices and $||\cdot||_F$ denote the Frobenius norm on M_n, that is, $||A||_F = (\sum_{i,j=1}^n |a_{ij}|^2)^{1/2}$ for $A = [a_{ij}] \in M_n$. Kittaneh and Manasrah [1, Theorem 3.4] proved that if $0 \leqslant v \leqslant 1$ and $A, B \in M_n$ are positive semidefinite, then

$$f(v) + 2r_0(\sqrt{||AX||_F} - \sqrt{||XB||_F})^2 \leqslant ||AX + XB||_F, \tag{1.1}$$

where $f(v) = ||A^v X B^{1-v} + A^{1-v} X B^v||_F$ and $r_0 = \min\{v, 1-v\}$. The inequality is a refinement of Heinz inequality

$$f(v) \leqslant ||AX + XB||_F.$$

Recently, the author of [2] showed the following refinement of Heinz inequality

$$f(v) + 4r_0 \left(\int_0^1 f(v)dv - 2||A^{1/2} X B^{1/2}||_F \right) \leqslant ||AX + XB||_F \tag{1.2}$$

and compared it with (1.1). In this paper, we show the following:

a) (1.2) follows directly from the proof of [1, Theorem 3.4].

b) Comparing (1.2) with (1.1) is not meaningful.

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2. Discussion

The two papers [1] and [2] use the functional notation $f(v)$ differently, but we will use $f(v)$ to denote $||AX+XB||_F - ||A^vXB^{1-v}+A^{1-v}XB^v||_F$ as in [1]. Then (1.1) and (1.2) can be written as

$$2r_0(\sqrt{||AX||_F} - \sqrt{||XB||_F})^2 \leqslant f(v) \tag{2.1}$$

and

$$4r_0\left(f(\frac{1}{2}) - \overline{f}\right) \leqslant f(v), \tag{2.2}$$

respectively, where $\overline{f} = \int_0^1 f(v)dv$.

The proof of [1, Theorem 3.4] shows

$$2r_0 f(\frac{1}{2}) \leqslant f(v) \tag{2.3}$$

and obtains (2.1) using the fact $(\sqrt{||AX||_F} - \sqrt{||XB||_F})^2 \leqslant f(\frac{1}{2})$ [1, Theorem 3.3]. By (2.3), it is clear $f(\frac{1}{2}) \leqslant 2\overline{f}$ and thus (2.2) follows from the relation

$$4r_0\left(f(\frac{1}{2}) - \overline{f}\right) \leqslant 2r_0 f(\frac{1}{2}).$$

Moreover, [2, Theorem 2.2] compares (2.1) and (2.2), but to show that (2.2) is a new kind of Heinz inequality, it needs to be compared with (2.3). Since

$$2r_0 f(\frac{1}{2}) \leqslant 4r_0\left(f(\frac{1}{2}) - \overline{f}\right) \leqslant f(v),$$

(2.3) is uniformly better than (2.2).

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