

## SYMMETRIZED CONVEXITY AND HERMITE–HADAMARD TYPE INEQUALITIES

S. S. DRAGOMIR

(Communicated by M. Klaričić Bakula)

*Abstract.* In this paper we extend the Hermite-Hadamard inequality to the class of symmetrized convex functions. The corresponding version for  $h$ -convex functions is also investigated. Some examples of interest are provided as well.

### 1. Introduction

The following inequality holds for any convex function  $f$  defined on  $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R}, \quad a \neq b. \quad (1.1)$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [43]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [43]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [10]–[19], [21]–[24], [31]–[34] and [46].

In this paper we show that the Hermite-Hadamard inequality can be extended to a larger class of functions containing the convex functions. The corresponding version for  $h$ -convex functions is also investigated. Some examples of interest are provided as well.

---

*Mathematics subject classification* (2010): 26D15, 25D10.

*Keywords and phrases:* Convex functions, Hermite-Hadamard inequality, integral inequalities,  $h$ -convex functions.

### 2. Symmetrized convexity

For a function  $f : [a, b] \rightarrow \mathbb{C}$  we consider the *symmetrical transform of  $f$*  on the interval  $[a, b]$ , denoted by  $\check{f}_{[a,b]}$  or simply  $\check{f}$ , when the interval  $[a, b]$  is implicit, which is defined by

$$\check{f}(t) := \frac{1}{2} [f(t) + f(a + b - t)], \quad t \in [a, b].$$

The *anti-symmetrical transform of  $f$*  on the interval  $[a, b]$  is denoted by  $\tilde{f}_{[a,b]}$ , or simply  $\tilde{f}$  and is defined by

$$\tilde{f}(t) := \frac{1}{2} [f(t) - f(a + b - t)], t \in [a, b].$$

It is obvious that for any function  $f$  we have  $\check{f} + \tilde{f} = f$ .

If  $f$  is convex on  $[a, b]$ , then for any  $t_1, t_2 \in [a, b]$  and  $\alpha, \beta \geq 0$  with  $\alpha + \beta = 1$  we have

$$\begin{aligned} \check{f}(\alpha t_1 + \beta t_2) &= \frac{1}{2} [f(\alpha t_1 + \beta t_2) + f(a + b - \alpha t_1 - \beta t_2)] \\ &= \frac{1}{2} [f(\alpha t_1 + \beta t_2) + f(\alpha(a + b - t_1) + \beta(a + b - t_2))] \\ &\leq \frac{1}{2} [\alpha f(t_1) + \beta f(t_2) + \alpha f(a + b - t_1) + \beta f(a + b - t_2)] \\ &= \frac{1}{2} \alpha [f(t_1) + f(a + b - t_1)] + \frac{1}{2} \beta [f(t_2) + f(a + b - t_2)] \\ &= \alpha \check{f}(t_1) + \beta \check{f}(t_2), \end{aligned}$$

which shows that  $\check{f}$  is convex on  $[a, b]$ .

Consider the real numbers  $a < b$  and define the function  $f_0 : [a, b] \rightarrow \mathbb{R}, f_0(t) = t^3$ . We have

$$\check{f}_0(t) := \frac{1}{2} [t^3 + (a + b - t)^3] = \frac{3}{2}(a + b)t^2 - \frac{3}{2}(a + b)^2t + \frac{1}{2}(a + b)^3$$

for any  $t \in \mathbb{R}$ .

Since the second derivative  $(\check{f}_0)''(t) = 3(a + b), t \in \mathbb{R}$ , then  $\check{f}_0$  is strictly convex on  $[a, b]$  if  $\frac{a+b}{2} > 0$  and strictly concave on  $[a, b]$  if  $\frac{a+b}{2} < 0$ . Therefore if  $a < 0 < b$  with  $\frac{a+b}{2} > 0$ , then we can conclude that  $f_0$  is not convex on  $[a, b]$  while  $\check{f}_0$  is convex on  $[a, b]$ .

We can introduce the following concept of convexity.

**DEFINITION 1.** We say that the function  $f : [a, b] \rightarrow \mathbb{R}$  is symmetrized convex (concave) on the interval  $[a, b]$  if the *symmetrical transform  $\check{f}$*  is convex (concave) on  $[a, b]$ .

Now, if we denote by  $Con[a, b]$  the closed convex cone of convex functions defined on  $[a, b]$  and by  $SCon[a, b]$  the class of symmetrized convex functions, then from the above remarks we can conclude that

$$Con[a, b] \subsetneq SCon[a, b]. \tag{2.1}$$

Also, if  $[c, d] \subset [a, b]$  and  $f \in SCon[a, b]$ , then this does not imply in general that  $f \in SCon[c, d]$ .

**THEOREM 1.** *Assume that  $f : [a, b] \rightarrow \mathbb{R}$  is symmetrized convex on the interval  $[a, b]$ . Then we have the Hermite-Hadamard inequalities*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{2.2}$$

*Proof.* Since  $f : [a, b] \rightarrow \mathbb{R}$  is symmetrized convex on the interval  $[a, b]$ , then by writing the Hermite-Hadamard inequality for the function  $\check{f}$  we have

$$\check{f}\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \check{f}(t)dt \leq \frac{\check{f}(a)+\check{f}(b)}{2}. \tag{2.3}$$

However

$$\check{f}\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right), \quad \frac{\check{f}(a)+\check{f}(b)}{2} = \frac{f(a)+f(b)}{2},$$

and

$$\int_a^b \check{f}(t)dt = \frac{1}{2} \int_a^b [f(t) + f(a+b-t)]dt = \int_a^b f(t)dt.$$

Then by (2.3) we get (2.2).  $\square$

For similar results see [36].

The following result holds:

**THEOREM 2.** *Assume that  $f : [a, b] \rightarrow \mathbb{R}$  is symmetrized convex on the interval  $[a, b]$ . Then for any  $x \in [a, b]$  we have the bounds*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2} [f(x) + f(a+b-x)] \leq \frac{f(a)+f(b)}{2}. \tag{2.4}$$

*Proof.* Since  $\check{f}$  is convex on  $[a, b]$  then for any  $x \in [a, b]$  we have

$$\frac{\check{f}(x) + \check{f}(a+b-x)}{2} \geq \check{f}\left(\frac{a+b}{2}\right)$$

and since

$$\frac{\check{f}(x) + \check{f}(a+b-x)}{2} = \frac{1}{2} [f(x) + f(a+b-x)]$$

while

$$\check{f}\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right),$$

we get the first inequality in (2.4).

Also, by the convexity of  $\check{f}$  we have for any  $x \in [a, b]$  that

$$\begin{aligned} \check{f}(x) &\leq \frac{x-a}{b-a} \cdot \check{f}(b) + \frac{b-x}{b-a} \cdot \check{f}(a) \\ &= \frac{x-a}{b-a} \cdot \frac{f(a)+f(b)}{2} + \frac{b-x}{b-a} \cdot \frac{f(a)+f(b)}{2} \\ &= \frac{f(a)+f(b)}{2}, \end{aligned}$$

which proves the second part of (2.4).  $\square$

REMARK 1. If  $f : [a, b] \rightarrow \mathbb{R}$  is symmetrized convex on the interval  $[a, b]$ , then we have the bounds

$$\inf_{x \in [a, b]} \check{f}(x) = \check{f}\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right)$$

and

$$\sup_{x \in [a, b]} \check{f}(x) = \check{f}(a) = \check{f}(b) = \frac{f(a)+f(b)}{2}.$$

COROLLARY 1. If  $f : [a, b] \rightarrow \mathbb{R}$  is symmetrized convex on the interval  $[a, b]$  and  $w : [a, b] \rightarrow [0, \infty)$  is integrable on  $[a, b]$ , then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \int_a^b w(t) dt &\leq \frac{1}{2} \int_a^b w(t) [f(t) + f(a+b-t)] dt \\ &\leq \frac{f(a)+f(b)}{2} \int_a^b w(t) dt. \end{aligned} \quad (2.5)$$

Moreover, if  $w$  is symmetric almost everywhere on  $[a, b]$ , i.e.  $w(t) = w(a+b-t)$  for almost every  $t \in [a, b]$ , then

$$f\left(\frac{a+b}{2}\right) \int_a^b w(t) dt \leq \int_a^b w(t) f(t) dt \leq \frac{f(a)+f(b)}{2} \int_a^b w(t) dt. \quad (2.6)$$

*Proof.* The inequality (2.5) follows by (2.4) written for  $x = t$ , multiplying by  $w(t) \geq 0$  and integrating over  $t$  on  $[a, b]$ .

By changing the variable, we have

$$\int_a^b w(t) f(a+b-t) dt = \int_a^b w(a+b-t) f(t) dt.$$

Since  $w$  is symmetric almost everywhere on  $[a, b]$ , then

$$\int_a^b w(a+b-t)f(t)dt = \int_a^b w(t)f(t)dt.$$

Therefore

$$\begin{aligned} & \frac{1}{2} \int_a^b w(t) [f(t) + f(a+b-t)] dt \\ &= \frac{1}{2} \left[ \int_a^b w(t)f(t)dt + \int_a^b w(t)f(a+b-t)dt \right] \\ &= \frac{1}{2} \left[ \int_a^b w(t)f(t)dt + \int_a^b w(t)f(t)dt \right] = \int_a^b w(t)f(t)dt \end{aligned}$$

and by (2.5) we get (2.6).  $\square$

REMARK 2. The inequality (2.6) was obtained by L. Fejér in 1906 for convex functions  $f$  and symmetric weights  $w$ . It has been shown now that this inequality remains valid for the larger class of symmetrized convex functions  $f$  on the interval  $[a, b]$ .

The following result also holds.

THEOREM 3. Assume that  $f : [a, b] \rightarrow \mathbb{R}$  is symmetrized convex on the interval  $[a, b]$ . Then for any  $x, y \in [a, b]$  with  $x \neq y$  we have the Hermite-Hadamard inequalities

$$\begin{aligned} & \frac{1}{2} \left[ f\left(\frac{x+y}{2}\right) + f\left(a+b-\frac{x+y}{2}\right) \right] \tag{2.7} \\ & \leq \frac{1}{2(y-x)} \left[ \int_x^y f(t)dt + \int_{a+b-y}^{a+b-x} f(t)dt \right] \\ & \leq \frac{1}{4} [f(x) + f(a+b-x) + f(y) + f(a+b-y)]. \end{aligned}$$

*Proof.* Since  $\check{f}_{[a,b]}$  is convex on  $[a, b]$ , then  $\check{f}_{[a,b]}$  is also convex on any subinterval  $[x, y]$  (or  $[y, x]$ ) where  $x, y \in [a, b]$ .

By Hermite-Hadamard inequalities for convex functions we have

$$\check{f}_{[a,b]}\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y \check{f}_{[a,b]}(t)dt \leq \frac{\check{f}_{[a,b]}(x) + \check{f}_{[a,b]}(y)}{2} \tag{2.8}$$

for any  $x, y \in [a, b]$  with  $x \neq y$ .

We have

$$\check{f}_{[a,b]}\left(\frac{x+y}{2}\right) = \frac{1}{2} \left[ f\left(\frac{x+y}{2}\right) + f\left(a+b-\frac{x+y}{2}\right) \right],$$

$$\begin{aligned} \int_x^y \check{f}_{[a,b]}(t)dt &= \frac{1}{2} \int_x^y [f(t) + f(a+b-t)] dt \\ &= \frac{1}{2} \int_x^y f(t)dt + \frac{1}{2} \int_x^y f(a+b-t) dt \\ &= \frac{1}{2} \int_x^y f(t)dt + \frac{1}{2} \int_{a+b-y}^{a+b-x} f(t)dt \end{aligned}$$

and

$$\frac{\check{f}_{[a,b]}(x) + \check{f}_{[a,b]}(y)}{2} = \frac{1}{4} [f(x) + f(a+b-x) + f(y) + f(a+b-y)].$$

Then by (2.8) we deduce the desired result (2.7).  $\square$

REMARK 3. If we take  $x = a$  and  $y = b$  in (2.7), then we get (2.2).

If, for a given  $x \in [a, b]$ , we take  $y = a + b - x$ , then from (2.7) we get

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2\left(\frac{a+b}{2}-x\right)} \int_x^{a+b-x} f(t)dt \leq \frac{1}{2} [f(x) + f(a+b-x)], \tag{2.9}$$

where  $x \neq \frac{a+b}{2}$ , provided that  $f : [a, b] \rightarrow \mathbb{R}$  is symmetrized convex on the interval  $[a, b]$ .

Integrating this inequality over  $x$  we get the following refinement of the first part of (2.2)

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2(b-a)} \int_a^b \left[ \frac{1}{\left(\frac{a+b}{2}-x\right)} \int_x^{a+b-x} f(t)dt \right] dx \\ &\leq \frac{1}{b-a} \int_a^b f(t)dt, \end{aligned} \tag{2.10}$$

provided that  $f : [a, b] \rightarrow \mathbb{R}$  is symmetrized convex on the interval  $[a, b]$ .

When the function is convex, we have the following inequalities as well:

REMARK 4. If  $f : [a, b] \rightarrow \mathbb{R}$  is convex, then from (2.7) we have the inequalities

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2} \left[ f\left(\frac{x+y}{2}\right) + f\left(a+b-\frac{x+y}{2}\right) \right] \\ &\leq \frac{1}{2(y-x)} \left[ \int_x^y f(t)dt + \int_{a+b-y}^{a+b-x} f(t)dt \right] \\ &\leq \frac{1}{4} [f(x) + f(a+b-x) + f(y) + f(a+b-y)] \end{aligned} \tag{2.11}$$

for any  $x, y \in [a, b]$ ,  $x \neq y$ .

If we integrate (2.11) over  $(x, y)$  on the square  $[a, b]^2$  and divide by  $(b-a)^2$ , then we get the following refinement of the first Hermite-Hadamard inequality for convex

functions

$$\begin{aligned}
 & f\left(\frac{a+b}{2}\right) \tag{2.12} \\
 & \leq \frac{1}{2(b-a)^2} \left[ \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dx dy + \int_a^b \int_a^b f\left(a+b-\frac{x+y}{2}\right) dx dy \right] \\
 & \leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b \frac{1}{y-x} \left[ \int_x^y f(t) dt + \int_{a+b-y}^{a+b-x} f(t) dt \right] dx dy \\
 & \leq \frac{1}{b-a} \int_a^b f(t) dt.
 \end{aligned}$$

We notice that, the second and the third inequalities also hold for the more general case of symmetrized convex functions on the interval  $[a, b]$ .

A concept of weaker symmetrized convexity can be introduced as follows:

DEFINITION 2. We say that the function  $f : [a, b] \rightarrow \mathbb{R}$  is weak symmetrized convex (concave) on the interval  $[a, b]$  if the *symmetrical transform*  $\check{f}$  is convex (concave) on the interval  $\left[a, \frac{a+b}{2}\right]$ .

We denote this class by  $W\mathit{SCon}[a, b]$ .

It is clear that any symmetrized convex function on  $[a, b]$  is weak symmetrized convex on that interval. Also, there are weak symmetrized convex functions on  $[a, b]$  that are not symmetrized convex on  $[a, b]$ .

If we consider the function  $f_0 : [a, b] \rightarrow \mathbb{R}$  defined by

$$f_0(t) = \begin{cases} t^2, & t \in \left[a, \frac{a+b}{2}\right], \\ (a+b-t)^2, & t \in \left(\frac{a+b}{2}, b\right], \end{cases}$$

then we observe that  $f_0$  is convex on  $\left[a, \frac{a+b}{2}\right]$  and  $\left[\frac{a+b}{2}, b\right]$  but not convex on the whole interval  $[a, b]$ . We also observe that  $\check{f}_0$  is a symmetrical function on  $[a, b]$  and then  $\check{f}_0 = f_0$ . Therefore  $f_0$  is weak symmetrized convex function on  $[a, b]$  but not symmetrized convex on that interval.

We have the following strict inclusion

$$\mathit{SCon}[a, b] \subsetneq W\mathit{SCon}[a, b]. \tag{2.13}$$

We also notice that  $f$  is weak symmetrized convex function on  $[a, b]$  if and only if  $\check{f}$  is convex on the second half of the interval  $[a, b]$ , namely  $\left[\frac{a+b}{2}, b\right]$ .

THEOREM 4. Assume that  $f : [a, b] \rightarrow \mathbb{R}$  is weak symmetrized convex on the interval  $[a, b]$ . Then for any  $x, y \in \left[a, \frac{a+b}{2}\right]$   $x \neq y$  we have the Hermite-Hadamard inequalities (2.7).

In particular, we have

$$\begin{aligned} \frac{1}{2} \left[ f \left( \frac{3a+b}{4} \right) + f \left( \frac{a+3b}{4} \right) \right] &\leq \frac{1}{b-a} \int_a^b f(t) dt \\ &\leq \frac{1}{2} \left[ f \left( \frac{a+b}{2} \right) + \frac{f(a)+f(b)}{2} \right]. \end{aligned} \tag{2.14}$$

*Proof.* The first part follows from the proof of Theorem 3 for  $x, y \in [a, \frac{a+b}{2}]$ . The second part follows from the inequality (2.7) by taking  $x = a$  and  $y = \frac{a+b}{2}$ .  $\square$

REMARK 5. We observe that if  $f : [a, b] \rightarrow \mathbb{R}$  is weak symmetrized convex on the interval  $[a, b]$ , then the inequality (2.9) holds for any  $x \in [a, \frac{a+b}{2}]$  and integrating on  $[a, \frac{a+b}{2}]$  we also have

$$\begin{aligned} f \left( \frac{a+b}{2} \right) &\leq \frac{1}{2(b-a)} \int_a^{\frac{a+b}{2}} \left[ \frac{1}{(\frac{a+b}{2} - x)} \int_x^{a+b-x} f(t) dt \right] dx \\ &\leq \frac{1}{b-a} \int_a^b f(t) dt. \end{aligned} \tag{2.15}$$

We can state in general the following result for symmetrized convex functions.

PROPOSITION 1. Any inequality that holds for convex functions  $f$  defined on the interval  $[a, b]$  will hold for symmetrized convex functions by replacing  $f$  with  $\check{f}_{[a,b]}$  and performing the required calculations.

We can illustrate this fact with two simple examples.

It is known that, see [19], if  $f : [a, b] \rightarrow \mathbb{R}$  is differentiable convex on  $(a, b)$ , then for any  $x, y \in (a, b)$  with  $x \neq y$  we have

$$0 \leq \frac{1}{y-x} \int_x^y f(t) - f \left( \frac{x+y}{2} \right) \leq \frac{1}{8} (f'(y) - f'(x)) (y-x). \tag{2.16}$$

Now, if  $f : [a, b] \rightarrow \mathbb{R}$  is differentiable and symmetrized convex on  $(a, b)$ , then by writing (2.16) for  $\check{f}_{[a,b]}$  we have

$$\begin{aligned} 0 &\leq \frac{1}{y-x} \int_x^y \check{f}_{[a,b]}(t) - \check{f}_{[a,b]} \left( \frac{x+y}{2} \right) \\ &\leq \frac{1}{8} \left( (\check{f}_{[a,b]})'(y) - (\check{f}_{[a,b]})'(x) \right) (y-x). \end{aligned} \tag{2.17}$$

However

$$\begin{aligned} \frac{1}{y-x} \int_x^y \check{f}_{[a,b]}(t) &= \frac{1}{2(y-x)} \left[ \int_x^y f(t) dt + \int_{a+b-y}^{a+b-x} f(t) dt \right], \\ \check{f}_{[a,b]} \left( \frac{x+y}{2} \right) &= \frac{1}{2} \left[ f \left( \frac{x+y}{2} \right) + f \left( a+b - \frac{x+y}{2} \right) \right] \end{aligned}$$



and

$$(\check{f}_{[a,b]})'(y) - (\check{f}_{[a,b]})'(x) = \frac{1}{2} (f'(y) - f'(a+b-y) - f'(x) + f'(a+b-x)).$$

Then by (2.17) we get

$$\begin{aligned} 0 &\leq \frac{1}{2(y-x)} \left[ \int_x^y f(t)dt + \int_{a+b-y}^{a+b-x} f(t)dt \right] \\ &\quad - \frac{1}{2} \left[ f\left(\frac{x+y}{2}\right) + f\left(a+b - \frac{x+y}{2}\right) \right] \\ &\leq \frac{1}{16} [f'(y) - f'(a+b-y) - f'(x) + f'(a+b-x)] (y-x) \end{aligned} \tag{2.18}$$

that holds for any  $x, y \in (a, b)$  with  $x \neq y$ .

From this inequality, by taking  $y = a + b - x$ , we get

$$\begin{aligned} 0 &\leq \frac{1}{2\left(\frac{a+b}{2} - x\right)} \int_x^{a+b-x} f(t) dt - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{4} [f'(a+b-x) - f'(x)] \left(\frac{a+b}{2} - x\right) \end{aligned} \tag{2.19}$$

for any  $x \in (a, b)$  with  $x \neq \frac{a+b}{2}$ .

If  $f : [a, b] \rightarrow \mathbb{R}$  is differentiable convex on  $(a, b)$ , then for any  $x, y \in (a, b)$  with  $x \neq y$  we also have [20]

$$0 \leq \frac{f(x) + f(y)}{2} - \frac{1}{y-x} \int_x^y f(t) \leq \frac{1}{8} (f'(y) - f'(x)) (y-x). \tag{2.20}$$

Now, if  $f : [a, b] \rightarrow \mathbb{R}$  is differentiable and symmetrized convex on  $(a, b)$ , then by a similar argument as above we have

$$\begin{aligned} 0 &\leq \frac{1}{4} [f(x) + f(a+b-x) + f(y) + f(a+b-y)] \\ &\quad - \frac{1}{2(y-x)} \left[ \int_x^y f(t)dt + \int_{a+b-y}^{a+b-x} f(t)dt \right] \\ &\leq \frac{1}{16} [f'(y) - f'(a+b-y) - f'(x) + f'(a+b-x)] (y-x) \end{aligned} \tag{2.21}$$

for any  $x, y \in (a, b)$  with  $x \neq y$ .

In particular, we have

$$\begin{aligned} 0 &\leq \frac{1}{2} [f(x) + f(a+b-x)] - \frac{1}{2\left(\frac{a+b}{2} - x\right)} \int_x^{a+b-x} f(t)dt \\ &\leq \frac{1}{4} [f'(a+b-x) - f'(x)] \left(\frac{a+b}{2} - x\right) \end{aligned} \tag{2.22}$$

for any  $x \in (a, b)$  with  $x \neq \frac{a+b}{2}$ .

### 3. Symmetrized $h$ -convexity

We recall here some concepts of convexity that are well known in the literature. Let  $I$  be an interval in  $\mathbb{R}$ .

DEFINITION 3. ([38]) We say that  $f : I \rightarrow \mathbb{R}$  is a Godunova-Levin function or that  $f$  belongs to the class  $\mathcal{Q}(I)$  if  $f$  is non-negative and for all  $x, y \in I$  and  $t \in (0, 1)$  we have

$$f(tx + (1-t)y) \leq \frac{1}{t}f(x) + \frac{1}{1-t}f(y). \quad (3.1)$$

Some further properties of this class of functions can be found in [27], [28], [30], [44], [47] and [48]. Among others, it has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

DEFINITION 4. ([30]) We say that a function  $f : I \rightarrow \mathbb{R}$  belongs to the class  $P(I)$  if it is nonnegative and for all  $x, y \in I$  and  $t \in [0, 1]$  we have

$$f(tx + (1-t)y) \leq f(x) + f(y). \quad (3.2)$$

Obviously  $\mathcal{Q}(I)$  contains  $P(I)$  and for applications it is important to note that also  $P(I)$  contain all nonnegative monotone, convex and *quasi convex functions*, i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\} \quad (3.3)$$

for all  $x, y \in I$  and  $t \in [0, 1]$ .

For some results on  $P$ -functions see [30] and [45] while for quasi convex functions, the reader can consult [29].

DEFINITION 5. ([7]) Let  $s$  be a real number,  $s \in (0, 1]$ . A function  $f : [0, \infty) \rightarrow [0, \infty)$  is said to be  $s$ -convex (in the second sense) or Breckner  $s$ -convex if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all  $x, y \in [0, \infty)$  and  $t \in [0, 1]$ .

For some properties of this class of functions see [1], [2], [7], [8], [25], [26], [39], [41] and [50].

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of  $h$ -convex functions as follows.

Assume that  $I$  and  $J$  are intervals in  $\mathbb{R}$ ,  $(0, 1) \subseteq J$  and functions  $h$  and  $f$  are real non-negative functions defined in  $J$  and  $I$ , respectively.

DEFINITION 6. ([53]) Let  $h : J \rightarrow [0, \infty)$  with  $h$  not identical to 0. We say that  $f : I \rightarrow [0, \infty)$  is an  $h$ -convex function if for all  $x, y \in I$  we have

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \quad (3.4)$$

for all  $t \in (0, 1)$ .

For some results concerning this class of functions see [53], [6], [42], [51], [49] and [52].

We can introduce now another class of functions.

DEFINITION 7. We say that the function  $f : I \rightarrow [0, \infty)$  is of  $s$ -Godunova-Levin type, with  $s \in [0, 1]$ , if

$$f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{3.5}$$

for all  $t \in (0, 1)$  and  $x, y \in I$ .

We observe that for  $s = 0$  we obtain the class of  $P$ -functions while for  $s = 1$  we obtain the class of Godunova-Levin. If we denote by  $Q_s(I)$  the class of  $s$ -Godunova-Levin functions defined on  $I$ , then we obviously have

$$P(I) = Q_0(I) \subseteq Q_{s_1}(I) \subseteq Q_{s_2}(I) \subseteq Q_1(I) = Q(I)$$

for  $0 \leq s_1 \leq s_2 \leq 1$ .

The following inequality of Hermite-Hadamard type holds [49].

THEOREM 5. Assume that the function  $f : I \rightarrow [0, \infty)$  is an  $h$ -convex function with  $h \in L[0, 1]$ . Let  $y, x \in I$  with  $y \neq x$  and assume that the mapping  $[0, 1] \ni t \mapsto f[(1-t)x + ty]$  is Lebesgue integrable on  $[0, 1]$ . Then

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(u) du \leq [f(x) + f(y)] \int_0^1 h(t) dt. \tag{3.6}$$

If we write (3.6) for  $h(t) = t$ , then we get the classical Hermite-Hadamard inequality for convex functions

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(u) du \leq \frac{f(x) + f(y)}{2}. \tag{3.7}$$

If we write (3.6) for the case of  $P$ -type functions  $f : I \rightarrow [0, \infty)$ , i.e.,  $h(t) = 1, t \in [0, 1]$ , then we get the inequality

$$\frac{1}{2} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(u) du \leq f(x) + f(y), \tag{3.8}$$

that has been obtained for functions of real variable in [30].

If  $f$  is Breckner  $s$ -convex on  $I$ , for  $s \in (0, 1)$ , then by taking  $h(t) = t^s$  in (3.6) we get

$$2^{s-1} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(u) du \leq \frac{f(x) + f(y)}{s+1}, \tag{3.9}$$

that was obtained for functions of a real variable in [25].

If  $f : I \rightarrow [0, \infty)$  is of  $s$ -Godunova-Levin type, with  $s \in [0, 1)$ , then

$$\frac{1}{2^{s+1}} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(u) du \leq \frac{f(x) + f(y)}{1-s}. \tag{3.10}$$

We notice that for  $s = 1$  the first inequality in (3.10) still holds [30], i.e.

$$\frac{1}{4} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(u) du. \tag{3.11}$$

We can introduce the following concept generalizing the notion of  $h$ -convexity.

**DEFINITION 8.** Assume that  $h$  is as in Definition 6. We say that the function  $f : [a, b] \rightarrow [0, \infty)$  is  $h$ -symmetrized convex (concave) on the interval  $[a, b]$  if the symmetrical transform  $\check{f}$  is  $h$ -convex (concave) on  $[a, b]$ .

Now, if we denote by  $Con_h[a, b]$  the closed convex cone of  $h$ -convex functions defined on  $[a, b]$  and by  $SCon_h[a, b]$  the class of  $h$ -symmetrized convex, then, as in the previous section, we can conclude in general that

$$Con_h[a, b] \subsetneq SCon_h[a, b]. \tag{3.12}$$

**DEFINITION 9.** Assume that  $h$  is as in Definition 6. We say that the function  $f : [a, b] \rightarrow \mathbb{R}$  is  $h$ -weak symmetrized convex (concave) on the interval  $[a, b]$  if the symmetrical transform  $\check{f}$  is  $h$ -convex (concave) on the interval  $[a, \frac{a+b}{2}]$ .

We denote this class by  $WCon_h[a, b]$ . As in the previous section, we can conclude in general that

$$SCon_h[a, b] \subsetneq WCon_h[a, b]. \tag{3.13}$$

Utilising Theorem 5 and a similar proof to that of Theorem 3, we can state the following result as well:

**THEOREM 6.** Assume that the function  $f : [a, b] \rightarrow [0, \infty)$  is  $h$ -symmetrized convex on the interval  $[a, b]$  with  $h$  integrable on  $[0, 1]$  and  $f$  integrable on  $[a, b]$ . Then for any  $x, y \in [a, b]$  we have the Hermite-Hadamard inequalities

$$\begin{aligned} & \frac{1}{4h\left(\frac{1}{2}\right)} \left[ f\left(\frac{x+y}{2}\right) + f\left(a+b - \frac{x+y}{2}\right) \right] \\ & \leq \frac{1}{2(y-x)} \left[ \int_x^y f(t) dt + \int_{a+b-y}^{a+b-x} f(t) dt \right] \\ & \leq \frac{1}{2} [f(x) + f(a+b-x) + f(y) + f(a+b-y)] \int_0^1 h(t) dt. \end{aligned} \tag{3.14}$$

In particular, we have

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq [f(a) + f(b)] \int_0^1 h(t) dt. \tag{3.15}$$

REMARK 6. If, for a given  $x \in [a, b]$ , we take  $y = a + b - x$ , then from (3.14) we get

$$\begin{aligned} \frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2\left(\frac{a+b}{2}-x\right)}\int_x^{a+b-x}f(t)dt \\ &\leq [f(x)+f(a+b-x)]\int_0^1h(t)dt, \end{aligned} \tag{3.16}$$

where  $x \neq \frac{a+b}{2}$ , provided that  $f : [a, b] \rightarrow \mathbb{R}$  is  $h$ -symmetrized convex and integrable on the interval  $[a, b]$ .

Integrating on  $[a, b]$  over  $x$  we get

$$\begin{aligned} \frac{1}{4h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right) &\leq \frac{1}{4(b-a)}\int_a^b\left[\frac{1}{\left(\frac{a+b}{2}-x\right)}\int_x^{a+b-x}f(t)dt\right]dx \\ &\leq \frac{1}{b-a}\int_a^bf(x)dx\int_0^1h(t)dt. \end{aligned} \tag{3.17}$$

We have the following result as well:

THEOREM 7. Assume that  $h$  is as in Definition 6. If the function  $f : [a, b] \rightarrow [0, \infty)$  is  $h$ -symmetrized convex on the interval  $[a, b]$ , then we have the bounds

$$\begin{aligned} \frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right) &\leq \frac{f(x)+f(a+b-x)}{2} \\ &\leq \left[h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right]\frac{f(a)+f(b)}{2} \end{aligned} \tag{3.18}$$

for any  $x \in [a, b]$ .

*Proof.* Since  $\check{f}$  is  $h$ -convex on  $[a, b]$  then for any  $x \in [a, b]$  we have

$$h\left(\frac{1}{2}\right)[\check{f}(x)+\check{f}(a+b-x)] \geq \check{f}\left(\frac{a+b}{2}\right)$$

and since

$$\frac{\check{f}(x)+\check{f}(a+b-x)}{2} = \frac{1}{2}[f(x)+f(a+b-x)]$$

while

$$\check{f}\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right),$$

we get the first inequality in (2.4).

Also, by the convexity of  $\check{f}$  we have for any  $x \in [a, b]$  that

$$\begin{aligned} \check{f}(x) &\leq h\left(\frac{x-a}{b-a}\right) \cdot \check{f}(b) + h\left(\frac{b-x}{b-a}\right) \cdot \check{f}(a) \\ &= h\left(\frac{x-a}{b-a}\right) \cdot \frac{f(a)+f(b)}{2} + h\left(\frac{b-x}{b-a}\right) \cdot \frac{f(a)+f(b)}{2} \\ &= \left[ h\left(\frac{b-x}{b-a}\right) + h\left(\frac{x-a}{b-a}\right) \right] \frac{f(a)+f(b)}{2}, \end{aligned}$$

which proves the second part of (3.18).  $\square$

**COROLLARY 2.** Assume that the function  $f : [a, b] \rightarrow [0, \infty)$  is  $h$ -symmetrized convex on the interval  $[a, b]$  with  $h$  integrable on  $[0, 1]$  and  $f$  integrable on  $[a, b]$ . If  $w : [a, b] \rightarrow [0, \infty)$  is integrable on  $[a, b]$ , then

$$\begin{aligned} &\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \int_a^b w(t) dt \\ &\leq \frac{1}{2} \int_a^b w(t) [f(t) + f(a+b-t)] dt \\ &\leq \frac{f(a)+f(b)}{2} \int_a^b h\left(\frac{t-a}{b-a}\right) [w(t) + w(a+b-t)] dt. \end{aligned} \tag{3.19}$$

Moreover, if  $w$  is symmetric almost everywhere on  $[a, b]$ , then

$$\begin{aligned} \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \int_a^b w(t) dt &\leq \int_a^b w(t) f(t) dt \\ &\leq [f(a) + f(b)] \int_a^b h\left(\frac{t-a}{b-a}\right) w(t) dt. \end{aligned} \tag{3.20}$$

*Proof.* From (3.18) we have

$$\begin{aligned} \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) &\leq \frac{f(t) + f(a+b-t)}{2} \\ &\leq \left[ h\left(\frac{b-t}{b-a}\right) + h\left(\frac{t-a}{b-a}\right) \right] \frac{f(a)+f(b)}{2} \end{aligned}$$

for any  $t \in [a, b]$ .

Multiplying with  $w(t) \geq 0$  and integrating over  $t \in [a, b]$  we get

$$\begin{aligned} &\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \int_a^b w(t) dt \\ &\leq \frac{1}{2} \int_a^b w(t) [f(t) + f(a+b-t)] dt \\ &\leq \frac{f(a)+f(b)}{2} \int_a^b \left[ h\left(\frac{b-t}{b-a}\right) + h\left(\frac{t-a}{b-a}\right) \right] w(t) dt. \end{aligned} \tag{3.21}$$

Observe that, by changing the variable  $t = a + b - s$ ,  $s \in [a, b]$ , we have

$$\int_a^b h\left(\frac{b-t}{b-a}\right) w(t) dt = \int_a^b h\left(\frac{s-a}{b-a}\right) w(a+b-s) ds,$$

then we get

$$\begin{aligned} & \int_a^b \left[ h\left(\frac{b-t}{b-a}\right) + h\left(\frac{t-a}{b-a}\right) \right] w(t) dt \\ &= \int_a^b h\left(\frac{t-a}{b-a}\right) [w(t) + w(a+b-t)] dt \end{aligned}$$

and by (3.21) we obtain the second part of (3.19).  $\square$

Utilising the previous examples of  $h$ -convex functions the reader may state various inequalities of Hermite-Hadamard type.

For instance, if we assume that the functions  $f : [a, b] \rightarrow [0, \infty)$  is integrable and of symmetrized Godunova-Levin type, then for the symmetric weight

$$w : [a, b] \rightarrow [0, \infty), \quad w(t) = (t-a)(b-t)$$

we have from (3.20) that

$$\begin{aligned} \frac{1}{4} f\left(\frac{a+b}{2}\right) \int_a^b (t-a)(b-t) dt &\leq \int_a^b (t-a)(b-t) f(t) dt \\ &\leq [f(a) + f(b)] (b-a) \int_a^b (b-t) dt \end{aligned}$$

and since

$$\int_a^b (t-a)(b-t) dt = \frac{1}{6} (b-a)^3, \quad \int_a^b (b-t) dt = \frac{1}{2} (b-a)^2,$$

then we get the following inequality of interest:

$$\frac{1}{24} f\left(\frac{a+b}{2}\right) (b-a)^3 \leq \int_a^b (t-a)(b-t) f(t) dt \leq \frac{f(a) + f(b)}{2} (b-a)^3. \quad (3.22)$$

Moreover, if we assume that the function  $f : [a, b] \rightarrow [0, \infty)$  is integrable and symmetrized Breckner  $s$ -convex with  $s \in (0, 1)$ , then for the symmetric weight

$$w : [a, b] \rightarrow [0, \infty), \quad w(t) = (t-a)(b-t)$$

we have from (3.20) that

$$\begin{aligned} & \frac{1}{2^{1-s}} f\left(\frac{a+b}{2}\right) \int_a^b (t-a)(b-t) dt \\ &\leq \int_a^b (t-a)(b-t) f(t) dt \\ &\leq \frac{f(a) + f(b)}{(b-a)^s} \int_a^b (t-a)^{s+1} (b-t) dt \end{aligned}$$

and since

$$\int_a^b (t-a)^{s+1} (b-t) dt = \frac{(b-a)^{s+3}}{(s+2)(s+3)}$$

then we get the following inequality of interest:

$$\begin{aligned} \frac{1}{2^{2-s}3} f\left(\frac{a+b}{2}\right) (b-a)^3 &\leq \int_a^b (t-a)(b-t) f(t) dt \\ &\leq \frac{f(a)+f(b)}{(s+2)(s+3)} (b-a)^3. \end{aligned} \quad (3.23)$$

*Acknowledgement.* The author would like to thank the anonymous referee for reading carefully the manuscript, discovering some typos and making some valuable comments that have been incorporated in the final version of the paper.

#### REFERENCES

- [1] M. ALOMARI AND M. DARUS, *The Hadamard's inequality for  $s$ -convex function*, Int. J. Math. Anal. (Ruse) **2** (2008), no. 13–16, 639–646.
- [2] M. ALOMARI AND M. DARUS, *Hadamard-type inequalities for  $s$ -convex functions*, Int. Math. Forum **3** (2008), no. 37–40, 1965–1975.
- [3] G. A. ANASTASSIOU, *Univariate Ostrowski inequalities, revisited*, Monatsh. Math., **135** (2002), no. 3, 175–189.
- [4] N. S. BARNETT, P. CERONE, S. S. DRAGOMIR, M. R. PINHEIRO AND A. SOFO, *Ostrowski type inequalities for functions whose modulus of the derivatives are convex and applications*, Inequality Theory and Applications, Vol. 2 (Chinju/Masan, 2001), 19–32, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint: RGMIA Res. Rep. Coll. **5** (2002), No. 2, Art. 1 [Online <http://rgmia.org/papers/v5n2/Paperwapp2q.pdf>].
- [5] E. F. BECKENBACH, *Convex functions*, Bull. Amer. Math. Soc. **54** (1948), 439–460.
- [6] M. BOMBARDELLI AND S. VAROŠANEC, *Properties of  $h$ -convex functions related to the Hermite-Hadamard-Fejér inequalities*, Comput. Math. Appl. **58** (2009), no. 9, 1869–1877.
- [7] W. W. BRECKNER, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen*, (German) Publ. Inst. Math. (Beograd) (N.S.) **23** (37) (1978), 13–20.
- [8] W. W. BRECKNER AND G. ORBÁN, *Continuity properties of rationally  $s$ -convex mappings with values in an ordered topological linear space*, Universitatea “Babeş-Bolyai”, Facultatea de Matematica, Cluj-Napoca, 1978. viii+92 pp.
- [9] P. CERONE AND S. S. DRAGOMIR, *Midpoint-type rules from an inequalities point of view*, Ed. G. A. Anastassiou, Handbook of Analytic-Computational Methods in Applied Mathematics, CRC Press, New York, 135–200.
- [10] P. CERONE AND S. S. DRAGOMIR, *New bounds for the three-point rule involving the Riemann-Stieltjes integrals*, in Advances in Statistics Combinatorics and Related Areas, C. Gulati, et al. (Eds.), World Science Publishing, 2002, 53–62.
- [11] P. CERONE, S. S. DRAGOMIR AND J. ROUMELIOTIS, *Some Ostrowski type inequalities for  $n$ -time differentiable mappings and applications*, Demonstratio Mathematica, **32** (2) (1999), 697–712.
- [12] G. CRISTESCU, *Hadamard type inequalities for convolution of  $h$ -convex functions*, Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity **8** (2010), 3–11.
- [13] S. S. DRAGOMIR, *Ostrowski's inequality for monotonous mappings and applications*, J. KSIAM, **3** (1) (1999), 127–135.
- [14] S. S. DRAGOMIR, *The Ostrowski's integral inequality for Lipschitzian mappings and applications*, Comp. Math. Appl., **38** (1999), 33–37.
- [15] S. S. DRAGOMIR, *On the Ostrowski's inequality for Riemann-Stieltjes integral*, Korean J. Appl. Math., **7** (2000), 477–485.



- [16] S. S. DRAGOMIR, *On the Ostrowski's inequality for mappings of bounded variation and applications*, Math. Ineq. & Appl., **4** (1) (2001), 33–40.
- [17] S. S. DRAGOMIR, *On the Ostrowski inequality for Riemann-Stieltjes integral  $\int_a^b f(t)du(t)$  where  $f$  is of Hölder type and  $u$  is of bounded variation and applications*, J. KSIAM, **5** (1) (2001), 35–45.
- [18] S. S. DRAGOMIR, *Ostrowski type inequalities for isotonic linear functionals*, J. Inequal. Pure & Appl. Math., **3** (5) (2002), Art. 68.
- [19] S. S. DRAGOMIR, *An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*, J. Inequal. Pure Appl. Math. **3** (2002), no. 2, Article 31, 8 pp.
- [20] S. S. DRAGOMIR, *An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*, J. Inequal. Pure Appl. Math. **3** (2002), no. 3, Article 35.
- [21] S. S. DRAGOMIR, *An Ostrowski like inequality for convex functions and applications*, Revista Math. Complutense, **16** (2) (2003), 373–382.
- [22] S. S. DRAGOMIR, *Operator Inequalities of Ostrowski and Trapezoidal Type*, Springer Briefs in Mathematics, Springer, New York, 2012.
- [23] S. S. DRAGOMIR, *Inequalities of Hermite-Hadamard type for  $h$ -convex functions on linear spaces*, Proyecciones **34**, 4 (2015), 323–341. Preprint RGMIA Res. Rep. Coll. **16** (2013), Art. 72 [Online <http://rgmia.org/papers/v16/v16a72.pdf>].
- [24] S. S. DRAGOMIR, P. CERONE, J. ROUMELIOTIS AND S. WANG, *A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis*, Bull. Math. Soc. Sci. Math. Romanic, **42** (90) (4) (1999), 301–314.
- [25] S. S. DRAGOMIR AND S. FITZPATRICK, *The Hadamard inequalities for  $s$ -convex functions in the second sense*, Demonstratio Math. **32** (1999), no. 4, 687–696.
- [26] S. S. DRAGOMIR AND S. FITZPATRICK, *The Jensen inequality for  $s$ -Breckner convex functions in linear spaces*, Demonstratio Math. **33** (2000), no. 1, 43–49.
- [27] S. S. DRAGOMIR AND B. MOND, *On Hadamard's inequality for a class of functions of Godunova and Levin*, Indian J. Math. **39** (1997), no. 1, 1–9.
- [28] S. S. DRAGOMIR AND C. E. M. PEARCE, *On Jensen's inequality for a class of functions of Godunova and Levin*, Period. Math. Hungar. **33** (1996), no. 2, 93–100.
- [29] S. S. DRAGOMIR AND C. E. M. PEARCE, *Quasi-convex functions and Hadamard's inequality*, Bull. Austral. Math. Soc. **57** (1998), 377–385.
- [30] S. S. DRAGOMIR, J. PEČARIĆ AND L. PERSSON, *Some inequalities of Hadamard type*, Soochow J. Math. **21** (1995), no. 3, 335–341.
- [31] S. S. DRAGOMIR AND TH. M. RASSIAS (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publisher, 2002.
- [32] S. S. DRAGOMIR AND S. WANG, *A new inequality of Ostrowski's type in  $L_1$ -norm and applications to some special means and to some numerical quadrature rules*, Tamkang J. of Math., **28** (1997), 239–244.
- [33] S. S. DRAGOMIR AND S. WANG, *Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules*, Appl. Math. Lett., **11** (1998), 105–109.
- [34] S. S. DRAGOMIR AND S. WANG, *A new inequality of Ostrowski's type in  $L_p$ -norm and applications to some special means and to some numerical quadrature rules*, Indian J. of Math., **40** (3) (1998), 245–304.
- [35] A. EL FARISSI, *Simple proof and refinement of Hermite-Hadamard inequality*, J. Math. Ineq. **4** (2010), No. 3, 365–369.
- [36] ABDALLAH EL FARISSI, MAAMAR BENBACHIR, MERIEM DAHMANE, *An extension of the Hermite-Hadamard inequality for convex symmetrized functions*, Real Anal. Exchange **38**, 2 (2012/13), 467–474.
- [37] L. FEJÉR, *Über die Fourierreihen, II*, Math. Naturwiss. Anz Ungar. Akad. Wiss. **24** (1906), 369–390, (in Hungarian).
- [38] E. K. GODUNOVA AND V. I. LEVIN, *Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions*, (Russian) Numerical mathematics and mathematical physics (Russian), 138–142, 166, Moskov. Gos. Ped. Inst., Moscow, 1985.

- [39] H. HUDZIK AND L. MALIGRANDA, *Some remarks on  $s$ -convex functions*, *Aequationes Math.* **48** (1994), no. 1, 100–111.
- [40] E. KIKIANTY AND S. S. DRAGOMIR, *Hermite-Hadamard's inequality and the  $p$ -HH-norm on the Cartesian product of two copies of a normed space*, *Math. Inequal. Appl.* **13**, 1 (2010), 1–32.
- [41] U. S. KIRMACI, M. KLARIČIĆ BAKULA, M. E. ÖZDEMİR AND J. PEČARIĆ, *Hadamard-type inequalities for  $s$ -convex functions*, *Appl. Math. Comput.* **193** (2007), no. 1, 26–35.
- [42] M. A. LATIF, *On some inequalities for  $h$ -convex functions*, *Int. J. Math. Anal. (Ruse)* **4** (2010), no. 29–32, 1473–1482.
- [43] D. S. MITRINOVIĆ AND I. B. LACKOVIĆ, *Hermite and convexity*, *Aequationes Math.* **28** (1985), 229–232.
- [44] D. S. MITRINOVIĆ AND J. E. PEČARIĆ, *Note on a class of functions of Godunova and Levin*, *C. R. Math. Rep. Acad. Sci. Canada* **12** (1990), no. 1, 33–36.
- [45] C. E. M. PEARCE AND A. M. RUBINOV,  *$P$ -functions, quasi-convex functions, and Hadamard-type inequalities*, *J. Math. Anal. Appl.* **240** (1999), no. 1, 92–104.
- [46] J. E. PEČARIĆ AND S. S. DRAGOMIR, *On an inequality of Godunova-Levin and some refinements of Jensen integral inequality*, *Itinerant Seminar on Functional Equations, Approximation and Convexity (Cluj-Napoca, 1989)*, 263–268, Preprint, 89–6, Univ. “Babeş-Bolyai”, Cluj-Napoca, 1989.
- [47] J. PEČARIĆ AND S. S. DRAGOMIR, *A generalization of Hadamard's inequality for isotonic linear functionals*, *Radovi Mat. (Sarajevo)* **7** (1991), 103–107.
- [48] M. RADULESCU, S. RADULESCU AND P. ALEXANDRESCU, *On the Godunova-Levin-Schur class of functions*, *Math. Inequal. Appl.* **12** (2009), no. 4, 853–862.
- [49] M. Z. SARIKAYA, A. SAGLAM, AND H. YILDIRIM, *On some Hadamard-type inequalities for  $h$ -convex functions*, *J. Math. Inequal.* **2** (2008), no. 3, 335–341.
- [50] E. SET, M. E. ÖZDEMİR AND M. Z. SARIKAYA, *New inequalities of Ostrowski's type for  $s$ -convex functions in the second sense with applications*, *Facta Univ. Ser. Math. Inform.* **27** (2012), no. 1, 67–82.
- [51] M. Z. SARIKAYA, E. SET AND M. E. ÖZDEMİR, *On some new inequalities of Hadamard type involving  $h$ -convex functions*, *Acta Math. Univ. Comenian. (N.S.)* **79** (2010), no. 2, 265–272.
- [52] M. TUNÇ, *Ostrowski-type inequalities via  $h$ -convex functions with applications to special means*, *J. Inequal. Appl.* **2013**, 2013:326.
- [53] S. VAROŠANEC, *On  $h$ -convexity*, *J. Math. Anal. Appl.* **326** (2007), no. 1, 303–311.

(Received April 21, 2015)

S. S. Dragomir  
 Mathematics, College of Engineering & Science, Victoria University  
 PO Box 14428, Melbourne City, MC 8001, Australia  
 and  
 School of Computer Science & Applied Mathematics  
 University of the Witwatersrand  
 Private Bag 3, Johannesburg 2050, South Africa  
 e-mail: sever.dragomir@vu.edu.au  
<http://rgmia.org/dragomir>