Journal of Mathematical Inequalities
Volume 10, Number 4 (2016), 919–931 doi:10.7153/jmi-10-75

REFINEMENTS OF THE CAUCHY–SCHWARZ INEQUALITY FOR τ–MEASURABLE OPERATORS

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(*Communicated by Y. Seo*)

Abstract. In this paper, we present the convexity of certain functions in noncommutative symmetric space generalizing the previous result of Hiai and Zhan. As an application, we gave some refinements of the Cauchy-Schwarz inequality for τ-measurable operators by using some integration techniques.

1. Introduction

Let $\mathbb{M}_n$ be the space of $n\times n$ complex matrices. A norm $\|\cdot\|$ on $\mathbb{M}_n$ is called unitarily invariant if $\|UAV\| = \|A\|$ for all $A\in\mathbb{M}_n$ and all unitary matrices $U,V\in\mathbb{M}_n$. For matrices A,B,X in $\mathbb{M}_n$ with A,B positive semidefinite and X arbitrary, Bhatia and Davis [1] gave the matrix Cauchy-Schwarz inequality

$$\||A^*XB|^r\|^2 \leqslant \||AA^*X|^r\|\cdot\||XBB^*|^r\| \text{ for all } r>0.$$

In 2002, for the above matrices A,B and X, Hiai and Zhan [10] proved that the function $f(t) = \||A^tXB^{1-t}|^r\|\cdot\||A^{1-t}XB^t|^r\|$ is convex on $[0,1]$ for each $r>0$. In particular, they gave a Cauchy-Schwarz type inequality as follows

$$\||A^{\frac12}XB^{\frac12}|^r\|^2 \leqslant \||A^sXB^{1-s}|^r\|\||A^{1-s}XB^s|^r\| \leqslant \||AX|^r\|\||XB|^r\|, \tag{1.1}$$

where $A,B\geqslant 0$ and $X\in\mathbb{M}_n$ and $s\in[0,1]$, $r>0$. Among other things, Bakherad [2] shows a further refinement of the Cauchy-Schwarz inequality as follows

$$\begin{aligned}\||A^{\frac12}XB^{\frac12}|^r\|^2 &\leqslant \||A^tXB^{1-s}|^r\|\||A^{1-t}XB^s|^r\|\\ &\leqslant \max\{\||AX|^r\|\||XB|^r\|, \||AXB|^r\|\||X|^r\|\},\end{aligned} \tag{1.2}$$

where $s,t\in[0,1]$ and $r>0$.

Let $E(\mathscr{M})$ be the noncommutative symmetric space of τ-measurable operators affiliated with a semifinite von Neumann algebra equipped with a normal faithful semifinite trace τ. In 2009, Zhou, Wang and Wu [16] gave the Cauchy-Schwarz inequality for τ-measurable operators

$$\||x^*zy|^r\|^2_{E(\mathscr{M})} \leqslant \||xx^*z|^r\|_{E(\mathscr{M})}\cdot\||zyy^*|^r\|_{E(\mathscr{M})} \text{ for all } r>0, \tag{1.3}$$

Mathematics subject classification (2010): 47A30, 47L05, 47L25.
Keywords and phrases: Symmetric space, inequality, τ-measurable operators.

where $x, y \in L_0(\mathscr{M})$ and $z \in \mathscr{M}$. We use the Cauchy-Schwarz inequality for τ-measurable operator and the method of Hiai and Zhan to obtain generalizations of the convexity of

$$\varphi(t) = \||x^t zy^{1-t}|^r\|_{E(\mathscr{M})} \cdot \||x^{1-t}zy^t|^r\|_{E(\mathscr{M})} \text{ for } t \in [0,1],$$

where $x,\ y \in E(\mathscr{M})_+^{(r)}$, $z \in \mathscr{M}$ and $r > 0$. As an application, we show a generalization of inequality (1.1) for the norm on noncommutative symmetric space. Finally, we show that inequality (1.2) holds for the norm on noncommutative symmetric space.

2. Preliminaries

Let L_0 be the set of all Lebesgue measurable functions on $(0,\infty)$. For $f \in L_0$ we define its non-increasing rearrangement as

$$f^*(t) = \inf\{s > 0 : d_f(s) = m\{r : |f(r)| > s\} \leqslant t\}, \quad t > 0,$$

where m denotes the Lebesgue measure on $(0,\infty)$. By a symmetric Banach space on $(0,\infty)$ we mean a Banach lattice E of measurable functions on $(0,\infty)$ satisfying the following properties: (a) E contains a simple function; (b) if $f \in L_0$ and $g \in E$ with $f^* = g^*$, then $f \in E$ and $\|f\|_E = \|g\|_E$. It is called fully symmetric if, in addition, for $f \in L_0$ and $g \in E$ with $\int_0^t f^*(s)ds \leqslant \int_0^t g^*(s)ds$ we have $f \in E$ and $\|f\|_E \leqslant \|g\|_E$. Let E be a symmetric Banach space on $(0,\infty)$. For $0 < r < \infty$, $E^{(r)}$ will denote the quasi-Banach spaces defined by

$$E^{(r)} := \{g \in L_0 : |g|^r \in E\} \text{ and } \|g\|_{E^{(r)}} = \||g|^r\|_E^{\frac{1}{r}}.$$

As is shown in [11, pp. 53], if E is a symmetric Banach space and $r \geqslant 1$, then $E^{(r)}$ is a symmetric Banach space. The symmetric Banach space E is called minimal if and only if $L^1 \cap L^\infty$ is dense in E. Further, if E is minimal, then $f^*(t) \to 0$ as $t \to 0$ for each $f \in E$. We say that E has order continuous norm if for every net $\{f_i\}_{i\in I}$ in E such that $f_i \downarrow 0$ we have $\|f_n\|_E \downarrow 0$. Moreover, a symmetric Banach space has order continuous norm if and only if it is separable, which is also equivalent to the statement $E' = E^*$, where E' is the köthe dual space of E given by

$$E' = \{f \in L_0 : \sup\{\int_0^\infty |f(t)g(t)|dt : \|g\|_E \leqslant 1\} < \infty\}.$$

In particular, a symmetric Banach space which has order continuous norm is automatically fully symmetric. Then there exists a family W of nonincreasing functions on $(0,\infty)$ such that

$$\|f\|_E = \sup\{\int_0^\infty f^*(t)\omega(t)dt : \omega \in W\}.$$

Consequently,

$$\|f\|_{E^{(r)}}^r = \||f|^r\|_E = \sup\{\int_0^\infty f^*(t)^r\omega(t)dt : \omega \in W\}. \tag{2.1}$$

We refer to [5, 11] for these spaces.

Unless stated otherwise, $\mathscr{M}$ will always denote a semifinite von Neumann algebra acting on the Hilbert space $\mathscr{H}$, with a normal faithful semifinite trace τ. We refer to [6, 12] for noncommutative integration. We denote the identity of $\mathscr{M}$ by 1 and let $\mathscr{P}$ denote the projection lattice of $\mathscr{M}$. For all $p,q \in \mathscr{P}$, the supremum $p \bigvee q$ is given by the orthogonal projection onto $\overline{ran(p) \cup ran(q)}$. A closed densely defined linear operator x in $\mathscr{H}$ with domain $D(x) \subseteq \mathscr{H}$ is said to be affiliated with $\mathscr{M}$ if $u^*xu = x$ for all unitary operators u which belong to the commutant $\mathscr{M}'$ of $\mathscr{M}$. If x is affiliated with $\mathscr{M}$, we define its distribution function by $\lambda_s(x) = \tau(e_s^{\perp}(|x|))$ and x will be called τ-measurable if and only if $\lambda_s(x) < \infty$ for some $s > 0$, where $e_s^{\perp}(|x|) = e_{(s,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (s,∞). The decreasing rearrangement of x is defined by $\mu_t(x) = \inf\{s > 0 : \lambda_s(x) \leqslant t\}$. We will denote simply by $\lambda(x)$ and $\mu(x)$ the functions $t \to \lambda_t(x)$ and $t \to \mu_t(x)$, respectively. See [8] for basic properties and detailed information on decreasing rearrangement of x.

The set of all τ-measurable operators will be denoted by $L_0(\mathscr{M})$. The set $L_0(\mathscr{M})$ is a $*$-algebra with sum and product being the respective closures of the algebraic sum and product. The measure topology in $L_0(\mathscr{M})$ is the vector space topology defined via the neighbourhood base $\{V(\varepsilon,\delta) : \varepsilon,\delta > 0\}$, where $V(\varepsilon,\delta) = \{x \in L_0(\mathscr{M}) : \tau(e_{(\varepsilon,\infty)}(|x|)) \leqslant \delta\}$ and $e_{(\varepsilon,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (ε,∞). With respect to the measure topology, $L_0(\mathscr{M})$ is a complete topological $*$-algebra. As usual, we denote by $\|\cdot\|$ $(= \|\cdot\|_\infty)$ the usual operator norm.

Let E be a symmetric Banach space on $(0,\infty)$. We define

$$E(\mathscr{M}) = \{x \in L_0(\mathscr{M}) : \mu(x) \in E\} \text{ and } \|x\|_{E(\mathscr{M})} = \|\mu(x)\|_E.$$

Then $(E(\mathscr{M}), \|\cdot\|_{E(\mathscr{M})})$ is a noncommutative symmetric Banach space (see, [6]). If $E = L^p$, then $E(\mathscr{M})$ is the usual noncommutative L_p spaces $L^p(\mathscr{M})$ (see, [12]). For $0 < r < \infty$, we define

$$E(\mathscr{M})^{(r)} = \{x \in L_0(\mathscr{M}) : |x|^r \in E(\mathscr{M})\} \text{ and } \|x\|_{E(\mathscr{M})^{(r)}} = \||x|^r\|_{E(\mathscr{M})}^{\frac{1}{r}}.$$

As is shown in Proposition 3.1 of [7], if E is a symmetric Banach space, then $E^{(r)}(\mathscr{M}) = E(\mathscr{M})^{(r)}$, where $E^{(r)}(\mathscr{M}) = \{x \in L_0(\mathscr{M}) : \mu(x) \in E^{(r)}\}$ and $\|x\|_{E^{(r)}(\mathscr{M})} = \|\mu(x)\|_{E^{(r)}}$. Further details may be found in [6, 7].

For every $x \in L_0(\mathscr{M})$, there is a unique polar decomposition $x = u|x|$ where $|x| \in L_0(\mathscr{M})_+$ (the positive part of $L_0(\mathscr{M})$) and u is a partial isometry operator. Let $r(x) = u^*u$ and $l(x) = uu^*$. We call $r(x)$ and $l(x)$ the right and left supports of x, respectively. Note that $l(x)$ (resp. $r(x)$) is the least projection e of $\mathscr{B}(\mathscr{H})$ such that $ex = x$ (resp. $xe = x$). If x is self-adjoint, then $r(x) = l(x)$. This common projection is then said to be the support of x and denoted by $s(x)$. Set $S(\mathscr{M})_+ = \{x \in \mathscr{M}_+ : \tau(s(x)) < \infty\}$ and let $S(\mathscr{M})$ be the linear span of $S(\mathscr{M})_+$.

Note the completion of $(S(\mathscr{M}), \|\cdot\|_p)$ is $L^p(\mathscr{M})$. If E is minimal, then $S(\mathscr{M}) \subseteq L^1(\mathscr{M}) \cap \mathscr{M} \subseteq E(\mathscr{M})$ and $\mu_t(x) \to 0$ as $t \to \infty$ for each $x \in E(\mathscr{M})$. Given $x, y \in L_0(\mathscr{M})$ and $0 < \alpha, q < \infty$, from Theorem 4.2 of [8], we have

$$\int_0^t \mu_s(xy)^{\alpha q} ds \leqslant \int_0^t \mu_s(y)^{\alpha q} \mu_s(x)^{\alpha q} ds, t > 0.$$

So the well-known rearrangement inequality of Hardy (see, Proposition 3.6 of Chapter II in [5]) implies that

$$\int_0^t \mu_s(xy)^{\alpha q}\omega(s)ds \leqslant \int_0^t \mu_s(y)^{\alpha q}\mu_s(x)^{\alpha q}\omega(s)ds, \quad t>0. \tag{2.2}$$

holds for all nonincreasing function $\omega(s)$. Therefore, inequality (2.2), (2.1) and the usual Hölder inequality imply that

$$\|xy\|_{E(\mathscr{M})^{(r)}} \leqslant \|x\|_{E(\mathscr{M})^{(p)}}\|y\|_{E(\mathscr{M})^{(q)}}, \quad x\in E(\mathscr{M})^{(p)},\ y\in E(\mathscr{M})^{(q)} \tag{2.3}$$

holds for $0<p,q,r<\infty$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$.

In what follows, E will always denotes a minimal symmetric Banach space with order continuous norm.

3. Main results

The following lemma, which is a refinement of inequality (2.3) and (1.3), plays a central role in the proof of the convexity of function

$$\varphi(t)=\||x^tzy^{1-t}|^r\|_{E(\mathscr{M})}\cdot\||x^{1-t}zy^t|^r\|_{E(\mathscr{M})} \text{ for } t\in[0,1],$$

where $x,\ y\in E(\mathscr{M})_+^{(r)}$, $z\in\mathscr{M}$ and $r>0$.

LEMMA 3.1. *Let $s,r,p,q>0$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{s}$. If $z\in\mathscr{M}$, $x\in E(\mathscr{M})^{(pr)}$, $y\in E(\mathscr{M})^{(qr)}$, then*

$$\||x^*zy|^r\|_{E(\mathscr{M})^{(s)}} \leqslant \||xx^*z|^r\|^{\frac{1}{2}}_{E(\mathscr{M})^{(\frac{p}{2})}}\cdot\||zyy^*|^r\|^{\frac{1}{2}}_{E(\mathscr{M})^{(\frac{q}{2})}}.$$

Proof. First we assume that $\tau(1)<\infty$. For $x,y,z\in\mathscr{M}$, we have

$$y^*z^*xx^*zy,\ zyy^*(xx^*z)^*\in\mathscr{M}\subseteq L^1(\mathscr{M}).$$

By Lemma 2.5(ii) and (iv) in [8] and Lemma 2 in [4], we obtain

$$\begin{aligned}\mu_s(|x^*zy|^r)^s &= \mu_s(|x^*zy|^2)^{\frac{rs}{2}}\\ &=\mu_s(zyy^*(xx^*z)^*)^{\frac{rs}{2}}\\ &=\mu_s(|zyy^*(xx^*z)^*|)^{\frac{rs}{2}}\\ &=\mu_s(|zyy^*(xx^*z)^*|^{\frac{r}{2}})^s,\end{aligned}$$

and so $\||x^*zy|^r\|_{E(\mathscr{M})^{(s)}} = \||zyy^*(xx^*z)^*|^{\frac{r}{2}}\|_{E(\mathscr{M})^{(s)}}$. Since $\frac{1}{p}+\frac{1}{q}=\frac{1}{s}$, then $\frac{1}{\frac{rp}{2}}+\frac{1}{\frac{rq}{2}}=\frac{1}{\frac{rs}{2}}$. Hence, by Lemma 2.5(ii) and (iv) in [8] and inequality (2.3), we have

$$\begin{aligned}
\||zyy^*(xx^*z)^*|^{\frac{r}{2}}\|_{E(\mathscr{M})^{(s)}} &= \||zyy^*(xx^*z)^*|^{\frac{rs}{2}}\|_{E(\mathscr{M})}^{\frac{1}{s}} \\
&= \|zyy^*(xx^*z)^*\|_{E(\mathscr{M})^{(\frac{rs}{2})}}^{\frac{r}{2}} \\
&\leqslant (\|zyy^*\|_{E(\mathscr{M})^{(\frac{rq}{2})}} \cdot \|(xx^*z)^*\|_{E(\mathscr{M})^{(\frac{rp}{2})}})^{\frac{r}{2}} \\
&= (\||zyy^*|^{\frac{rq}{2}}\|_{E(\mathscr{M})}^{\frac{2}{rq}} \cdot \||(xx^*z)^*|^{\frac{rp}{2}}\|_{E(\mathscr{M})}^{\frac{2}{rp}})^{\frac{r}{2}} \\
&= \||zyy^*|^{\frac{rq}{2}}\|_{E(\mathscr{M})}^{\frac{1}{q}} \cdot \||xx^*z|^{\frac{rp}{2}}\|_{E(\mathscr{M})}^{\frac{1}{p}} \\
&= \||zyy^*|^r\|_{E(\mathscr{M})^{(\frac{q}{2})}}^{\frac{1}{2}} \cdot \||xx^*z|^r\|_{E(\mathscr{M})^{(\frac{p}{2})}}^{\frac{1}{2}}.
\end{aligned}$$

For the general case, namely, for any $x \in E(\mathscr{M})^{(pr)}$ and $y \in E(\mathscr{M})^{(qr)}$, there exist $u_1, u_2 \in \mathscr{M}$ such that $x = u_1|x|$ and $y = u_2|y|$ are the polar decomposition of x and y, respectively. We write $x_n = u_1|x|e_{[0,n]}(|x|)$, $y_n = u_2|y|e_{[0,n]}(|y|)$, $n = 1,2\cdots$, where $e_{[0,n]}(|x|)$ and $e_{[0,n]}(|y|)$ are the spectral projection of $|x|$ and $|y|$ associated with the interval $[0,n]$, respectively. According to Lemma 2.6 in [8], we obtain

$$\mu_t(x-x_n) = \mu_t(xe_{(n,\infty)}(|x|)) \leqslant \mu_t(x)\chi_{[0,\tau(e_{(n,\infty)}(|x|))]}.$$

Then x_n, $y_n \in \mathscr{M}$ and

$$\|x-x_n\|_{E(\mathscr{M})^{(pr)}} = \|xe_{(n,\infty)}(|x|)\|_{E(\mathscr{M})^{(pr)}} \leqslant \|\mu(x)^{pr}\chi_{[0,\tau(e_{(n,\infty)}(|x|))]}\|_E^{\frac{1}{pr}}.$$

Since $\tau(e_{(n,\infty)}(x))$ decreases to zero as $n \to \infty$ (cf, Proposition 21 of Chapter I in [14]) and E has order continuous norm, we obtain $x_n \to x$ in $E(\mathscr{M})^{(pr)}$. Similarly, $y_n \to y$ in $E(\mathscr{M})^{(qr)}$. Moreover,

$$\||x_n^*zy_n|^r\|_{E(\mathscr{M})^{(s)}} \leqslant \||x_nx_n^*z|^{\frac{pr}{2}}\|_{E(\mathscr{M})}^{\frac{1}{p}} \cdot \||zy_ny_n^*|^{\frac{qr}{2}}\|_{E(\mathscr{M})}^{\frac{1}{q}}.$$

Hence,

$$\||x^*zy|^r\|_{E(\mathscr{M})^{(s)}} \leqslant \||xx^*z|^{\frac{pr}{2}}\|_{E(\mathscr{M})}^{\frac{1}{p}} \cdot \||zyy^*|^{\frac{qr}{2}}\|_{E(\mathscr{M})}^{\frac{1}{q}}.$$

In the general case when τ is semifinite, for any $x \in E(\mathscr{M})^{(pr)}$ and $y \in E(\mathscr{M})^{(qr)}$, there exist $u_1, u_2 \in \mathscr{M}$ such that $x = u_1|x|$, $y = u_2|y|$ are the polar decomposition of x and y, respectively. We put

$$x_n = u_1|x|e_{(\frac{1}{n},\infty)}(|x|), \quad y_n = u_2|y|e_{(\frac{1}{n},\infty)}(|y|), \quad n = 1,2\cdots,$$

where $e_{(\frac{1}{n},\infty)}(|x|)$ and $e_{(\frac{1}{n},\infty)}(|y|)$ are the spectral projection of $|x|$ and $|y|$ associated with the interval $(\frac{1}{n},\infty)$, respectively. Then $x_n \in E(P_n\mathscr{M}P_n)^{(pr)}$, $y_n \in E(P_n\mathscr{M}P_n)^{(qr)}$, where $P_n = e_{(\frac{1}{n},\infty)}(|x|) \bigvee e_{(\frac{1}{n},\infty)}(|y|)$. Since E is minimal, then $\lim_{t\to\infty}\mu_t(|x|^{pr}) = \lim_{t\to\infty}\mu_t(|y|^{qr}) = 0$. According to Proposition 3.2 in [8], we obtain $\tau(e_{(\frac{1}{n},\infty)}(|x|)) < \infty$ and $\tau(e_{(\frac{1}{n},\infty)}(|y|)) < \infty$ for any $n \in \mathbb{N}_+$. Hence,

$$\tau(P_n) \leqslant \tau(e_{(\frac{1}{n},\infty)}(|x|)) + \tau(e_{(\frac{1}{n},\infty)}(|y|)) < \infty.$$

Thus, $P_n\mathscr{M}P_n$ is finite and so

$$\||x_n^*zy_n|^r\|_{E^{(s)}(\mathscr{M})} \leqslant \||x_nx_n^*z|^{\frac{pr}{2}}\|_{E(\mathscr{M})}^{\frac{1}{p}} \cdot \||zy_ny_n^*|^{\frac{qr}{2}}\|_{E(\mathscr{M})}^{\frac{1}{q}}.$$

Since E has order continuous norm and

$$\mu_t(|x|e_{[0,\frac{1}{n+1}]}(|x|)) \leqslant \mu_t(|x|e_{[0,\frac{1}{n}]}(|x|)) \leqslant \frac{1}{n}, \quad n = 1,2,\cdots,$$

by a simple computation we derive $x_n \to x$ in $E(\mathscr{M})^{(pr)}$. Similarly, $y_n \to y$ in $E(\mathscr{M})^{(qr)}$. Thus,

$$\||x^*zy|^r\|_{E(\mathscr{M})^{(s)}} \leqslant \||xx^*z|^r\|_{E(\mathscr{M})^{(\frac{p}{2})}}^{\frac{1}{2}} \cdot \||zyy^*|^r\|_{E(\mathscr{M})^{(\frac{q}{2})}}^{\frac{1}{2}}. \quad \square$$

REMARK 3.2. It is clear that $E(\mathscr{M})^{(1)} = E(\mathscr{M})$. If we replace s,p,q by $1,2,2$, respectively, in Lemma 3.1, then we obtain the Cauchy-Schwarz inequality

$$\||x^*zy|^r\|_{E(\mathscr{M})}^2 \leqslant \||xx^*z|^r\|_{E(\mathscr{M})} \cdot \||zyy^*|^r\|_{E(\mathscr{M})} \text{ for all } r > 0.$$

THEOREM 3.3. *Let $x,\ y \in E(\mathscr{M})_+^{(r)}$, $z \in \mathscr{M}$ and $r > 0$. Then the function*

$$\varphi(t) = \||x^tzy^{1-t}|^r\|_{E(\mathscr{M})} \cdot \||x^{1-t}zy^t|^r\|_{E(\mathscr{M})}$$

is convex on the interval $[0,1]$ and attains its minimum at $t = \frac{1}{2}$. Consequently, it is decreasing on $[0,\frac{1}{2}]$ and increasing on $[\frac{1}{2},1]$.

Proof. Let $x,\ y \in E(\mathscr{M})_+^{(r)}$. By Lemma 2.5(iv) of [8], we have $\mu(x)^r = \mu(x^r)$, $\mu(y)^r = \mu(y^r)$, which means that $\mu(x),\mu(y) \in E^{(r)}$. Therefore, Lemma 2.5 (iv) and (vi) in [8] and inequality (2.3) imply that

$$\begin{aligned}
\||x^tzy^{1-t}|^r\|_{E(\mathscr{M})}^{\frac{1}{r}} &= \|x^tzy^{1-t}\|_{E(\mathscr{M})^{(r)}} \\
&\leqslant \|x^tz\|_{E(\mathscr{M})^{(\frac{r}{t})}} \|y^{1-t}\|_{E(\mathscr{M})^{(\frac{r}{1-t})}} \\
&\leqslant \|z\| \|\mu(x)^r\|_E^{\frac{t}{r}} \|\mu(y)^r\|_E^{\frac{1-t}{r}} \\
&= \|z\| \|\mu(x)\|_{E^{(r)}}^t \|\mu(y)\|_{E^{(r)}}^{1-t}.
\end{aligned}$$

Thus, $x,\ y \in E(\mathscr{M})_+^{(r)}$ imply that $|x^tzy^{1-t}|^r \in E(\mathscr{M})$. Similarly, $|x^{1-t}zy^t|^r \in E(\mathscr{M})$.

First we assume that τ is finite and $x,\ y \in \mathscr{M}_+$. Since φ is continuous and symmetric with respect to $t = \frac{1}{2}$, all the conclusions will follow after we show that

$$\varphi(t) \leqslant \frac{1}{2}\{\varphi(t+s) + \varphi(t-s)\}, \quad t \pm s \in [0,1].$$

By Lemma 3.1 (or, Theorem 3 of [16]), we have

$$\begin{aligned}
\||x^t zy^{1-t}|^r\|_{E(\mathscr{M})} &= \||x^s(x^{t-s}zy^{1-t-s})y^s|^r\|_{E(\mathscr{M})} \\
&\leqslant \{\||x^{t+s}zy^{1-(t+s)}|^r\|_{E(\mathscr{M})} \cdot \||x^{t-s}zy^{1-(t-s)}|^r\|_{E(\mathscr{M})}\}^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
\||x^{1-t}zy^t|^r\|_{E(\mathscr{M})} &= \||x^s(x^{1-t-s}zy^{t-s})y^s|^r\|_{E(\mathscr{M})} \\
&\leqslant \{\||x^{1-(t-s)}zy^{t-s}|^r\|_{E(\mathscr{M})} \cdot \||x^{1-(t+s)}zy^{t+s}|^r\|_{E(\mathscr{M})}\}^{\frac{1}{2}}.
\end{aligned}$$

Multiplying the above two inequalities we obtain

$$\varphi(t) \leqslant \frac{1}{2}\{\varphi(t+s) + \varphi(t-s)\}, \quad t \pm s \in [0,1].$$

For the general case, for any $x,\ y \in E(\mathscr{M})_+^{(r)}$, we write $x_n = xe_{[0,n]}(x)$ and $y_n = ye_{[0,n]}(y) \in \mathscr{M}_+$, $n = 1,2\cdots$. According to Lemma 2.6 in [8], we obtain

$$\mu_t(x - x_n) = \mu_t(xe_{(n,\infty)}(|x|)) \leqslant \mu_t(x)\chi_{[0,\tau(e_{(n,\infty)}(|x|))]}.$$

Hence,

$$\|x - x_n\|_{E(\mathscr{M})^{(r)}} = \|xe_{(n,\infty)}(x)\|_{E(\mathscr{M})^{(r)}} = \|\mu(x)^r\chi_{[0,\tau(e_{(n,\infty)}(x))]}\|_E^{\frac{1}{r}}.$$

Since $\tau(e_{(n,\infty)}(x))$ decreases to zero as $n \to \infty$ (cf, Proposition 21 of Chapter I in [14]) and E has order continuous norm we infer that $x_n \to x$ in $E(\mathscr{M})^{(r)}$ as $n \to \infty$. Similarly, $y_n \to y$ in $E(\mathscr{M})^{(r)}$. It follows from the above case that $\varphi_n(t) = \||x_n^t zy_n^{1-t}|^r\|_{E(\mathscr{M})} \cdot \||x_n^{1-t}zy_n^t|^r\|_{E(\mathscr{M})}$ is convex for all $t \in [0,1]$ and attains its minimum at $t = \frac{1}{2}$. On the other hand, by inequality (2.3) and the fact $E(\mathscr{M})^{(r)}$ is a quasi-Banach space, there exists some constant $C > 0$ such that

$$\begin{aligned}
\|x_n^t zy_n^{1-t} - x^t zy^{1-t}\|_{E(\mathscr{M})^{(r)}} &= \|x_n^t zy_n^{1-t} - x^t zy_n^{1-t} + x^t zy_n^{1-t} - x^t zy^{1-t}\|_{E(\mathscr{M})^{(r)}} \\
&\leqslant C(\|(x_n^t - x^t)zy_n^{1-t}\|_{E(\mathscr{M})^{(r)}} \\
&\quad + \|x^t z(y_n^{1-t} - y^{1-t})\|_{E(\mathscr{M})^{(r)}}) \\
&\leqslant C\|z\|(\|x^t - x_n^t\|_{E(\mathscr{M})^{(\frac{r}{t})}}\|y_n^{1-t}\|_{E(\mathscr{M})^{(\frac{r}{1-t})}} \\
&\quad + \|x^t\|_{E(\mathscr{M})^{(\frac{r}{t})}}\|y^{1-t} - y_n^{1-t}\|_{E(\mathscr{M})^{(\frac{r}{1-t})}}) \\
&\leqslant C\|z\|(\|\mu(x)^r\chi_{[0,\tau(e_{(n,\infty)}(x))]}\|_E^{\frac{t}{r}}\|y_n^r\|_{E(\mathscr{M})}^{\frac{1-t}{r}} \\
&\quad + \|x^r\|_E^{\frac{t}{r}}\|\mu(y)^r\chi_{[0,\tau(e_{(n,\infty)}(y))]}\|_{E(\mathscr{M})}^{\frac{1-t}{r}}).
\end{aligned}$$

This implies that $x_n^t z y_n^{1-t} \to x^t z y^{1-t}$ in $E(\mathscr{M})^{(r)}$ as $n \to \infty$. Similarly, $x_n^{1-t} z y_n^t \to x^{1-t} z y^t$ in $E(\mathscr{M})^{(r)}$ as $n \to \infty$, and so $\varphi_n(t) \to \varphi(t), n \to \infty$. Therefore, $\varphi(t)$ is convex on $[0,1]$ and attains its minimum at $t = \frac{1}{2}$.

In the general case when τ is semifinite, for $x,\ y \in E(\mathscr{M})_+^{(r)}$, we write $x_n = xe_{(\frac{1}{n},\infty)}(x)$ and $y_n = xe_{(\frac{1}{n},\infty)}(y)$, $n = 1,2\cdots$. Then $x_n,\ y_n \in E(P_n\mathscr{M}P_n)_+^{(r)}$, where $P_n = e_{(\frac{1}{n},\infty)}(x) \bigvee e_{(\frac{1}{n},\infty)}(y)$, $n = 1,2\cdots$. Since E is minimal, then $\lim_{t\to\infty}\mu_t(|x|^r) = \lim_{t\to\infty}\mu_t(|y|^r) = 0$. According to Proposition 3.2 in [8], we obtain $\tau(e_{(\frac{1}{n},\infty)}(|x|)) < \infty$ and $\tau(e_{(\frac{1}{n},\infty)}(|y|)) < \infty$ for any $n \in \mathbb{N}_+$. Hence,

$$\tau(P_n) \leqslant \tau(e_{(\frac{1}{n},\infty)}(|x|)) + \tau(e_{(\frac{1}{n},\infty)}(|y|)) < \infty.$$

Therefore, $P_n\mathscr{M}P_n$ is finite and the function

$$\varphi_n(t) = \||x_n^t z y_n^{1-t}|^r\|_E \cdot \||x_n^{1-t} z y_n^t|^r\|_{E(\mathscr{M})}$$

is convex on $[0,1]$ and attains its minimum at $t = \frac{1}{2}$. On the other hand, since $xe_{[0,\frac{1}{n}]}(x) \geqslant xe_{[0,\frac{1}{n+1}]}(x)$ and $\mu(xe_{[0,\frac{1}{n}]}(x)) \leqslant \frac{1}{n}$, we obtain

$$\|x - x_n\|_{E(\mathscr{M})^{(r)}} = \|xe_{[0,\frac{1}{n}]}(x)\|_{E(\mathscr{M})^{(r)}} \to 0, n \to \infty.$$

Similarly, $y_n \to y$ in $E(\mathscr{M})^{(r)}$. By a simple computation we derive $\lim_n \varphi_n(t) = \varphi(t)$. Therefore, $\varphi(t)$ is convex on $[0,1]$ and attains its minimum at $t = \frac{1}{2}$. This completes the proof. $\square$

Based on Theorem 3.3, we obtain the generalizations of inequality (1.1) for the norm on noncommutative symmetric space.

COROLLARY 3.4. *Let $x,\ y \in E(\mathscr{M})_+^{(r)}$, $z \in \mathscr{M}$ and $r > 0$. Then*

$$\begin{aligned} \||x^{\frac{1}{2}} z y^{\frac{1}{2}}|^r\|_{E(\mathscr{M})}^2 &\leqslant \||x^t z y^{1-t}|^r\|_{E(\mathscr{M})} \cdot \||x^{1-t} z y^t|^r\|_{E(\mathscr{M})} \\ &\leqslant \||xz|^r\|_{E(\mathscr{M})} \cdot \||zy|^r\|_{E(\mathscr{M})}. \end{aligned}$$

holds for $0 \leqslant t \leqslant 1$.

Proof. It follows immediately from Theorem 3.3. $\square$

In view of the result of Theorem 3.3, we obtain our refinement of the first inequality in Corollary 3.4.

COROLLARY 3.5. *Let $x,\ y \in E(\mathscr{M})_+^{(r)}$, $z \in \mathscr{M}$ and $r > 0$. Then*

$$\begin{aligned} \||x^{\frac{1}{2}} z y^{\frac{1}{2}}|^r\|_{E(\mathscr{M})}^2 &\leqslant \frac{1}{|1-2\alpha|}\left|\int_\alpha^{1-\alpha} \||x^s z y^{1-s}|^r\|_{E(\mathscr{M})}\||x^{1-s} z y^s|^r\|_{E(\mathscr{M})} ds\right| \\ &\leqslant \frac{1}{2}[\||x^{\frac{1}{2}} z y^{\frac{1}{2}}|^r\|_{E(\mathscr{M})}^2 + \||x^\alpha z y^{1-\alpha}|^r\|_{E(\mathscr{M})}\||x^{1-\alpha} z y^\alpha|^r\|_{E(\mathscr{M})}] \\ &\leqslant \||x^\alpha z y^{1-\alpha}|^r\|_{E(\mathscr{M})}\||x^{1-\alpha} z y^\alpha|^r\|_{E(\mathscr{M})} \end{aligned}$$

for all $0 \leqslant \alpha \leqslant 1$ *and* $\alpha \neq \frac{1}{2}$.

Proof. Let $\varphi(t) = \||x^t zy^{1-t}|^r\|_{E(\mathscr{M})}\||x^{1-t}zy^t|^r\|_{E(\mathscr{M})}$ and let $0 \leqslant \alpha < \frac{1}{2}$. By Theorem 1.1 of [9] and Theorem 3.3, we have

$$\begin{aligned}\varphi\left(\frac{\alpha+1-\alpha}{2}\right) &\leqslant \frac{1}{1-2\alpha}\int_{\alpha}^{1-\alpha}\varphi(s)ds \\ &\leqslant L(h) \leqslant \frac{\varphi(1-\alpha)+\varphi(\alpha)}{2}, \quad h \in [0,1],\end{aligned}$$

where

$$L(h) = \frac{1}{2}[\varphi(h(1-\alpha)+(1-h)\alpha)+h\varphi(\alpha)+(1-h)\varphi(1-\alpha)].$$

Thus,

$$\varphi\left(\frac{1}{2}\right) \leqslant \frac{1}{1-2\alpha}\int_{\alpha}^{1-\alpha}\varphi(s)ds \leqslant L\left(\frac{1}{2}\right) \leqslant \frac{\varphi(1-\alpha)+\varphi(\alpha)}{2}.$$

It follows that

$$\begin{aligned}\||x^{\frac{1}{2}}zy^{\frac{1}{2}}|^r\|_{E(\mathscr{M})}^2 &\leqslant \frac{1}{1-2\alpha}\int_{\alpha}^{1-\alpha}\||x^{1-s}zy^s|^r\|_{E(\mathscr{M})}\||x^s zy^{1-s}|^r\|_{E(\mathscr{M})}ds \\ &\leqslant \frac{1}{2}[\||x^{\frac{1}{2}}zy^{\frac{1}{2}}|^r\|_{E(\mathscr{M})}+\||x^{\alpha}zy^{1-\alpha}|^r\|_{E(\mathscr{M})}\||x^{1-\alpha}zy^{\alpha}|^r\|_{E(\mathscr{M})}] \\ &\leqslant \||x^{\alpha}zy^{1-\alpha}|^r\|_{E(\mathscr{M})}\||x^{1-\alpha}zy^{\alpha}|^r\|_{E(\mathscr{M})}.\end{aligned}$$

Let $\frac{1}{2} < \alpha \leqslant 1$. By the symmetry property of φ with respect to $\alpha = \frac{1}{2}$, if we replace α by $1-\alpha$, then

$$\begin{aligned}\||x^{\frac{1}{2}}zy^{\frac{1}{2}}|^r\|_{E(\mathscr{M})}^2 &\leqslant \frac{1}{2\alpha-1}\int_{1-\alpha}^{\alpha}\||x^{1-s}zy^s|^r\|_{E(\mathscr{M})}\||x^s zy^{1-s}|^r\|_{E(\mathscr{M})}ds \\ &\leqslant \frac{1}{2}[\||x^{\frac{1}{2}}zy^{\frac{1}{2}}|^r\|_{E(\mathscr{M})}+\||x^{\alpha}zy^{1-\alpha}|^r\|_{E(\mathscr{M})}\||x^{1-\alpha}zy^{\alpha}|^r\|_{E(\mathscr{M})}] \\ &\leqslant \||x^{\alpha}zy^{1-\alpha}|^r\|_{E(\mathscr{M})}\||x^{1-\alpha}zy^{\alpha}|^r\|_{E(\mathscr{M})}. \quad \square\end{aligned}$$

The following result is a refinement of the second inequality in Corollary 3.4.

THEOREM 3.6. *Let* $x,\ y \in E(\mathscr{M})_+^{(r)}$, $z \in \mathscr{M}$, $r > 0$ *and* $s \in [0,1]$. *For every* $t \in (0,1)$, *we have*

$$\begin{aligned}\||xz|^r\|_{E(\mathscr{M})}\||zy|^r\|_{E(\mathscr{M})} &- \||x^s zy^{1-s}|^r\|_{E(\mathscr{M})}\||x^{1-s}zy^s|^r\|_{E(\mathscr{M})} \\ &\geqslant \frac{1}{t}\left(\varphi\left(\frac{1-t}{2}\right)-\varphi\left(\frac{1-t}{2}+ts\right)\right) \geqslant 0,\end{aligned}$$

where $\varphi(s) = \||x^s zy^{1-s}|^r\|_{E(\mathscr{M})}\||x^{1-s}zy^s|^r\|_{E(\mathscr{M})}$.

Proof. Let $\varphi(s) = \||x^s zy^{1-s}|^r\|_{E(\mathcal{M})}\||x^{1-s}zy^s|^r\|_{E(\mathcal{M})}$ and $f(s) = (1-t)\varphi(\frac{1}{2}) + t\varphi(s) - \varphi(\frac{1-t}{2} + st)$. Lemma 1 of [3] and Theorem 3.3 imply that $f(s)$ is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$. Since $f(s)$ is decreasing on $[0, \frac{1}{2}]$, we have $f(0) \geqslant f(s)$, $s \in [0, \frac{1}{2}]$. This means that

$$\varphi(0) - \varphi(s) \geqslant \frac{1}{t}\left[\varphi\left(\frac{1-t}{2}\right) - \varphi\left(\frac{1-t}{2} + st\right)\right]. \tag{3.1}$$

Thus

$$\begin{aligned}\||xz|^r\|_{E(\mathcal{M})}\||zy|^r\|_{E(\mathcal{M})} - \||x^s zy^{1-s}|^r\|_{E(\mathcal{M})}\||x^{1-s}zy^s|^r\|_{E(\mathcal{M})} \\ \geqslant \frac{1}{t}\left(\varphi\left(\frac{1-t}{2}\right) - \varphi\left(\frac{1-t}{2} + ts\right)\right) \geqslant 0,\end{aligned}$$

where $\varphi(\frac{1-t}{2}) - \varphi(\frac{1-t}{2} + ts) \geqslant 0$ follows immediately from that $\frac{1}{2} \geqslant \frac{1-t}{2} + st \geqslant \frac{1-t}{2}$ and Theorem 3.3. Now, let $s \in [\frac{1}{2}, 1]$. By the symmetry property of (3.1) with respect to $s = \frac{1}{2}$, if we replace s by $1-s$, we obtain

$$\varphi(0) - \varphi(1-s) \geqslant \frac{1}{t}\left[\varphi\left(\frac{1-t}{2}\right) - \varphi\left(\frac{1-t}{2} + (1-s)t\right)\right],$$

which reduces to the desire result since $\varphi(1-s) = \varphi(s)$, $s \in [0,1]$. $\square$

Recall that a real valued function $F(s,t)$ defined on $[a,b] \times [c,d]$ is called convex if

$$F(\lambda s_1 + (1-\lambda)s_2, \lambda t_1 + (1-\lambda)t_2) \leqslant \lambda F(s_1,t_1) + (1-\lambda)F(s_2,t_2)$$

for all $s_1, s_2 \in [a,b]$, $t_1, t_2 \in [c,d]$ and $0 < \lambda < 1$. Now, we show the convexity of the function

$$\varphi(s,t) = \||x^{1-t}zy^{1+s}|^r\|_{E(\mathcal{M})} \cdot \||x^{1+t}zy^{1-s}|^r\|_{E(\mathcal{M})}$$

and we use the convexity of φ to prove some Cauchy-Schwarz type inequalities.

To achieve one of our main results, we state for easy reference the following fact, obtained from [15], which will be applied below.

LEMMA 3.7. *1. An operator $x \in \mathcal{M}$ belongs to $S(\mathcal{M})$ if and only if there exists a projection e of finite trace such that $l(x) \bigvee r(x) \leqslant e$(or equivalently, $exe = x$).*

2. Let $x \in S(\mathcal{M})$. Then $|x|^p \in S(\mathcal{M})$ for any $0 < p < \infty$. More generally, let h be a bounded Borel function on the spectrum $\sigma(|x|)$ of $|x|$. Then $h(|x|) \in S(\mathcal{M})$.

THEOREM 3.8. *Let $z \in \mathcal{M}$, $r > 0$ and x, $y \in S(\mathcal{M})_+$. Then the function*

$$\varphi(s,t) = \||x^{1-t}zy^{1+s}|^r\|_{E(\mathcal{M})} \cdot \||x^{1+t}zy^{1-s}|^r\|_{E(\mathcal{M})}$$

is convex on $[-1,1] \times [-1,1]$ and attains its minimum at $(0,0)$.

Proof. Let $x,\ y \in S(\mathscr{M})_+$. By Lemma 3.7(2), we have $x^{1-t},\ y^{1+s},\ x^{1+t},\ y^{1-s} \in S(\mathscr{M})$. We write $e = s(x) \bigvee s(y)$. It is clear that

$$ex^{1-t}zy^{1+s}e = x^{1-t}zy^{1+s}, \quad ex^{1+t}zy^{1-s}e = x^{1+t}zy^{1-s}.$$

From Lemma 3.7(1), we deduce $x^{1-t}zy^{1+s},\ x^{1+t}zy^{1-s} \in S(\mathscr{M})$. By Lemma 3.7(2), we obtain

$$|x^{1-t}zy^{1+s}|^r,\ |x^{1+t}zy^{1-s}|^r \in S(\mathscr{M}) \subseteq E(\mathscr{M})$$

for all $r > 0$ and $s,t \in [-1,1]$. Since $E(\mathscr{M})^{(r)}$ is a quasi-Banach space and $S(\mathscr{M}) \subseteq \mathscr{M}$, by Lemma 2.5 (vi) of [8], there exists some constant $C > 0$ such that

$$\begin{aligned}
\||x^{1\mp t}zy^{1\pm s} - x^{1\mp t_1}zy^{1\pm s_1}|^r\|_{E(\mathscr{M})} &= \|x^{1\mp t}zy^{1\pm s} - x^{1\mp t_1}zy^{1\pm s_1}\|^r_{E(\mathscr{M})^{(r)}} \\
&\leqslant C(\|(x^{1\mp t} - x^{1\mp t_1})zy^{1\pm s}\|_{E(\mathscr{M})^{(r)}} \\
&\quad + \|x^{1\mp t_1}z(y^{1\pm s} - y^{1\pm s_1})\|_{E(\mathscr{M})^{(r)}})^r \\
&\leqslant C\|z\|^r(\|x^{1\mp t} - x^{1\mp t_1}\|\|y^{1\pm s}\|\|s(y)\|_{E(\mathscr{M})^{(r)}} \\
&\quad + \|y^{1\pm s} - y^{1\pm s_1}\|\|x^{1\mp t_1}\|\|s(x)\|_{E(\mathscr{M})^{(r)}})^r.
\end{aligned}$$

This implies that $\varphi(s,t)$ is continuous. A similar argument to the proof of Theorem 3.3 shows that it suffices to prove the following inequality

$$\varphi(s_1,t_1) \leqslant \frac{1}{2}[\varphi(s_1+s_2,t_1+t_2) + \varphi(s_1-s_2,t_1-t_2)],\ s_1 \pm s_2,\ t_1 \pm t_2 \in [-1,1] \times [-1,1].$$

By Lemma 3.1 (or, Theorem 3 of [16]), we obtain

$$\begin{aligned}
\||x^{1-t_1}zy^{1+s_1}|^r\|_{E(\mathscr{M})} &= \||x^{t_2}(x^{1-t_1-t_2}zy^{1+s_1-s_2})y^{s_2}|^r\|_{E(\mathscr{M})} \\
&\leqslant \||x^{1-t_1+t_2}zy^{1+s_1-s_2}|^r\|^{\frac{1}{2}}_{E(\mathscr{M})}\||x^{1-t_1-t_2}zy^{1+s_1+s_2}|^r\|^{\frac{1}{2}}_{E(\mathscr{M})}
\end{aligned}$$

and

$$\begin{aligned}
\||x^{1+t_1}zy^{1-s_1}|^r\|_{E(\mathscr{M})} &= \||x^{t_2}(x^{1+t_1-t_2}zy^{1-s_1-s_2})y^{s_2}|^r\|_{E(\mathscr{M})} \\
&\leqslant \||x^{1+t_1+t_2}zy^{1-s_1-s_2}|^r\|^{\frac{1}{2}}_{E(\mathscr{M})}\||x^{1+t_1-t_2}zy^{1-s_1+s_2}|^r\|^{\frac{1}{2}}_{E(\mathscr{M})}.
\end{aligned}$$

Multiplying the above two inequalities we have

$$\begin{aligned}
\varphi(s_1,t_1) &= \||x^{1-t_1}zy^{1+s_1}|^r\|_{E(\mathscr{M})} \cdot \||x^{1+t_1}zy^{1-s_1}|^r\|_{E(\mathscr{M})} \\
&\leqslant [\varphi(s_1+s_2,t_1+t_2)\varphi(s_1-s_2,t_1-t_2)]^{\frac{1}{2}} \\
&\leqslant \frac{1}{2}[\varphi(s_1+s_2,t_1+t_2) + \varphi(s_1-s_2,t_1-t_2)].
\end{aligned}$$

This implies the desired result. $\square$

From Theorem 3.8, we obtain the generalization of inequality (1.2) for the norm on noncommutative symmetric space.

COROLLARY 3.9. *Let $x,\ y\in S(\mathscr{M})_+$, $z\in\mathscr{M}$ and $r>0$. Then*

$$\begin{aligned}|||x^{\frac{1}{2}}zy^{\frac{1}{2}}|^r\|_{E(\mathscr{M})}^2 &\leqslant \||x^tzy^{1-s}|^r\|_{E(\mathscr{M})}\cdot\||x^{1-t}zy^s|^r\|_{E(\mathscr{M})}\\ &\leqslant \max\{\||xz|^r\|_{E(\mathscr{M})}\||zy|^r\|_{E(\mathscr{M})},\||xzy|^r\|_{E(\mathscr{M})}\||z|^r\|_{E(\mathscr{M})}\},\end{aligned}$$

where $s,t\in[0,1]$.

Proof. If we replace s, t, x, y by $2s-1$, $2t-1$, $x^{\frac{1}{2}}$, $y^{\frac{1}{2}}$, respectively, in Theorem 3.8, we deduce that the function $\psi(s,t)=\||x^tzy^{1-s}|^r\|_{E(\mathscr{M})}\cdot\||x^{1-t}zy^s|^r\|_{E(\mathscr{M})}$ is convex on $[0,1]\times[0,1]$ and attains its minimum at $(\frac{1}{2},\frac{1}{2})$. Hence

$$\||x^{\frac{1}{2}}zy^{\frac{1}{2}}|^r\|_{E(\mathscr{M})}^2\leqslant\||x^tzy^{1-s}|^r\|_{E(\mathscr{M})}\cdot\||x^{1-t}zy^s|^r\|_{E(\mathscr{M})}.$$

Since $x,y\in S(\mathscr{M})_+\subseteq\mathscr{M}_+$ and $z\in\mathscr{M}$, then ψ is continuous and convex on $[0,1]\times[0,1]$, and so ψ attains its maximum at the vertices of the square. Moreover, due to the symmetry there are two possibilities for the maximum. $\square$

We conclude this section with a series of inequalities that lead to another refinements of the inequality in Corollary 3.5 and 3.9. From Theorem 3.8 and the proof of Corollary 3.9, we obtain that the functions

$$\varphi(s,t)=\||x^{1-t}zy^{1+s}|^r\|_{E(\mathscr{M})}\cdot\||x^{1+t}zy^{1-s}|^r\|_{E(\mathscr{M})}\ \text{ for }\ s,t\in[-1,1]$$

and

$$\psi(s,t)=\||x^tzy^{1-s}|^r\|_{E(\mathscr{M})}\cdot\||x^{1-t}zy^s|^r\|_{E(\mathscr{M})}\ \text{ for }\ s,t\in[0,1]$$

are convex functions. Applying Theorem 1 of [13] to the convex function $\varphi(s,t)$ and $\psi(s,t)$, we obtain the following series of inequalities.

PROPOSITION 3.10. *Let $x,\ y\in S(\mathscr{M})_+$ and $z\in\mathscr{M}$.*

1. *For $r>0$, we have*

$$\begin{aligned}\||xzy|^r\|_{E(\mathscr{M})}^2 &\leqslant \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\||x^{1-t}zy^{1+s}|^r\|_{E(\mathscr{M})}\cdot\||x^{1+t}zy^{1-s}|^r\|_{E(\mathscr{M})}dsdt\\ &\leqslant\frac{1}{2}[\||x^2z|^r\|_{E(\mathscr{M})}\||zy^2|^r\|_{E(\mathscr{M})}+\||x^2zy^2|^r\|_{E(\mathscr{M})}\||z|^r\|_{E(\mathscr{M})}].\end{aligned}$$

2. *If $r>0$, $p,q\in[0,1]$ and $p\neq\frac{1}{2}$, $q\neq\frac{1}{2}$, then we have*

$$\begin{aligned}\||x^{\frac{1}{2}}zy^{\frac{1}{2}}|^r\|_{E(\mathscr{M})}^2 &\leqslant\Big|\frac{1}{1-2p}\frac{1}{1-2q}\\ &\qquad\int_p^{1-p}\int_q^{1-q}\||x^tzy^{1-s}|^r\|_{E(\mathscr{M})}\cdot\||x^{1-t}zy^s|^r\|_{E(\mathscr{M})}dsdt\Big|\\ &\leqslant\frac{1}{2}[\||x^pzy^{1-q}|^r\|_{E(\mathscr{M})}\cdot\||x^{1-p}zy^q|^r\|_{E(\mathscr{M})}\\ &\qquad+\||x^{1-p}zy^{1-q}|^r\|_{E(\mathscr{M})}\cdot\||x^pzy^q|^r\|_{E(\mathscr{M})}].\end{aligned}$$

Acknowledgement. The author is grateful to the reviewer and the editor for their valuable comments and suggestions. This research is supported by the National Natural Science Foundation of China (No. 11401507) and the Natural Science Foundation of Xinjiang University (Starting Fund for Doctors, Grant No. BS150202).

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(Received April 18, 2015)

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