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# ON APPROXIMATION IN BA SPACES FOR JACKSON–MATSUOKA POLYNOMIALS ON THE SPHERE

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*Abstract.* We consider the best approximation by Jackson-Matsuoka polynomials on the unit sphere of $\mathbb{R}^d$ in the *Ba* space. Establish and use the relations between $K$-functionals and modulus of smoothness on the sphere, we obtain the direct and inverse estimate of approximation by these polynomials for the $h$-spherical harmonics.

## 1. Introduction and notations

Let $\mathbb{S} := \mathbb{S}^{d-1} = \{x : \|x\| = 1\}$ denote the unit sphere in $\mathbb{R}^d$ $(d \geqslant 3)$, $d \in \mathbb{N}$, where $\|x\|$ denotes the usual Euclidean norm, $\mathbb{R}$ the set of real numbers. For a nonzero vector $v \in \mathbb{R}^d$, let $\sigma_v$ denote the reflection with respect to the hyperplane perpendicular to $v$, $x\sigma_v := x - 2(\langle x,v\rangle/\|v\|^2)v$, $x \in \mathbb{R}^d$, where $\langle x,v\rangle$ denote the usual Euclidean inner product. Let $G$ be a finite reflection group on $\mathbb{R}^d$ with a fixed real halfline $\mathbb{R}_+$, normalized so that $\langle v,v\rangle = 2$ for all $v \in \mathbb{R}_+$. Then $G$ is a subgroup of the orthogonal group generated by the reflections $\{\sigma_v : v \in \mathbb{R}_+\}$. Let $\kappa$ be a nonnegative multiplicity function $v \mapsto \kappa_v$ defined on $\mathbb{R}_+$ with the property that $\kappa_u = \kappa_v$ whenever $\sigma_u$ is conjugate to $\sigma_v$ in $G$, then $v \mapsto \kappa_v$ is a $G$-invariant function.

We denote by $L_p(h_\kappa^2)$, $1 \leqslant p \leqslant \infty$, the space of functions defined on $\mathbb{S}$ with the finite norm

$$\|f\|_{\kappa,p} := \left(a_\kappa \int_{\mathbb{S}} |f(y)|^p h_\kappa^2(y) d\omega(y)\right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,$$

and for $p = \infty$ we assume that $L_\infty$ is replaced by $C(\mathbb{S})$ the space of continuous functions on $\mathbb{S}$ with the usual uniform norm $\|f\|_\infty$, where $h_\kappa$ is defined by (see [3])

$$h_\kappa = \prod_{v \in \mathbb{R}_+} |\langle x,v\rangle|^{\kappa_v}, \quad x \in \mathbb{R}^d,$$

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$d\omega$ is the surface (Lebesgue) measure on $\mathbb{S}$, and $a_\kappa$ the normalization constant of $h_\kappa$, $a_\kappa^{-1} = \int_{\mathbb{S}} h_\kappa^2(y)d\omega$. The function $h_\kappa$ is a positive homogeneous function of degree $\gamma_\kappa := \sum_{v\in R_+} \kappa_v$, and it is invariant under the reflection group.

The spherical $h$-harmonics are the restriction of $h$-harmonics on the unit sphere. Throughout this paper, we fie the value of $\lambda$ as

$$\lambda = \gamma_\kappa + \frac{d-1}{2}. \tag{1.1}$$

In terms of the polar coordinates $y = ry'$, $r = \|y\|$, the $h$-Laplacian operator $\Delta_h$ takes the form (see [3])

$$\Delta_h = \frac{\partial^2}{\partial r^2} + \frac{2\lambda+1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{h,0} \tag{1.2}$$

$\Delta_{h,0}$ is the Laplace-Beltrami operator on the sphere. Hence, applying $\Delta_h$ to $h$-harmonics $Y \in \mathscr{H}_n^d(h_\kappa^2)$ with $Y(y) = r^n Y(y')$ shows that spherical $h$-harmonics are eigenfunctions of $\Delta_{h,0}$; that is,

$$\Delta_{h,0}Y(x) = -n(n+2\lambda)Y(x), \quad x\in\mathbb{S}, \quad Y\in\mathscr{H}_n^d(h_\kappa^2). \tag{1.3}$$

It is known that $\dim \mathscr{H}_n^d(h_\kappa^2) = \dim \mathscr{P}_n^d - \dim \mathscr{P}_{n-2}^d$ with $\dim \mathscr{P}_n^d = \binom{n+d-1}{d}$.

The standard Hilbert space theory shows that $L^2(h_\kappa^2) = \Sigma_{n=0}^\infty \bigoplus \mathscr{H}_n^{d+1}(h_\kappa^2)$. That is, with each $f\in L^2(h_\kappa^2)$ we can associate its $h$-harmonic expansion

$$f(x) = \Sigma_{n=0}^\infty Y_n(h_\kappa^2; f, x), \quad x\in\mathbb{S},$$

in $L^2(h_\kappa^2)$ norm. For the surface measure $(\kappa = 0)$, such a series is called the Laplace series (see [10, Chap, [2])]. The orthogonal projection $Y_n(h_\kappa^2) : L_2(h_\kappa^2) \to \mathscr{H}_n^d(h_\kappa^2)$ takes the form

$$Y_n(h_\kappa^2; f, x) := \int_{\mathbb{S}} f(y)P_n(h_\kappa^2; x, y)h_\kappa^2(y)d\omega(y), \tag{1.4}$$

where $P_n(h_\kappa^2; x, y)$ is the reproducing kernel of the space of $h$-harmonics $\mathscr{H}_n^d(h_\kappa^2)$, which is given by (see [3])

$$P_n(h_\kappa^2; x, y) = \frac{n+\lambda}{\lambda} V_\kappa[C_n^\lambda(\langle\cdot, y\rangle)](x), \tag{1.5}$$

$C_n^\lambda$ is the ultraspherical polynomial of degree $n$, the intertwining operator $V_\kappa$ is a linear operator uniquely determined by

$$V_\kappa \mathscr{P}_n \subset \mathscr{P}_n, \quad V_\kappa 1 = 1, \quad \mathscr{D}_i V_\kappa = V_\kappa \partial_i, \quad 1 \leqslant i \leqslant d.$$

The spherical means denotes by

$$T_\theta(f) = \frac{1}{|\mathbb{S}^{d-2}|(\sin\theta)^{d-2}} \int_{\langle x,y\rangle=\cos\theta} f(y)d\omega(y), \tag{1.6}$$

where $|\mathbb{S}^{d-2}| = \int_{\mathbb{S}^{d-2}} d\omega = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$.

The spherical means associated with $h_\kappa^2 d\omega$, which $T_\theta^\kappa(f)$ is defined by

$$c_\lambda \int_0^\pi T_\theta^\kappa(f,x) g(\cos\theta)(\sin\theta)^{2\lambda} d\theta = a_\kappa \int_{\mathbb{S}} f(y) V_\kappa g(\langle x,y\rangle) h_\kappa^2(y) d\omega(y), \tag{1.7}$$

where $g$ is any function $[-1,1] \mapsto \mathbb{R}$ such that the integral in the right-hand side is finite, $c_\lambda^{-1} = \int_{-1}^1 (1-t^2)^{\lambda-\frac{1}{2}} dt = \frac{\Gamma(\lambda+\frac{1}{2})\sqrt{\pi}}{\Gamma(\lambda+1)}$. $T_\theta^\kappa(f)$ is a proper extension of $T_\theta(f)$, since $T_\theta(f)$ satisfies $T_\theta^\kappa(f)$ when $\kappa = 0$ and $V_\kappa = id$, and the properties of $T_\theta^\kappa$ are well known (see [3]).

Based on the classical Jackson-Matsuoka kernel (see [4]), we define a new kernel

$$M_{n;j,i,s}(\theta) := \frac{1}{\Omega_{n;j,i,s}} \left( \frac{\sin^{2j} n\theta/2}{\sin^{2i}\theta/2} \right)^{2s}, \quad n = 1,2,\ldots, \theta \in \mathbb{R},$$

where $j,i,s \in \mathbb{N}, \Omega_{n;j,i,s}$ is a constant chosen such that $c_\lambda \int_0^\pi M_{n;j,i,s}(\theta) \sin^{2\lambda}\theta d\theta = 1$. It is known that $M_{n;j,i,s}(\theta)$ is an even nonnegative operator. In particular, it is an even nonnegative trigonometric polynomial of degree at most $2s(nj+2j-2i)$ for $j > i$ and the Jackson polynomial for $j = i$. Using $M_{n;j,i,s}(\theta)$ we consider the spherical convolution

$$J_{n;j,i,s}(f;x) := (f * M_{n;j,i,s})(x) := c_\lambda \int_0^\pi T_\theta^\kappa(f;x) M_{n;j,i,s}(\theta)(\theta) \sin^{2\lambda}\theta d\theta. \tag{1.8}$$

It is called the Jackson-Matsuoka polynomials on the sphere based on the Jackson-Matsuoka kernel. In particular, $(f_0 * M_{n;j,i,s})(x) = 1$ for $f_0(x) = 1$. The classical Jackson-Matsuoka polynomials in the classical $L_p$ space have been studied by many authors (see [6, 4]). In [7], we got the equivalent theorem of the approximation for the Jackson-Matsuoka polynomials on the sphere by means of $h$-harmonic polynomials in the weighted Lebesgus space.

The conception of Ba spaces was first put forward by Ding (see [1]) in his discussion on the prior estimate of Laplace operator in some classical domains and in his study of the embedding theorem of Orlicz-Sobolev spaces, higher dimensional singular integrals and harmonic function etc.

DEFINITION 1.1. [1] Let $B = \{B_1, B_2, \cdots, B_m, \cdots\}$ be a sequence of linear normed function spaces, $a = \{a_1, a_2, \cdots, a_m, \cdots\}$ be a sequence of nonnegative numbers. For $f \in \bigcap_{m=1}^\infty B_m$, we form the power series of

$$I(f,\alpha) := \sum_{m=1}^\infty a_m \alpha^m \|f\|_{B_m}^m. \tag{1.9}$$

If $I(f,\alpha)$ has a non-zero radius of convergence, we say $f \in Ba$.

The norm in Ba is defined by

$$\|f\|_{Ba} := \inf_{\alpha>0} \{\frac{1}{\alpha} : I(f,\alpha) \leqslant 1\}. \tag{1.10}$$

As proved in [1], Ba is a Banach space if $B_m$ is a Banach space. Evidently, if $B_m = L_m$, then Ba space is an Orlicz space. If $B_m = L_p$, $a = \{1, 0, \cdots, 0, \cdots\}$, then a Ba space is a classical Lebesgue space (see [8]).

The purpose of this paper is to consider the best approximation by Jackson-Matsuoka polynomials on the unit sphere of $\mathbb{R}^d$ in the Ba space. The results presented in this paper extend and unify the results of the weighted Lebesgus space $L_p(h_\kappa^2)$ and the weighted Orlicz space. The approximation of $h$-harmonix polynomials, in the $L_p$ metric can be viewed as weighted approximation, in which the measure $d\omega$ on the sphere is replaced by $h_\kappa^2 d\omega$. It is well known that the situation can be quite different from that of ordinary harmonics, the weighted approximation is not a simple extension. Since the orthogonal group acts transitively on the point, the reflection groups do not act transitively on the sphere.

The paper is organized as follows. Section 2 introduces new $K$-functional and new modulus of smoothness and establishes their relations. Some properties of the Jackson-Matsuoka polynomials on the sphere are discussed in section 3. The direct and inverse estimate for the best approximation by the Jackson-Matsuoka polynomials on the unit sphere of $\mathbb{R}^d$ in the $Ba$ space in section 4.

## 2. $K$-functionals and modulus of smoothness

DEFINITION 2.1. For $f \in Ba$, the modulus of smoothness on the sphere is given by

$$\omega(f;t)_{Ba} := \sup_{0<\theta\leqslant t} \|f - T_\theta^\kappa(f)\|_{Ba}. \tag{2.1}$$

The $K$-functional of the sphere is given by

$$K(f;t^2)_{Ba} := \inf_{g\in W_{Ba}(h_\kappa^2)} \left\{\|f-g\|_{Ba} + t^2\|\Delta_{h,0}g\|_{Ba}\right\}, \tag{2.2}$$

where $W_{Ba}(h_\kappa^2) := \left\{f : f \in Ba, -k(k+2\lambda)P_k(h_\kappa^2;f) = P_k(h_\kappa^2;g) \text{ for some } g \in Ba\right\}$, $0 < t < t_0$, $t_0$ is a positive constant.

To prove the equivalence between the $K$-functional and the modulus of smoothness on the sphere, we need the following Lemma.

LEMMA 2.2. *Let* $B = \{L_{p_1}(h_\kappa^2), L_{p_2}(h_\kappa^2), \cdots, L_{p_m}(h_\kappa^2), \cdots\}$ *be a sequence of Lebesgue spaces,* $p_m \geqslant 1$, $m = 1, 2, \ldots,$ $a = \{a_1, a_2, \cdots, a_m, \cdots\}$ *be a sequence of nonnegative numbers,* $\{a_m^{\frac{1}{m}}\} \in l^\infty$. *If* $f \in Ba := \bigcap_{m=1}^{\infty} L_{p_m}(h_\kappa^2)$, *then*

$$\|f\|_{\kappa,p_m} \leqslant \frac{1}{\mu}\|f\|\|_{Ba}. \tag{2.3}$$

*where* $\mu = \inf_{m\geqslant 1}\{a_m^{\frac{1}{m}}\}$.

*Proof.* Since $\{a_m^{\frac{1}{m}}\} \in l^\infty$, we may let $0 < q = \sup_{m\geqslant 1}\{a_m^{\frac{1}{m}}\}$. From $\{a_m^{-\frac{1}{m}}\} \in l^\infty$, we may let $\mu = \inf_{m\geqslant 1}\{a_m^{\frac{1}{m}}\}$. Then $0 < \mu < \infty$.

In view of the $\sum\limits_{m=1}^{\infty} a_m \alpha^m \|f\|_{\kappa,p_m}^m \leqslant 1$, the $\sup\limits_{m\geqslant 1} \|f\|_{\kappa,p_m}$ exists. Let

$$u = \sup_{m\geqslant 1}\{\|f\|_{\kappa,p_m}\}.$$

By the definition of supremum, for any $\delta > 0$, there exists $K \geqslant 1$, such that $\|f\|_{\kappa,p_K} > u - \delta$. By the definition of $\|f\|_{Ba} = \inf\limits_{\alpha>0}\{\frac{1}{\alpha} : I(f,\alpha) \leqslant 1\}$,for any $\varepsilon > 0$, there exists $\frac{1}{\alpha_1}$, such that $\sum\limits_{m=1}^{\infty} a_m \alpha_1^m \|f\|_{\kappa,p_m}^m \leqslant 1$ holds. Therefore $\|f\|_{Ba} = \inf\limits_{\alpha>0}\{\frac{1}{\alpha} : I(f,\alpha)\} > \frac{1}{\alpha_1} - \varepsilon$. Namely

$$1 \geqslant \sum_{m=1}^{\infty} a_m \alpha_1^m \|f\|_{\kappa,p_m}^m \geqslant a_K \alpha_1^K \|f\|_{\kappa,p_K}^K > [a_K^{\frac{1}{K}}(u-\delta)]^K \geqslant [s\alpha_1(u-\delta)]^K.$$

By the arbitrariness of $\delta$,

$$\frac{1}{\alpha_1} \geqslant \mu \cdot u = \mu \cdot \sup_{m\geqslant 1}\{\|f\|_{\kappa,p_m}\},$$

$$\|f\|_{\kappa,p_m} > \frac{1}{\alpha_1} - \varepsilon \geqslant \mu \cdot \sup_{m\geqslant 1}\{\|f\|_{\kappa,p_m}\} - \varepsilon$$

and also, $\varepsilon$ is arbitrary, therefore

$$\sup_{m\geqslant 1}\{\|f\|_{\kappa,p_m}\} \leqslant \frac{1}{\mu}\|f\|_{Ba},$$

which implies that for any $p_m$, we have

$$\|f\|_{\kappa,p_m} \leqslant \frac{1}{\mu}\|f\|_{Ba}.$$

The proof is completed. □

We will establish the weak equivalence between the $K$-functional and the modulus of smoothess on the unit sphere in the Ba space.

THEOREM 2.3. *Let* $B = \{L_{p_1}(h_\kappa^2), L_{p_2}(h_\kappa^2), \cdots, L_{p_m}(h_\kappa^2), \cdots\}$ *be a sequence of Lebesgue spaces,* $p_m \geqslant 1$, $m = 1,2,\ldots$, $a = \{a_1, a_2, \cdots, a_m, \cdots\}$ *be a sequence of nonnegative numbers. If* $\{a_m^{\frac{1}{m}}\} \in l^\infty$. *Then, for* $f \in Ba$, $0 < t < \frac{\pi}{2}$, *then there exist constants* $C_1(\mu,q)$ *and* $C_2(\mu,q)$ *such that*

$$C_1(\mu,q)\omega(f;t)_{Ba} \leqslant K(f;t^2)_{Ba} \leqslant C_2(\mu,q)\omega(f;t)_{Ba}. \tag{2.4}$$

*where* $\mu = \inf\limits_{m\geqslant 1}\{a_m^{\frac{1}{m}}\}, q = \sup\limits_{m\geqslant 1}\{a_m^{\frac{1}{m}}\}$.

Throughout this paper, $C$ denotes a positive constant independent on $n$ and $f$ and $C(a)$ denotes a positive constant dependent on $a$, which may be different in according to the circumstances.

*Proof.* For $m=1,2,\ldots,\ g\in W_{Ba}(h_\kappa^2)$, note that [3]

$$\|T_\theta^\kappa g-g\|_{\kappa,p_m}\leqslant C\theta^2\|\Delta_{h,0}g\|_{\kappa,p_m}.$$

$$\|T_\theta^\kappa f\|_{\kappa,p_m}\leqslant \|f\|_{\kappa,p_m}.$$

By the definition of the Ba-norm $\|\cdot\|_{Ba}$ and (2.3), we have

$$\begin{aligned}\|T_\theta^\kappa g-g\|_{Ba}&=\inf_{\alpha>0}\left\{\alpha:\sum_{m=1}^{\infty}\frac{a_m}{\alpha^m}\|T_\theta^\kappa g-g\|_{\kappa,p_m}^m\leqslant 1\right\}\\&\leqslant\inf_{\alpha>0}\left\{\alpha:\sum_{m=1}^{\infty}\frac{a_m}{\alpha^m}C^m\theta^{2m}\|\Delta_{h,0}g\|_{\kappa,p_m}^m\leqslant 1\right\}\\&\leqslant\inf_{\alpha>0}\left\{\alpha:\sum_{m=1}^{\infty}\frac{C^mq^m}{\alpha^m}\theta^{2m}\|\Delta_{h,0}g\|_{\kappa,p_m}^m\leqslant 1\right\}\\&\leqslant\inf_{\alpha>0}\left\{\alpha:\sum_{m=1}^{\infty}\frac{1}{\alpha^m}\left(\frac{C\cdot q\cdot\theta^2}{\mu}\|\Delta_{h,0}g\|_{Ba}\right)^m\leqslant 1\right\}.\end{aligned}\tag{2.5}$$

Let $\alpha=2\frac{C\cdot q\cdot\theta^2}{\mu}\|\Delta_{h,0}g\|_{Ba}$, then $\sum_{m=1}^{\infty}\frac{1}{\alpha^m}\left(\frac{C\cdot q\cdot\theta^2}{\mu}\|\Delta_{h,0}g\|_{Ba}\right)^m=1$. Consequently $\sum_{m=1}^{\infty}\frac{a_m}{\alpha^m}\|T_\theta^\kappa g-g\|_{\kappa,p_m}^m\leqslant 1$.

Therefore, we have

$$\|T_\theta^\kappa g-g\|_{Ba}\leqslant C(q,\mu)\theta^2\|\Delta_{h,0}g\|_{Ba}.\tag{2.6}$$

The proof is similar to that of (2.6), we get

$$\|T_\theta^\kappa(f-g)\|_{Ba}\leqslant C(q,\mu)\|f-g\|_{Ba}.\tag{2.7}$$

The triangle inequality gives

$$\|T_\theta^\kappa f-f\|_{Ba}\leqslant 2\|f-g\|_{Ba}+C(q,\mu)\theta^2\|\Delta_{h,0}g\|_{Ba},$$

which shows that $\omega(f;t)_{Ba}\leqslant C(q,\mu)K(f;t^2)_{Ba}$. On the other hand, define

$$g(x)=v_0\int_0^\theta(\sin u)^{-2\lambda}du\int_0^u T_t^\kappa f(x)(\sin t)^{2\lambda}dt$$

with $v_\theta^{-1}=\int_0^\theta(\sin u)^{-2\lambda}du\int_0^u(\sin t)^{2\lambda}dt$. Then $\Delta_{h,0}g=v_\theta(T_\theta^\kappa f-f)$, this also give

$$\|\Delta_{h,0}g\|_{\kappa,p_m}\leqslant C\theta^{-2}\|T_\theta^\kappa f-f\|_{\kappa,p_m}.\tag{2.8}$$

Since for $\theta \leqslant \frac{\pi}{2}$, the inequality $\frac{2}{\pi}\theta \leqslant \sin\theta \leqslant \theta$ shows that $v_\theta^{-1} \sim \theta^2$. Moreover,

$$f-g = v_\theta^{-1}\int_0^\theta (\sin u)^{-2\lambda}du\int_0^u (T_t^\kappa - f)(\sin t)^{2\lambda}dt,$$

Consequently, we get

$$\|f-g\|_{\kappa,p_m} \leqslant C\|T_\theta^\kappa f - f\|_{\kappa,p_m}. \tag{2.9}$$

By (2.8) and (2.9), similar to the proof of (2.6), we obtain that

$$\|\Delta_{h,0}g\|_{Ba} \leqslant C\theta^{-2}\|T_\theta^\kappa f - f\|_{Ba}. \tag{2.10}$$

and

$$\|f-g\|_{Ba} \leqslant C\|T_\theta^\kappa f - f\|_{Ba}. \tag{2.11}$$

Combining (2.10),(2.11) and the definition of $K$-functional, we have

$$\begin{aligned} K(f;\theta^2)_{Ba} &\leqslant \|f-g\|_{Ba} + \theta^2\|\Delta_{h,0}g\|_{Ba} \\ &\leqslant C\|T_\theta^\kappa f - f\|_{Ba} + C\theta^{-2}\theta^2\|T_\theta^\kappa f - f\|_{Ba} \\ &\leqslant C\|T_\theta^\kappa f - f\|_{Ba}. \end{aligned} \tag{2.12}$$

Thus

$$K(f;t^2)_{Ba} \leqslant C\omega(f;t)_{Ba}. \quad \square$$

COROLLARY 2.4. *For $t \geqslant 0$, there is a constant $C$ such that*

$$\omega(f;t\delta)_{Ba} \leqslant C\max\{1,t^2\}\omega(f;\delta)_{Ba}. \tag{2.13}$$

*Proof.* By the equivalent relation between the modulus of smoothness and $K$-functional, and the definition of $K(f;t^2)_{Ba}$, we have

$$\begin{aligned} \omega(f;t\delta)_{Ba} &\leqslant CK(f;(t\delta)^2)_{Ba} \leqslant C(\|f-g\|_{Ba} + t^2\delta^2\|\Delta_{h,0}g\|_{Ba}) \\ &\leqslant C\max\{1,t^2\}(\|f-g\|_{Ba} + \delta^2\|\Delta_{h,0}g\|_{Ba}) \\ &\leqslant C\max\{1,t^2\}K(f;\delta^2)_{Ba} \leqslant C\max\{1,t^2\}\omega(f;\delta)_{Ba} \end{aligned}$$

The Corollary 2.4 has been proved. $\square$

## 3. Auxiliary Lemmas

We need the following auxiliary Lemmas.

LEMMA 3.1. *Let $\Omega_{n;,j,i,s} = \int_0^\pi \left(\frac{\sin^{2j}\frac{n\theta}{2}}{\sin^{2i}\frac{\theta}{2}}\right)^{2s}\sin^{2\lambda}\theta d\theta$. Then, the weak equivalence*

$$\Omega_{n;j,i,s} \asymp n^{4is-2\lambda-1} \tag{3.1}$$

*holds for $4si > 2\lambda+1$, $j \geqslant i$, where the weak equivalent relation $A(n) \asymp B(n)$ means that $A(n) \ll B(n)$ and $B(n) \ll A(n)$, and relation $A_n \ll B_n$ means that there is a positive constant $C$ independent on $n$ such that $A(n) \leqslant CB(n)$ holds.*

*Proof.* Since $\frac{\theta}{\pi} \leqslant \sin\frac{\theta}{2} \leqslant \frac{\theta}{2}$, and $\sin\theta \leqslant \theta$ for $0 \leqslant \theta \leqslant \pi$, we have

$$\begin{aligned}\Omega_{n;,j,i,s} &= \int_0^{\pi} \left(\frac{\sin^{2j}\frac{n\theta}{2}}{\sin^{2i}\frac{\theta}{2}}\right)^{2s} \sin^{2\lambda}\theta d\theta \\ &\asymp n^{4is-2\lambda-1}\int_0^{n\pi/2} t^{2\lambda}\left(\frac{\sin^{2j}t}{t^{2i}}\right)^{2s} dt \\ &\asymp n^{4is-2\lambda-1}\left(\int_0^{\pi/2} t^{2\lambda}\left(\frac{\sin^{2j}t}{t^{2i}}\right)^{2s} dt + \int_{\pi/2}^{\infty} t^{2\lambda}\left(\frac{\sin^{2j}t}{t^{2i}}\right)^{2s} dt\right) \\ &\asymp n^{4is-2\lambda-1}, \end{aligned} \tag{3.2}$$

as $4si > 2\lambda+1$, $j \geqslant i$. The Lemma 3.1 has been proved. □

LEMMA 3.2. *For* $4is > r+2\lambda+1$, $j \geqslant i$, $r \in \mathbb{R}$, *there is a constant* $C(\lambda, j, i, s)$ *such that*

$$\int_0^{\pi} \theta^r M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta \leqslant C(\lambda, j, i, s) n^{-r}. \tag{3.3}$$

*Proof.* Since $\frac{\theta}{\pi} \leqslant \sin\frac{\theta}{2} \leqslant \frac{\theta}{2}$, and $\sin\theta \leqslant \theta$ for $0 \leqslant \theta \leqslant \pi$, by $\Omega_{n;j,i,s} \asymp n^{4is-2\lambda-1}$, we have

$$\begin{aligned}&\int_0^{\pi} \theta^r M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta \\ &\leqslant C(\lambda,i,j,s) n^{-4is+2\lambda+1}\int_0^{\pi} \theta^r \left(\frac{\sin^{2j}\frac{n\theta}{2}}{\sin^{2i}\frac{\theta}{2}}\right)^{2s} \sin^{2\lambda}\theta d\theta \\ &\leqslant C(\lambda,i,j,s) n^{-4is+2\lambda+1} n^{4is-r-2\lambda-1}\int_0^{n\pi/2} t^{r+2\lambda}\left(\frac{\sin^{2j}t}{t^{2i}}\right)^{2s} dt \\ &\leqslant C(\lambda,i,j,s) n^{-r}\left(\int_0^{\pi/2} t^{r+2\lambda}\left(\frac{\sin^{2j}t}{t^{2i}}\right)^{2s} dt + \int_{\pi/2}^{\infty} t^{r+2\lambda}\left(\frac{\sin^{2j}t}{t^{2i}}\right)^{2s} dt\right) \\ &\leqslant C(\lambda,j,i,s) C_2 n^{\lambda} \leqslant C(\lambda,j,i,s) n^{\lambda},\end{aligned}$$

where

$$C_2 = \int_0^{\pi/2} t^{\lambda}\left(\frac{\sin^{2j}t}{t^{2i}}\right)^{2s} dt + \int_{\pi/2}^{\infty} t^{\lambda}\left(\frac{\sin^{2j}t}{t^{2i}}\right)^{2s} dt, \ \ 4is > r+2\lambda+1, \ \ j \geqslant i. \quad \square$$

LEMMA 3.3. [3] *For* $0 \leqslant \theta \leqslant \pi$, *we have*

$$\begin{aligned} T_\theta^\kappa(g;x) - g(x) &= \int_0^{\theta} \sin^{-2\lambda} t dt \int_0^t T_u^\kappa(\Delta_{h,0}g)\sin^{2\lambda} u du \\ &= \int_0^{\theta} \sin^{-2\lambda} t \Phi(t) B_t(\Delta_{h,0}g, x) dt, \end{aligned} \tag{3.4}$$

*where*

$$B_t(\Delta_{h,0}g,x)=\frac{1}{\Phi(t)}\int_0^t T_u^\kappa(\Delta_{h,0}g)\sin^{2\lambda}u\,du,$$

*and* $\Phi(t)=c_\lambda^{-1}\int_0^t\sin^{2\lambda}u\,du$.

LEMMA 3.4. *Let* $g,\ \Delta_{h,0}g,\ \Delta_{h,0}^2g\in Ba:=\bigcap_{m=1}^{\infty}L_{p_m}(h_\kappa^2)$, $1\leqslant p_m\leqslant\infty$, $m=1,2,\ldots$, $J_{n;j,i,s}(f;x)$ *be the Jackson-Matsuoka polynomials on the sphere based on the Jackson-Matsuoka kernel,* $4is>2\lambda+5$, $j\geqslant i$. *Then, there is a constant* $C(\lambda,j,i,s)$ *such that*

$$\|J_{n;j,i,s}g-g-\alpha(n)\Delta_{h,0}g\|_{Ba}\leqslant C(\lambda,j,i,s,q,\mu)n^{-4}\|\Delta_{h,0}^2g\|_{Ba},\tag{3.5}$$

*where* $\alpha(n)\asymp n^{-2}$.

*Proof.* By Lemma 3.3, we have

$$\begin{aligned}
&J_{n;j,i,s}(g;x)-g(x)\\
=&\ c_\lambda\int_0^\pi M_{n;j,i,s}(\theta)(T_\theta^\kappa(g;x)-g(x))\sin^{2\lambda}\theta d\theta\\
=&\ c_\lambda\int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta\int_0^\theta\frac{\Phi(t)}{\sin^{2\lambda}t}B_t(\Delta_{h,0}g,x)dt\\
=&\ c_\lambda\Delta_{h,0}g(x)\int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta\int_0^\theta\frac{\Phi(t)}{\sin^{2\lambda}t}dt\\
&+c_\lambda\int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta\int_0^\theta\frac{\Phi(t)}{\sin^{2\lambda}t}(B_t(\Delta_{h,0}g,x)-\Delta_{h,0}g(x))dt\\
=&\ \Delta_{h,0}g(x)\int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta\int_0^\theta\frac{dt}{\sin^{2\lambda}t}\int_0^t\sin^{2\lambda}u\,du\\
&+\int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta\int_0^\theta\frac{dt}{\sin^{2\lambda}t}\int_0^t\sin^{2\lambda}u(B_t(\Delta_{h,0}g,x)-\Delta_{h,0}g(x))du\\
:=&\ \alpha(n)\Delta_{h,0}g(x)+\int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta\Psi_\theta(g,x)d\theta,
\end{aligned}\tag{3.6}$$

where

$$\alpha(n):=\int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta\int_0^\theta\frac{dt}{\sin^{2\lambda}t}\int_0^t\sin^{2\lambda}u\,du,$$

and

$$\Psi_\theta(g,x):=\int_0^\theta\frac{dt}{\sin^{2\lambda}t}\int_0^t\sin^{2\lambda}u(B_t(\Delta_{h,0}g,x)-\Delta_{h,0}g(x))du.$$

By Lemma3.2, we have

$$\begin{aligned}
\alpha(n)&=\int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta\int_0^\theta\frac{dt}{\sin^{2\lambda}t}\int_0^t\sin^{2\lambda}u\,du\\
&\asymp\int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta\int_0^\theta\frac{t\sin^{2\lambda}\xi}{\sin^{2\lambda}t}dt\\
&\asymp\int_0^\pi\theta^2M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta\asymp n^{-2},\quad(0<\xi<t).
\end{aligned}\tag{3.7}$$

Using Lemma 3.3 and the expression of $B_t(\Delta_{h,0}g,x)-\Delta_{h,0}g(x)$, and obtain

$$\|\Psi_\theta(g)\|_{\kappa,p_m}\leqslant C(\lambda,j,i,s)\theta^4\|\Delta_{h,0}^2 g\|_{\kappa,p_m}.$$

By Lemma3.2, and Hölder-Minkowski inequality shows that

$$\begin{aligned}
&\|\int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta\Psi_\theta(g,x)d\theta\|_{\kappa,p_m}\\
&\leqslant C(\lambda,j,i,s)\|\Delta_{h,0}^2 g\|_{\kappa,p_m}\int_0^\pi \theta^4 M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta\\
&\leqslant C(\lambda,j,i,s)n^{-4}\|\Delta_{h,0}^2 g\|_{\kappa,p_m}.
\end{aligned}\tag{3.8}$$

Consequently, by (3.6), (3.7) and (3.8), we have

$$\|J_{n;j,i,s}g-g-\alpha(n)\Delta_{h,0}g\|_{\kappa,p_m}\leqslant C(\lambda,j,i,s)n^{-4}\|\Delta_{h,0}^2 g\|_{\kappa,p_m}.\tag{3.9}$$

By Lemma 3.2, we get

$$\begin{aligned}
&\|J_{n;j,i,s}g-g-\alpha(n)\Delta_{h,0}g\|_{Ba}\\
&=\inf_{\alpha>0}\left\{\alpha:\sum_{m=1}^\infty \frac{a_m}{\alpha^m}\|J_{n;j,i,s}g-g-\alpha(n)\Delta_{h,0}g\|_{\kappa,p_m}^m\leqslant 1\right\}\\
&\leqslant \inf_{\alpha>0}\left\{\alpha:\sum_{m=1}^\infty \frac{a_m}{\alpha^m}C(\lambda,j,i,s)n^{-4}\|\Delta_{h,0}^2 g\|_{\kappa,p_m}^m\leqslant 1\right\}\\
&\leqslant \inf_{\alpha>0}\left\{\alpha:\sum_{m=1}^\infty \frac{q^m\cdot C^m}{\alpha^m}C(\lambda,j,i,s)n^{-4}\|\Delta_{h,0}^2 g\|_{Ba}^m\leqslant 1\right\}\\
&\leqslant C(\lambda,j,i,s,q,\mu)n^{-4}\|\Delta_{h,0}^2 g\|_{Ba}.
\end{aligned}$$

The Lemma 3.4 has been proved. □

## 4. Main results

The main result of this paper is the following:

THEOREM 4.1. *Suppose that* $f\in Ba:=\bigcap_{m=1}^{\infty}L_{p_m}(h_\kappa^2)$, $1\leqslant p_m\leqslant\infty$, $m=1,2,\ldots,$ $J_{n;j,i,s}(f;x)$ *be the Jackson-Matsuoka polynomials on the sphere based on the Jackson-Matsuoka kernel,* $4is>2\lambda+5$, $j\geqslant i$. *Then*

$$\|J_{n;j,i,s}(f)-f\|_{Ba}\asymp\omega(f;n^{-1})_{Ba}.\tag{4.1}$$

*Proof.* First we prove $\|J_{n;j,i,s}(f)-f\|_{Ba} \ll \omega(f;n^{-1})_{Ba}$. Since $(f_0 * M_{n;j,i,s})(x) = 1$ for $f_0(x)=1$, therefore, by Lemma 2.2, we have that

$$\begin{aligned}
&\|J_{n;j,i,s}(f)-f\|_{Ba} \\
&= \inf_{\alpha>0}\left\{\alpha : \sum_{m=1}^{\infty}\frac{a_m}{\alpha^m}\|J_{n;j,i,s}(f)-f\|_{\kappa,p_m}^m \leqslant 1\right\} \\
&\leqslant \inf_{\alpha>0}\left\{\alpha : \sum_{m=1}^{\infty}\frac{a_m}{\alpha^m}\|\int_0^{\pi}M_{n;j,i,s}(\theta)\left(f(x)-T_\theta^\kappa(f;x)\right)\sin^{2\lambda}\theta d\theta\|_{\kappa,p_m}^m \leqslant 1\right\} \\
&\leqslant \inf_{\alpha>0}\left\{\alpha : \sum_{m=1}^{\infty}\frac{a_m}{\alpha^m}\left(\int_0^{\pi}\|f-T_\theta^\kappa(f)\|_{\kappa,p_m}M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta\right)^m \leqslant 1\right\} \\
&\leqslant \inf_{\alpha>0}\left\{\alpha : \sum_{m=1}^{\infty}\frac{q^m\cdot C^m}{\alpha^m}\left(\int_0^{\pi}\|f-T_\theta^\kappa(f)\|_{Ba}M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta\right)^m \leqslant 1\right\}. \qquad (4.2)
\end{aligned}$$

Splitting the integral over $[0,\pi]$ into two integrals over $[0,1/n]$ and $[1/n,\pi]$, respectively, and using the definition of $\omega(f;t)_{Ba}$, we conclude that

$$\|f-T_\theta^\kappa(f)\|_{Ba} \leqslant \omega(f;n^{-1})_{Ba} + \int_{1/n}^{\pi}\omega(f;\theta)_{Ba}M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta. \qquad (4.3)$$

From Corollary 2.4 it follows that, for $\theta \geqslant n^{-1}$,

$$\omega(f;\theta)_{Ba} = \omega(f;n\theta\frac{1}{n})_{Ba} \leqslant C\max\{1,n^2\theta^2\}\omega\left(f;\frac{1}{n}\right)_{Ba} \leqslant Cn^2\theta^2\omega\left(f;\frac{1}{n}\right)_{Ba}. \qquad (4.4)$$

By (4.3), (4.4) and Lemma 3.1, we get

$$\|f-T_\theta^\kappa(f)\|_{Ba} \leqslant C\omega\left(f;\frac{1}{n}\right)_{Ba}. \qquad (4.5)$$

Therefore, by (4.2), (4.5), it follows that

$$\begin{aligned}
&\|J_{n;j,i,s}(f)-f\|_{Ba} \\
&\leqslant \inf_{\alpha>0}\left\{\alpha : \sum_{m=1}^{\infty}\frac{q^m\cdot C^m}{\alpha^m}\left(\int_0^{\pi}\omega\left(f;\frac{1}{n}\right)_{Ba}M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta\right)^m \leqslant 1\right\} \\
&= \inf_{\alpha>0}\left\{\alpha : \sum_{m=1}^{\infty}\frac{q^m\cdot C^m}{\alpha^m}\left(\omega\left(f;\frac{1}{n}\right)_{Ba}\right)^m \leqslant 1\right\} \\
&\leqslant C(\lambda,j,i,s,q,\mu)\omega\left(f;\frac{1}{n}\right)_{Ba}. \qquad (4.6)
\end{aligned}$$

Next we prove $\omega(f;n^{-1})_{Ba} \ll \|J_{n;j,i,s}(f)-f\|_{Ba}$. Let $r$ be a fixed positive integer, Denote by

$$J_{n;j,i,s}^r(f) := \sum_{k=0}^{r}\left(\int_0^{\pi}M_{n;j,i,s}(\theta)Q_k^\lambda(\cos\theta)\sin^{2\lambda}\theta d\theta\right)^r Y_k(h_\kappa^2;f).$$

By orthogonality of the orthogonal projector $Y_k$, we have that

$$J^{r+l}(f) = \sum_{k=0}^{r}\left(\int_0^{\pi} M_{n;j,i,s}(\theta)Q_k^{\lambda}(\cos\theta)\sin^{2\lambda}\theta d\theta\right)^r \times Y_k\left(h_\kappa^2;\sum_{v=0}^{r}\left(\int_0^{\pi} M_{n;j,i,s}(\theta)Q_v^{\lambda}(\cos\theta)\sin^{2\lambda}\theta d\theta\right)^l Y_v(h_\kappa^2;f)\right) = J^r_{n;j,i,s}(J^l_{n;j,i,s}(f)). \tag{4.7}$$

Let $g = J^r_{n;j,i,s}(f)$, by (4.7) we get

$$\begin{aligned}
\|f-g\|_{Ba} &= \inf_{\alpha>0}\left\{\alpha : \sum_{m=1}^{\infty}\frac{a_m}{\alpha^m}|f-g\|^m_{\kappa,p}\leqslant 1\right\} \\
&= \inf_{\alpha>0}\left\{\alpha : \sum_{m=1}^{\infty}\frac{a_m}{\alpha^m}\|f-J^r_{n;j,i,s}(f)\|^m_{\kappa,p}\leqslant 1\right\} \\
&\leqslant \inf_{\alpha>0}\left\{\alpha : \sum_{m=1}^{\infty}\frac{a_m}{\alpha^m}\left(\sum_{k=1}^{r}\|J^{k-1}_{n;j,i,s}(f)-J^k_{n;j,i,s}(f)\|_{\kappa,p}\right)^m\leqslant 1\right\} \\
&\leqslant \inf_{\alpha>0}\left\{\alpha : \sum_{m=1}^{\infty}\frac{a_m}{\alpha^m}\left(C(\lambda,j,i,s)\sum_{k=1}^{r}\|J^{k-1}_{n;j,i,s}(f-J_{n;j,i,s}(f))\|_{\kappa,p}\right)^m\leqslant 1\right\} \\
&\leqslant \inf_{\alpha>0}\left\{\alpha : \sum_{m=1}^{\infty}\frac{a_m}{\alpha^m}C(\lambda,j,i,s)r\|f-J_{n;j,i,s}(f)\|^m_{\kappa,p}\leqslant 1\right\} \\
&\leqslant \inf_{\alpha>0}\left\{\alpha : \sum_{m=1}^{\infty}\frac{q^m\cdot C_1^m(\lambda,j,i,s,r)}{\alpha^m}\|f-J_{n;j,i,s}(f)\|^m_{Ba}\leqslant 1\right\} \\
&\leqslant C(\lambda,j,i,s,r,q,\mu)\|f-J_{n;j,i,s}(f)\|_{Ba},
\end{aligned} \tag{4.8}$$

where $J^0_{n;j,i,sq,\mu}(f) = f$.

On the other hand,

$$\|\Delta_{h,0}J^r_{n;j,i,s}(f)\|_{\kappa,p_m} \leqslant \sum_{k=0}^{r}k(k+2\lambda)\Big(\int_0^{\pi}M_{n;j,i,s}(\theta)|Q_k^{\lambda}(\cos\theta)|\sin^{2\lambda}\theta d\theta\Big)^r Y_k(h_\kappa^2;f).$$

Note that [5]

$$|Q_k^{\lambda}(\cos\theta)| \equiv |\frac{C_k^{\lambda}(\cos\theta)}{C_k^{\lambda}(1)}| \leqslant C\min\{(k\theta)^{-1},1\}.$$

For $k\theta \geqslant 1$, from (3.3) it follows that

$$\begin{aligned}
&\|\Delta_{h,0}J^r_{n;j,i,s}(f)\|_{\kappa,p_m} \\
&\leqslant C(\lambda,j,i,s)\|\sum_{k=0}^{r}k(k+2\lambda)k^{-r\lambda}\left(\int_0^{\pi}M_{n;j,i,s}(\theta)\theta^{-\lambda}\sin^{2\lambda}\theta d\theta\right)^r Y_k(h_\kappa^2;f)\|_{\kappa,p_m} \\
&\leqslant C(\lambda,j,i,s)n^{r\lambda}\|f\|_{\kappa,p_m}\sum_{k=0}^{\infty}k^{2-r\lambda} \leqslant C(\lambda,j,i,s)n^{r\lambda}\|f\|_{\kappa,p_m}
\end{aligned} \tag{4.9}$$

holds for $r > \frac{3}{\lambda}$. For $k\theta < 1$, by (3.3), we get

$$\begin{aligned}
&\|\Delta_{h,0}J^r_{n;j,i,s}(f)\|_{\kappa,p_m}\\
&\leqslant \|\sum_{k=0}^{r}\left(\int_0^\pi M_{n;j,i,s}(\theta)\theta^{-\frac{2}{r}}(\theta^2k(k+2\lambda))^{\frac{1}{r}}|Q_k^\lambda(\cos\theta)|\sin^{2\lambda}\theta d\theta\right)^r Y_k(h_\kappa^2;f)\|_{\kappa,p_m}\\
&\leqslant C(\lambda,j,i,s)\|\sum_{k=0}^{r}\left(\int_0^\pi M_{n;j,i,s}(\theta)\theta^{-\frac{2}{r}}((k\theta)^2)^{\frac{2}{r}}\sin^{2\lambda}\theta d\theta\right)^r Y_k(h_\kappa^2;f)\|_{\kappa,p_m}\\
&\leqslant C(\lambda,j,i,s)\|\sum_{k=0}^{r}\left(\int_0^\pi M_{n;j,i,s}(\theta)\theta^{-\frac{2}{r}}\sin^{2\lambda}\theta d\theta\right)^r Y_k(h_\kappa^2;f)\|_{\kappa,p_m}\\
&\leqslant C(\lambda,j,i,s)n^2\|\sum_{k=0}^{\infty}Y_k(h_\kappa^2;f)\|_{\kappa,p_m}\leqslant C(\lambda,j,i,s)n^2\|f\|_{\kappa,p_m}.
\end{aligned} \tag{4.10}$$

Consequently, the inequality

$$\|\Delta_{h,0}J^r_{n;j,i,s}(f)\|_{\kappa,p_m}\leqslant C(\lambda,j,i,s)n^2\|f\|_{\kappa,p_m} \tag{4.11}$$

holds uniformly for $r > \frac{3}{\lambda}$. thereby

$$\begin{aligned}
&\|\Delta_{h,0}J^r_{n;j,i,s}(f)\|_{Ba}\\
&=\inf_{\alpha>0}\left\{\alpha:\sum_{m=1}^{\infty}\frac{a_m}{\alpha^m}\|\Delta_{h,0}J^r_{n;j,i,s}(f)\|_{\kappa,p_m}\leqslant 1\right\}\\
&\leqslant \inf_{\alpha>0}\left\{\alpha:\sum_{m=1}^{\infty}\frac{a_m}{\alpha^m}C(\lambda,j,i,s)n^2\|f\|_{\kappa,p_m}\leqslant 1\right\}\\
&\leqslant \inf_{\alpha>0}\left\{\alpha:\sum_{m=1}^{\infty}\frac{q^m\cdot C^m}{\alpha^m}C(\lambda,j,i,s)n^2\|f\|_{Ba}\leqslant 1\right\}\\
&\leqslant C(\lambda,j,i,s,q,\mu)n^2\|f\|_{Ba}.
\end{aligned} \tag{4.12}$$

Without loss of generality, we may assume $r_1 > \frac{3}{\lambda}$, $r > r_1+\frac{3}{\lambda}$. Using Lemma 3.4 and (4.7), we have

$$\begin{aligned}
&\alpha(n)\|\Delta_{h,0}J^r_{n;j,i,s}(f)\|_{Ba}\\
&=\|\alpha(n)\Delta_{h,0}J^r_{n;j,i,s}(f)\|_{Ba}\\
&\leqslant \|J^r_{n;j,i,s}(f)-f\|_{Ba}+C(\lambda,j,i,s)n^{-2}\|\Delta^2_{h,0}J^r_{n;j,i,s}(f)\|_{Ba}\\
&\leqslant r\|J_{n;j,i,s}(f)-f\|_{Ba}+C(\lambda,j,i,s)n^{-2}\|\Delta^2_{h,0}J^{r-r_1}_{n;j,i,s}(f)\|_{Ba}\\
&\leqslant r\|J_{n;j,i,s}(f)-f\|_{Ba}+C(\lambda,j,i,s)\\
&\quad\times\left(n^{-2}\|\Delta_{h,0}J^r_{n;j,i,s}(f)\|_{Ba}+n^{-2}\|J^r_{n;j,i,s}(f)-J^{r-r_1}_{n;j,i,s}(f)\|_{Ba}\right)\\
&\leqslant r\|J_{n;j,i,s}(f)-f\|_{Ba}+C(\lambda,j,i,s)\left(n^{-2}\|\Delta_{h,0}J^r_{n;j,i,s}(f)\|_{Ba}+\|J^{r_1}_{n;j,i,s}(f)-f\|_{Ba}\right)\\
&\leqslant C(\lambda,j,i,s,r)\left(\|J_{n;j,i,s}(f)-f\|_{Ba}+n^{-2}\|\Delta_{h,0}J^r_{n;j,i,s}(f)\|_{Ba}\right)
\end{aligned}$$

$$\leqslant C(\lambda, j, i, s, r, q, \mu)\Big(\|J_{n;j,i,s}(f) - f\|_{Ba} + \|f\|_{Ba}\Big). \tag{4.13}$$

Consequently, $n^{-2}\|\Delta_{h,0}J^r_{n;j,i,s}(f)\|_{Ba} \leqslant C(\lambda, j, i, s, r, q, \mu)\|f - J_{n;j,i,s}(f)\|_{Ba}$, by the definition of $K(f;t^2)_{Ba}$ and (2.4) shows that

$$\omega(f;n^{-1})_{Ba} \leqslant CK(f;n^{-2})_{Ba} \leqslant C\left(\|f - J^r_{n;j,i,s}(f)\|_{Ba} + n^{-2}\|\Delta_{h,0}J^r_{n;j,i,s}(f)\|_{Ba}\right)$$
$$\leqslant C(\lambda, j, i, s, r, q, \mu)\|f - J_{n;j,i,s}(f)\|_{Ba}. \tag{4.14}$$

i.e., $\omega(f;n^{-1})_{Ba} \ll \|f - J_{n;j,i,s}(f)\|_{Ba}$. The proof is completed. □

REMARK 4.2. Since Ba space is a weighted Lebesgue space if $L_{p_m}(h^2_\kappa) = L_p(h^2_\kappa)$, $a = \{1, 0, \cdots\}$, and Ba space is a weighted Orlicz space if $L_{p_m}(h^2_\kappa) = L_m(h^2_\kappa)$. Thus, the results presented in the Theorem 4.1 improve, extend and unify the results of the weighted Lebesgue space and weighted Orlicz space.

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