Journal of Mathematical Inequalities
Volume 10, Number 4 (2016), 947–970 doi:10.7153/jmi-10-77

COMMUTATORS OF MARCINKIEWICZ INTEGRALS ASSOCIATED WITH SCHRÖDINGER OPERATOR ON GENERALIZED WEIGHTED MORREY SPACES

VAGIF S. GULIYEV, ALI AKBULUT, VUGAR H. HAMZAYEV AND OKAN KUZU

(*Communicated by R. Oinarov*)

Abstract. Let $L=-\Delta+V$ be a Schrödinger operator, where Δ is the Laplacian on $\mathbb{R}^n$, while nonnegative potential V belongs to the reverse Hölder class. Let also $\Omega\in L_q(S^{n-1})$ be a homogeneous function of degree zero with $q>1$ and have a mean value zero on S^{n-1}. In this paper, we study the boundedness of the Marcinkiewicz operators $\mu^L_{j,\Omega}$ and their commutators $\mu^L_{j,\Omega,b}$ with rough kernels associated with Schrödinger operator on generalized weighted Morrey spaces $M_{p,\varphi}(w)$. We find the sufficient conditions on the pair (φ_1,φ_2) with $q'<p<\infty$ and $w\in A_{p/q'}$ or $1<p<q$ and $w^{1-p'}\in A_{p'/q'}$ which ensures the boundedness of the operators $\mu^L_{j,\Omega}$ from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$ for $1<p<\infty$. We find the sufficient conditions on the pair (φ_1,φ_2) with $b\in BMO(\mathbb{R}^n)$ and $q'<p<\infty$, $w\in A_{p/q'}$ or $1<p<q$, $w^{1-p'}\in A_{p'/q'}$ which ensures the boundedness of the operators $\mu^L_{j,\Omega,b}$, $j=1,\dots,n$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $1<p<\infty$. In all cases the conditions for the boundedness of the operators $\mu^L_{j,\Omega}$, $\mu^L_{j,\Omega,b}$, $j=1,\dots,n$ are given in terms of Zygmund-type integral inequalities on (φ_1,φ_2) and w, which do not assume any assumption on monotonicity of $\varphi_1(x,r)$, $\varphi_2(x,r)$ in r.

1. Introduction

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderon [5, 6] studied a kind of commutators, appearing in Cauchy integral problems of Lip-line. Let K be a Calderón-Zygmund singular integral operator and $b\in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [9] states that the commutator operator $[b,K]f=K(bf)-bKf$ is bounded on $L_p(\mathbb{R}^n)$ for $1<p<\infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [7]–[8], [10], [17], [18]).

The classical Morrey spaces were originally introduced by Morrey in [37] to study the local behavior of solutions to second order elliptic partial differential equations.

Mathematics subject classification (2010): 42B20, 42B35.
Keywords and phrases: Marcinkiewicz operator, rough kernel, Schrödinger operator, generalized weighted Morrey spaces, commutator, A_p weights.
The research of V. Guliyev was partially supported by the grant of Science Development Foundation under the President of the Republic of Azerbaijan, Grant EIF-2013-9(15)-46/10/1 and by the grant of Presidium Azerbaijan National Academy of Science 2015. The research of A. Akbulut was partially supported by the grant of Ahi Evran University Scientific Research Projects (PYO.FEN.4003.13.004) and (PYO.FEN.4001.14.006).

Paper JMI-10-77

For the properties and applications of classical Morrey spaces, we refer the readers to [17, 18, 22, 37]. Mizuhara [36] introduced generalized Morrey spaces. Later, Guliyev [22] defined the generalized Morrey spaces $M_{p,\varphi}$ with normalized norm. Recently, Komori and Shirai [33] considered the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Guliyev [23] gave a concept of generalized weighted Morrey space $M_{p,\varphi}(w)$ which could be viewed as extension of both generalized Morrey space $M_{p,\varphi}$ and weighted Morrey space $L^{p,\kappa}(w)$. In [23] Guliyev also studied the boundedness of the classical operators and its commutators in these spaces $M_{p,\varphi}(w)$, see also Guliyev et al. [28, 29, 32].

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ the unit sphere of $\mathbb{R}^n$ $(n \geqslant 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$.

Suppose that Ω satisfies the following conditions.

(i) Ω is a homogeneous function of degree zero on $\mathbb{R}^n$. That is,

$$\Omega(tx) = \Omega(x) \tag{1.1}$$

for all $t > 0$ and $x \in \mathbb{R}^n$.

(ii) Ω has mean zero on S^{n-1}. That is,

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.2}$$

where $x' = x/|x|$ for any $x \neq 0$.

The Marcinkiewicz integral operator of higher dimension μ_Ω is defined by

$$\mu_\Omega f(x) = \left(\int_0^\infty |F_{\Omega,t} f(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t} f(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

It is well known that the Littlewood-Paley g-function is a very important tool in harmonic analysis and the Marcinkiewicz integral is essentially a Littlewood-Paley g-function. In this paper, we will also consider the commutator $\mu_{\Omega,b}$ which is given by the following expression

$$\mu_{\Omega,b} f(x) = \left(\int_0^\infty |F^b_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F^b_{\Omega,t}(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)] f(y)dy.$$

On the other hand, the study of Schrödinger operator $L = -\Delta + V$ recently attracted much attention. In particular, Shen [39] considered Lp estimates for Schrödinger operators L with certain potentials which include Schrödinger Riesz transforms $R^L_j =$

$\frac{\partial}{\partial x_j}L^{-\frac{1}{2}}$, $j = 1,\ldots,n$. Then, Dziubański and Zienkiewicz [16] introduced the Hardy type space $H^1_L(\mathbb{R}^n)$ associated with the Schrödinger operator L, which is larger than the classical Hardy space $H^1(\mathbb{R}^n)$.

Similar to the classical Marcinkiewicz function, we define the Marcinkiewicz functions $\mu_{j,\Omega}$ associated with the Schrödinger operator L by

$$\mu^L_{j,\Omega}f(x) = \left(\int_0^\infty \left|\int_{|x-y|\leqslant t} |\Omega(x-y)|K^L_j(x,y)f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2},$$

where $K^L_j(x,y) = \widetilde{K^L_j}(x,y)|x-y|$ and $\widetilde{K^L_j}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j}L^{-\frac{1}{2}}$, $j = 1,\ldots,n$. In particular, when $V=0$, $K^\Delta_j(x,y) = \widetilde{K^\Delta_j}(x,y)|x-y| = \frac{(x-y)_j/|x-y|}{|x-y|^{n-1}}$ and $\widetilde{K^\Delta_j}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j}\Delta^{-\frac{1}{2}}$, $j = 1,\ldots,n$. In this paper, we write $K_j(x,y) = K^\Delta_j(x,y)$ and

$$\mu_{j,\Omega}f(x) = \left(\int_0^\infty \left|\int_{|x-y|\leqslant t} |\Omega(x-y)|K_j(x,y)f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2}.$$

Obviously, $\mu_{j,\Omega}$ are classical Marcinkiewicz functions with rough kernel. Therefore, it will be an interesting thing to study the property of $\mu^L_{j,\Omega}$. The main purpose of this paper is to show that Marcinkiewicz operators with rough kernel associated with Schrödinger operators $\mu^L_{j,\Omega}$, $j = 1,\ldots,n$ are bounded from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$, $1 < p < \infty$.

The commutator of the classical Marcinkiewicz function with rough kernel is defined by

$$\mu_{j,\Omega,b}f(x) = \left(\int_0^\infty \left|\int_{|x-y|\leqslant t} |\Omega(x-y)|K_j(x,y)[b(x)-b(y)]f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2}.$$

The commutator $\mu^L_{j,\Omega,b}$ formed by $b \in BMO(\mathbb{R}^n)$ and the Marcinkiewicz function with rough kernel $\mu^L_{j,\Omega}$ is defined by

$$\mu^L_{j,\Omega,b}f(x) = \left(\int_0^\infty \left|\int_{|x-y|\leqslant t} |\Omega(x-y)|K^L_j(x,y)[b(x)-b(y)]f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2}.$$

Let $f \in L^{\mathrm{loc}}_1(\mathbb{R}^n)$. The maximal operator with rough kernel M_Ω is defined by

$$M_\Omega f(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |\Omega(x-y)|\,|f(y)|dy.$$

It is obvious that when $\Omega \equiv 1$, M_Ω is the Hardy-Littlewood maximal operator M. For $b \in L^{\mathrm{loc}}_1(\mathbb{R}^n)$ the commutator of the maximal operator $M_{\Omega,b}$ is defined by

$$M_{\Omega,b}f(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |b(x)-b(y)|\,|\Omega(x-y)|\,|f(y)|dy.$$

We find the sufficient conditions on the pair (φ_1, φ_2) with $b \in BMO(\mathbb{R}^n)$ and $q' < p < \infty$, $w \in A_{p/q'}$ or $1 < p < q$, $w^{1-p'} \in A_{p'/q'}$ which ensures the boundedness of the operators $\mu^L_{j,\Omega,b}$, $j = 1, \ldots, n$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $1 < p < \infty$. Note that, in [25] was studied the boundedness of the parametric Marcinkiewicz operator and its commutators on generalized Morrey spaces $M_{p,\varphi}$.

2. Preliminaries

We say that $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \leqslant 1$ if there exists a constant $C > 0$ such that $|\Omega(x') - \Omega(y')| \leqslant C|x' - y'|^\alpha$ for all $x', y' \in S^{n-1}$.

The operator μ_Ω was first defined by Stein [41]. And Stein proved that if is continuous and satisfies a $\mathrm{Lip}_\alpha(S^{n-1})$ $(0 < \alpha \leqslant 1)$ condition, then μ_Ω is an operator of type (p,p) $(1 < p \leqslant 2)$ and of weak type $(1,1)$. In [4], Benedek, Calderón and Panzone proved that if $\Omega \in C^1(S^{n-1})$, then μ_Ω is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The L_p boundedness of μ_Ω has been studied extensively. See [4, 30, 41, 42], among others. A survey of past studies can be found in [11]. Ding, Fan and Pan [12] proved the weighted $L_p(\mathbb{R}^n)$ boundedness with A_p weighs for a class of rough Marcinkiewicz integrals. Recently, Ding, Fan and Pan [13] improved the results mentioned above and showed that if Ω belongs to the Hardy space on the unit sphere, that is $\Omega \in H^1(S^{n-1})$, then μ_Ω is still a bounded operator on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. In [43], Xu, Chen and Ying proved the same result as [13] using a different method.

THEOREM 2.1. ([15]) *Suppose that Ω satisfies the conditions (1.1) and $\Omega \in L_q(S^{n-1})$, $1 < q \leqslant \infty$. Then for every $q' \leqslant p < \infty$, $p \neq 1$ and $w \in A_{p/q'}$ or $1 < p \leqslant q$ and $w^{1-p'} \in A_{p'/q'}$, there is a constant C independent of f such that*

$$\|M_\Omega f\|_{L_{p,w}} \leqslant C\|f\|_{L_{p,w}}.$$

THEOREM 2.2. ([3]) *Suppose that Ω satisfies the conditions (1.1) and $\Omega \in L_q(S^{n-1})$, $1 < q \leqslant \infty$. Let also $b \in BMO(\mathbb{R}^n)$. Then for every $q' \leqslant p < \infty$, $p \neq 1$ and $w \in A_{p/q'}$ or $1 < p \leqslant q$ and $w^{1-p'} \in A_{p'/q'}$, there is a constant C independent of f such that*

$$\|M_{\Omega,b} f\|_{L_{p,w}} \leqslant C\|f\|_{L_{p,w}}.$$

THEOREM 2.3. ([12]) *Suppose that Ω satisfies the conditions (1.1), (1.2) and $\Omega \in L_q(S^{n-1})$, $1 < q \leqslant \infty$. Then for every $q' < p < \infty$ and $w \in A_{p/q'}$ or $1 < p < q$ and $w^{1-p'} \in A_{p'/q'}$, there is a constant C independent of f such that*

$$\|\mu_\Omega f\|_{L_{p,w}} \leqslant C\|f\|_{L_{p,w}}.$$

THEOREM 2.4. ([14]) *Suppose that Ω satisfies the conditions (1.1), (1.2) and $\Omega \in L_q(S^{n-1})$, $1 < q \leqslant \infty$. Let also $b \in BMO(\mathbb{R}^n)$. Then for $q' < p < \infty$ and $w \in A_{p/q'}$ or $1 < p < q$ and $w^{1-p'} \in A_{p'/q'}$, there is a constant $C > 0$ independent of f such that*

$$\left\|\mu_{\Omega,b} f\right\|_{L_{p,w}} \leqslant C\|f\|_{L_{p,w}}.$$

Note that a nonnegative locally L_q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to B_q $(1<q<\infty)$ if there exists $C>0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q(y)dy\right)^{1/q} \leqslant C\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)dy\right) \tag{2.1}$$

holds for every $x\in\mathbb{R}^n$ and $r>0$, where $B(x,r)$ denotes the open ball centered at x with radius r; see [39]. It is worth pointing out that, if $V\in B_q$ for some $q>1$, then there exists $\varepsilon>0$, which depends only on n and the constant C in (2.1), such that $V\in B_{q+\varepsilon}$. Throughout this paper, we always assume that $0\neq V\in B_n$.

We will use the following statements on the boundedness of the weighted Hardy operators

$$H_w g(r) := \int_r^\infty g(t)w(t)dt,\ 0<t<\infty$$

and

$$H_w^* g(r) := \int_r^\infty \left(1+\ln\frac{t}{r}\right) g(t)w(t)dt,\ 0<t<\infty,$$

where w is a fixed function non-negative and measurable on $(0,\infty)$.
The following theorem was proved in [26, 27].

THEOREM 2.5. ([26, 27]) *Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0,\infty)$. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t)H_w g(t) \leqslant C \operatorname*{ess\,sup}_{t>0} v_1(t)g(t) \tag{2.2}$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t)\int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Moreover, the value $C=B$ is the best constant for (2.2).

The following theorem was proved in [23].

THEOREM 2.6. ([23]) *Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0,\infty)$. The inequality*

$$\operatorname*{ess\,sup}_{r>0} v_2(r)H_w^* g(r) \leqslant C \operatorname*{ess\,sup}_{r>0} v_1(r)g(r) \tag{2.3}$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{r>0} v_2(r)\int_r^\infty \left(1+\ln\frac{t}{r}\right)\frac{w(t)dt}{\sup_{t<s<\infty} v_1(s)} < \infty. \tag{2.4}$$

Moreover, the value $C=B$ is the best constant for (2.2).

REMARK 2.1. In (2.2)–(2.4) it is assumed that $0\cdot\infty=0$.

By $A \lesssim D$ we mean that $A \leqslant CD$ with some positive constant C independent of appropriate quantities. If $A \lesssim D$ and $D \lesssim A$, we write $A \approx D$ and say that A and D are equivalent.

3. Generalized weighted Morrey spaces

The classical Morrey spaces $M_{p,\lambda}$ were originally introduced by Morrey in [37] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [20, 34].

We denote by $M_{p,\lambda} \equiv M_{p,\lambda}(\mathbb{R}^n)$ the Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $1 \leqslant p < \infty$ and $0 \leqslant \lambda \leqslant n$.

Note that $M_{p,0}=L_p(\mathbb{R}^n)$ and $M_{p,n}=L_\infty(\mathbb{R}^n)$. If $\lambda<0$ or $\lambda>n$, then $M_{p,\lambda}=\Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We recall that a weight function w is in the Muckenhoupt class A_p [38], $1<p<\infty$, if

$$\begin{aligned}[w]_{A_p} &:= \sup_B [w]_{A_p(B)} \\ &= \sup_B \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}dx\right)^{p-1} \end{aligned} \tag{3.1}$$

where the sup is taken with respect to all the balls B and $\frac{1}{p}+\frac{1}{p'}=1$. Note that, for all balls B using Hölder's inequality, we have that

$$[w]_{A_p(B)}^{1/p} = |B|^{-1}\|w\|_{L_1(B)}^{1/p}\,\|w^{-1/p}\|_{L_{p'}(B)} \geqslant 1. \tag{3.2}$$

For $p=1$, the class A_1 is defined by the condition $Mw(x) \leqslant Cw(x)$ with $[w]_{A_1} = \sup\limits_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)}$, and for $p=\infty$ $A_\infty = \bigcup_{1\leqslant p<\infty} A_p$ and $[w]_{A_\infty} = \inf\limits_{1\leqslant p<\infty} [w]_{A_p}$.

REMARK 3.2. It is known that

$$w^{1-p'} \in A_{p'/q'} \Rightarrow [w^{1-p'}]_{A_{p'/q'}(B)}^{q'/p'} = |B|^{-1}\|w^{1-p'}\|_{L_1(B)}^{q'/p'}\,\|w^{q'/p}\|_{L_{(p'/q')'}(B)}.$$

Moreover, we can write $w^{1-p'} \in A_{p'/q'} \Rightarrow w^{1-p'} \in A_{p'}$ because of $w^{1-p'} \in A_{p'/q'} \subset A_{p'}$. Therefore, we get

$$\begin{aligned} w^{1-p'} \in A_{p'/q'} &\Rightarrow w^{1-p'} \in A_{p'} \\ &\Rightarrow [w^{1-p'}]_{A_{p'}(B)}^{1/p'} = |B|^{-1}\|w^{1-p'}\|_{L_1(B)}^{1/p'}\,\|w^{1/p}\|_{L_p(B)}. \end{aligned} \tag{3.3}$$

But the opposite is not true.

REMARK 3.3. Let's write $w^{1-p'} \in A_{p'/q'}$ and used the definitions A_p classes we get the following

$$w^{1-p'} \in A_{p'/q'} \Rightarrow [w^{1-p'}]_{A_{p'/q'}}^{\frac{q(p-1)}{p(q-1)}} = |B|^{-1} \|w^{1-p'}\|_{L_1(B)}^{\frac{q(p-1)}{p(q-1)}} \|w^{q'/p}\|_{L_{(p'/q')'}(B)}$$
$$\Rightarrow [w^{1-p'}]_{A_{p'/q'}}^{1/p'} = |B|^{-\frac{q-1}{q}} \|w^{1-p'}\|_{L_1(B)}^{1/p'} \|w\|_{L_{\frac{q}{q-p}}(B)}^{1/p}, \tag{3.4}$$

where the following equalities are provided.

$$1-p' = -\frac{p'}{p}, \ \frac{q'}{p} = \frac{q}{p(q-1)}, \ \frac{q'}{p'} = \frac{q(p-1)}{p(q-1)}, \ \left(\frac{q}{p}\right)' = \frac{q}{q-p}, \ \left(\frac{p'}{q'}\right)' = \frac{p(q-1)}{q-p}.$$

Then from eq.(3.3) and eq.(3.4) we have

$$w^{1-p'} \in A_{p'/q'} \Rightarrow [w^{1-p'}]_{A_{p'/q'}}^{1/p'}$$
$$= |B|^{\frac{1}{q}} [w^{1-p'}]_{A_{p'}(B)}^{1/p'} \|w^{1/p}\|_{L_p(B)}^{-1} \|w\|_{L_{\frac{q}{q-p}}(B)}^{1/p}. \tag{3.5}$$

DEFINITION 3.1. ([22]) Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \leqslant p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space consisting of all measurable functions f for which

$$\|f\|_{WL_p(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{R}^n)} < \infty.$$

Also the spaces $L_p^{\mathrm{loc}}(\mathbb{R}^n)$ and $WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ endowed with the natural topology are defined as the sets of all functions f such that $f\chi_B \in L_p(\mathbb{R}^n)$ and $f\chi_B \in WL_p(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$, respectively.

According to this definition, we recover the space $M_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}},$$
$$WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

We define the generalized weighed Morrey spaces as follows.

DEFINITION 3.2. ([23]) Let $1 \leqslant p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_{p,w}(B(x,r))},$$

where $L_{p,w}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{L_{p,w}(\mathbb{R}^n)} = \left(\int_{B(x,r)} |f(y)|^p w(y)dy\right)^{\frac{1}{p}}.$$

Furthermore, by $WM_{p,\varphi}(w)$ we denote the weak generalized weighted Morrey space of all functions $f \in WL^{\rm loc}_{p,w}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{WL_{p,w}(B(x,r))} < \infty,$$

where $WL_{p,w}(B(x,r))$ denotes the weak $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,w}(\mathbb{R}^n)} = \sup_{t>0} t \left(\int_{\{y\in B(x,r):\,|f(y)|>t\}} w(y)dy\right)^{\frac{1}{p}}.$$

REMARK 3.4. (1) If $w \equiv 1$, then $M_{p,\varphi}(1) = M_{p,\varphi}$ is the generalized Morrey space.

(2) If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ is the weighted Morrey space.

(3) If $\varphi(x,r) \equiv v(B(x,r))^{\frac{\kappa}{p}} w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(v,w)$ is the two weighted Morrey space.

(4) If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M_{p,\varphi}(w) = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM_{p,\varphi}(w) = WL_{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.

(5) If $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,w}(\mathbb{R}^n)$ is the weighted Lebesgue space.

Suppose that T_Ω represents a linear or a sublinear operator, such that that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

$$|T_\Omega f(x)| \leqslant c_0 \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)|dy, \tag{3.6}$$

where c_0 is independent of f and x.

For a function b, suppose that the commutator operator $T_{\Omega,b}$ represents a linear or a sublinear operator, such that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

$$|T_{\Omega,b} f(x)| \leqslant c_0 \int_{\mathbb{R}^n} |b(x)-b(y)| \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)|dy, \tag{3.7}$$

where c_0 is independent of f and x.

We point out that the condition (3.6) in the case $\Omega \equiv 1$ was first introduced by Soria and Weiss in [40]. The condition (3.6) are satisfied by many interesting operators in harmonic analysis, such as the Calderón-Zygmund operators, Carleson's maximal operator, Hardy–Littlewood maximal operator, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci-Stein's oscillatory singular integrals, the Bochner–Riesz means and so on (see [35], [40] for details).

The following statement, was proved in [32], see also [23, 28].

THEOREM 3.7. *Let* $1 \leqslant p < \infty$, $w \in A_p$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \leqslant C \varphi_2(x,r), \tag{3.8}$$

where C *does not depend on* x *and* r. *Let* $T \equiv T_1$ *be a sublinear operator satisfying condition* (3.6) *with* $\Omega \equiv 1$ *bounded on* $L_{p,w}(\mathbb{R}^n)$ *for* $p > 1$, *and bounded from* $L_{1,w}(\mathbb{R}^n)$ *to* $WL_{1,w}(\mathbb{R}^n)$. *Then the operator* T *is bounded from* $M_{p,\varphi_1}(w)$ *to* $M_{p,\varphi_2}(w)$ *for* $p > 1$ *and from* $M_{1,\varphi_1}(w)$ *to* $WM_{1,\varphi_2}(w)$.

The following statement, was proved in [28], see also [23].

THEOREM 3.8. *Let* $1 < p < \infty$, $w \in A_p$, $b \in BMO(\mathbb{R}^n)$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \leqslant C \varphi_2(x,r), \tag{3.9}$$

where C *does not depend on* x *and* r. *Let* $T_b \equiv T_{1,b}$ *be a sublinear commutator operator satisfying condition* (3.7) *with* $\Omega \equiv 1$ *bounded on* $L_{p,w}(\mathbb{R}^n)$. *Then the operator* T_b *is bounded from* $M_{p,\varphi_1}(w)$ *to* $M_{p,\varphi_2}(w)$.

Note that, in the case $w = 1$ Theorem 3.7 was proved in [24] and for the operators M and K in [1].

4. Sublinear operator with rough kernels T_Ω in the spaces $M_{p,\varphi}(w)$

In the following lemma we get local estimate (see, for example, [21, 22] in the case $w = 1$ and [23] in the case $w \in A_p$) for the operator T_Ω.

LEMMA 4.1. *Suppose that* Ω *be satisfies the conditions* (1.1), (1.2) *and* $\Omega \in L_q(S^{n-1})$, $1 < q \leqslant \infty$. *Let* T_Ω *be a sublinear operator satisfying condition* (3.6), *and bounded on* $L_p(\mathbb{R}^n)$ *for* $1 < p < \infty$.

If $q' < p < \infty$ *and* $w \in A_{p/q'}$, *then the inequality*

$$\|T_\Omega f\|_{L_{p,w}(B(x_0,r))} \lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$*, and for all* $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$.

If $1<p<q$ *and* $w^{1-p'} \in A_{p'/q'}$*, then the inequality*

$$\|T_\Omega f\|_{L_{p,w}(B(x_0,r))} \lesssim \|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x_0,r))} \int_{2r}^\infty \|f\|_{L_{p,w}(B(x_0,t))} \|w\|^{-1/p}_{L_{\frac{q}{q-p}}(B(x_0,t))} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$*, and for all* $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$.

Proof. Let Ω be satisfies the conditions (1.1), (1.2) and $\Omega \in L_q(S^{n-1})$, $1<q\leqslant\infty$. Note that

$$\begin{aligned} \|\Omega(x-\cdot)\|_{L_q(B(x_0,t))} &= \left(\int_{B(x-x_0,t)} |\Omega(y)|^q dy\right)^{\frac{1}{q}} \\ &\leqslant \left(\int_{B(0,t+|x-x_0|)} |\Omega(y)|^q dy\right)^{\frac{1}{q}} \\ &= \left(\int_0^{t+|x-x_0|} r^{n-1} dr \int_{S^{n-1}} |\Omega(y')|^q d\sigma(y')\right)^{\frac{1}{q}} \\ &= c_0 \, \|\Omega\|_{L_q(S^{n-1})} \, |B(0,t+|x-x_0|)|^{\frac{1}{q}}, \end{aligned} \tag{4.1}$$

where $c_0=(nv_n)^{-1/q}$ and $v_n=|B(0,1)|$.

For arbitrary $x_0 \in \mathbb{R}^n$, set $B=B(x_0,r)$ for the ball centered at x_0 and of radius r, $2B=B(x_0,2r)$. We represent f as

$$f=f_1+f_2, \;\; f_1(y)=f(y)\chi_{2B}(y), \;\; f_2(y)=f(y)\chi_{\complement_{(2B)}}(y), \;\; r>0 \tag{4.2}$$

and have

$$\|T_\Omega f\|_{L_{p,w}(B)} \leqslant \|T_\Omega f_1\|_{L_{p,w}(B)} + \|T_\Omega f_2\|_{L_{p,w}(B)}.$$

Since $f_1 \in L_{p,w}(\mathbb{R}^n)$, $T_\Omega f_1 \in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of T_Ω in $L_{p,w}(\mathbb{R}^n)$ for $w \in A_{p/q'}$ and $q'<p<\infty$ (see Theorem 2.3) it follows that

$$\begin{aligned} \|T_\Omega f_1\|_{L_{p,w}(B)} &\leqslant \|T_\Omega f_1\|_{L_{p,w}(\mathbb{R}^n)} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})} [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \|f_1\|_{L_{p,w}(\mathbb{R}^n)} \\ &\approx \|\Omega\|_{L_q(S^{n-1})} [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \|f\|_{L_{p,w}(2B)}. \end{aligned}$$

It's clear that $x \in B$, $y \in {}^{\complement}(2B)$ implies $\frac{1}{2}|x_0-y| \leqslant |x-y| \leqslant \frac{3}{2}|x_0-y|$. Then by the Minkowski inequality and conditions on Ω, we get

$$T_\Omega f_2(x) \lesssim \int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n} dy.$$

By Fubini's theorem we have

$$\begin{aligned}\int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n}dy &\approx \int_{\complement_{(2B)}} |\Omega(x-y)|\,|f(y)| \int_{|x-y|}^{\infty} \frac{dt}{t^{n+1}}dy \\ &= \int_{2r}^{\infty}\int_{2r\leqslant |x_0-y|<t} |\Omega(x-y)|\,|f(y)|dy\frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty}\int_{B(x_0,t)} |\Omega(x-y)|\,|f(y)|dy\frac{dt}{t^{n+1}}\ .\end{aligned}$$

By applying Hölder's inequality for $q'<p<\infty$ and $w\in A_{p/q'}$, we get

$$\begin{aligned}&\int_{\complement_{(2B)}} \frac{|\Omega(x-y)|\,|f(y)|}{|x_0-y|^n}dy \lesssim \int_{2r}^{\infty} \|\Omega(x-\cdot)\|_{L_q(B(x_0,t))}\,\|f\|_{L_{q'}(B(x_0,t))}\,\frac{dt}{t^{n+1}} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\,\|w^{-q'/p}\|^{\frac{1}{q'}}_{L_{(p/q')'}(B(x_0,t))}\,|B(0,t+|x-x_0|)|^{\frac{1}{q}}\,\frac{dt}{t^{n+1}} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,|B(x_0,t)|^{\frac{1}{q'}}\,|B(0,t)|^{\frac{1}{q}}\,\frac{dt}{t^{n+1}} \\ &\approx \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}. \qquad (4.3)\end{aligned}$$

Moreover, for all $p\in(1,\infty)$ the inequality

$$\|T_\Omega f_2\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\,w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}.$$

is valid. Thus

$$\begin{aligned}\|T_\Omega f\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\Big(&\|f\|_{L_{p,w}(2B)} \\ &+w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}\Big).\end{aligned}$$

On the other hand,

$$\begin{aligned}\|f\|_{L_{p,w}(2B)} &\approx |B|\|f\|_{L_{p,w}(2B)}\int_{2r}^{\infty}\frac{dt}{t^{n+1}} \\ &\lesssim |B|\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\,\frac{dt}{t^{n+1}} \\ &\lesssim w(B)^{\frac{1}{p}}\|w^{-1/p}\|_{L_{p'}(B)}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\,\frac{dt}{t^{n+1}} \\ &\lesssim w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\,\|w^{-1/p}\|_{L_{p'}(B(x_0,t))}\,\frac{dt}{t^{n+1}} \\ &\lesssim [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\,w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}. \qquad (4.4)\end{aligned}$$

Thus

$$\begin{aligned}&\|T_\Omega f\|_{L_{p,w}(B)}\\&\lesssim \|\Omega\|_{L_q(S^{n-1})}[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}\\&\lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}.\end{aligned}$$

Let also $1<p<q$ and $w^{1-p'}\in A_{p'/q'}$. Since $f_1\in L_{p,w}(\mathbb{R}^n)$, $T_\Omega f_1\in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of T_Ω in $L_{p,w}(\mathbb{R}^n)$ for $w^{1-p'}\in A_{p'/q'}$ and $1<p<q$ (see Theorem 2.3) it follows that

$$\begin{aligned}\|T_\Omega f_1\|_{L_{p,w}(B)} \leqslant \|T_\Omega f_1\|_{L_{p,w}(\mathbb{R}^n)} &\lesssim \|\Omega\|_{L_q(S^{n-1})}[w^{1-p'}]^{\frac{1}{p'}}_{A_{\frac{p'}{q'}}} \|f_1\|_{L_{p,w}(\mathbb{R}^n)}\\&\approx \|\Omega\|_{L_q(S^{n-1})}[w^{1-p'}]^{\frac{1}{p'}}_{A_{\frac{p'}{q'}}} \|f\|_{L_{p,w}(2B)}.\end{aligned}$$

If $1<p<q$ and $w^{1-p'}\in A_{p'/q'}$, then Minkowski theorem and Hölder inequality,

$$\begin{aligned}&\|T_\Omega f_2\|_{L_{p,w}(B)} \leqslant \left(\int_B \left(\int_{2r}^{\infty}\int_{B(x_0,t)} |\Omega(x-y)||f(y)|dy \frac{dt}{t^{n+1}}\right)^p w(x)dx\right)^{\frac{1}{p}}\\&\leqslant \int_{2r}^{\infty}\int_{B(x_0,t)} \|\Omega(\cdot-y)\|_{L_{p,w}(B)}|f(y)|dy \frac{dt}{t^{n+1}}\\&\lesssim \int_{2r}^{\infty}\int_{B(x_0,t)} \|\Omega(\cdot-y)\|_{L_q(B)} \|w\|^{\frac{1}{p}}_{L_{(q/p)'}(B)} |f(y)|dy \frac{dt}{t^{n+1}}\\&\lesssim \|\Omega\|_{L_q(S^{n-1})} \|w\|^{\frac{1}{p}}_{L_{(q/p)'}(B)} \int_{2r}^{\infty}\int_{B(x_0,t)} |B(0,r+|x_0-y|)|^{\frac{1}{q}} |f(y)|dy \frac{dt}{t^{n+1}}\\&\lesssim \|\Omega\|_{L_q(S^{n-1})} \|w\|^{\frac{1}{p}}_{L_{(q/p)'}(B)} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} |B(0,r+t)|^{\frac{1}{q}} \frac{dt}{t^{n+1}}\\&\lesssim \|\Omega\|_{L_q(S^{n-1})} \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w^{-p'/p}\|^{\frac{1}{p'}}_{L_1(B(x_0,t))} |B(x_0,t)|^{\frac{1}{q}} \frac{dt}{t^{n+1}}\\&\lesssim \|\Omega\|_{L_q(S^{n-1})} |B|^{\frac{1}{q}} \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w^{1-p'}\|^{\frac{1}{p'}}_{L_1(B(x_0,t))} |B(x_0,t)|^{\frac{1}{q}} \frac{dt}{t^{n+1}}\end{aligned}$$

is obtained. By applying (3.3) for $\|w^{1-p'}\|^{\frac{1}{p'}}_{L_1(B(x_0,t))}$ and (3.5) for $\|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)}$ we have the following inequality

$$\begin{aligned}&\|T_\Omega f_2\|_{L_{p,w}(B)}\\&\lesssim \|\Omega\|_{L_q(S^{n-1})}[w^{1-p'}]^{\frac{1}{p'}}_{A_{\frac{p'}{q'}}} \|w\|_{L^{\frac{1}{p}}_{\frac{q}{q-p}}(B)} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x_0,t))} \frac{dt}{t}\end{aligned}$$

is valid. Thus

$$\|T_\Omega f\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_q(S^{n-1})} [w^{1-p'}]_{A_{\frac{p'}{q'}}}^{\frac{1}{p'}} \Big(\|f\|_{L_{p,w}(2B)} + \|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w\|_{L_{\frac{q}{q-p}}(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t} \Big).$$

On the other hand,

$$\begin{aligned}
&\|f\|_{L_{p,w}(2B)} \approx |B| \|f\|_{L_{p,w}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \\
&\lesssim |B| \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&= [w^{1-p'}]_{A_{p'}(B)}^{-\frac{1}{p'}} |B|^{\frac{1}{q}} \|w^{1-p'}\|_{L_1(B)}^{\frac{1}{p'}} \|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,\omega}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim [w^{1-p'}]_{A_{p'}(B)}^{-\frac{1}{p'}} \|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,\omega}(B(x_0,t))} |B(x_0,t)|^{\frac{1}{q}} \|w^{1-p'}\|_{L_1(B(x_0,t))}^{\frac{1}{p'}} \frac{dt}{t^{n+1}} \\
&\lesssim \|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,\omega}(B(x_0,t))} \|w\|_{L_{\frac{q}{q-p}}(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|T_\Omega f\|_{L_{p,w}(B)} \\
&\lesssim \|\Omega\|_{L_q(S^{n-1})} [w^{1-p'}]_{A_{\frac{p'}{q'}}}^{\frac{1}{p'}} \|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w\|_{L_{\frac{q}{q-p}}(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t}.
\end{aligned}$$

Thus we complete the proof of Lemma 4.1. □

THEOREM 4.9. *Suppose that Ω be satisfies the conditions (1.1), (1.2) and $\Omega \in L_q(S^{n-1})$, $1 < q \leqslant \infty$. Let T_Ω be a sublinear operator satisfying condition (3.6), and bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. Let also, for $q' < p < \infty$, $w \in A_{p/q'}$ the pair (φ_1, φ_2) satisfies the condition (3.8) and for $1 < p < q$, $w^{1-p'} \in A_{p'/q'}$ the pair (φ_1, φ_2) satisfies the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) \|w\|_{L_{\frac{q}{q-p}}(B(x,\tau))}^{1/p}}{\|w\|_{L_{\frac{q}{q-p}}(B(x,t))}^{1/p}} \frac{dt}{t} \leqslant C \varphi_2(x,r) \frac{w(B(x,r))^{\frac{1}{p}}}{\|w\|_{L_{\frac{q}{q-p}}(B(x,r))}^{\frac{1}{p}}}, \tag{4.5}$$

where C does not depend on x and r.

Then the operator T_Ω is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$. Moreover

$$\|T_\Omega f\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)}.$$

Proof. When $q' < p < \infty$, $w \in A_{p/q'}$, by Lemma 4.1 and Theorem 2.5 with $\nu_2(r) = \varphi_2(x,r)^{-1}$, $\nu_1(r) = \varphi_1(x,r)^{-1}w(B(x,r))^{-\frac{1}{p}}$, $g(r) = \|f\|_{L_{p,w}(B(x,r))}$ and $w(r) = w(B(x,r))^{-\frac{1}{p}}r^{-1}$ we have

$$\begin{aligned}\|T_\Omega f\|_{M_{p,\varphi_2}(w)} &= \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|\mu_\Omega f\|_{L_{p,w}(B(x,r))} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L_{p,w}(B(x,t))}\, w(B(x,t))^{-\frac{1}{p}} \frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))} \\ &= \|f\|_{M_{p,\varphi_1}(w)}.\end{aligned}$$

For the case of $1 < p < q$, $w^{1-p'} \in A_{p'/q'}$, by Lemma 4.1 and Theorem 2.5 with $\nu_2(r) = \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))}$, $\nu_1(r) = \varphi_1(x,r)^{-1}w(B(x,r))^{-\frac{1}{p}}$, $g(r) = \|f\|_{L_{p,w}(B(x,r))}$ and $w(r) = \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))} r^{-1}$ we have

$$\begin{aligned}\|T_\Omega f\|_{M_{p,\varphi_2}(w)} &= \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|\mu_\Omega f\|_{L_{p,w}(B(x,r))} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)} \int_r^\infty \|f\|_{L_{p,w}(B(x_0,t))}\, \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x_0,t))} \frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))} \\ &= \|f\|_{M_{p,\varphi_1}(w)}. \quad \square\end{aligned}$$

5. Commutator of sublinear operator with rough kernels $T_{\Omega,b}$ in the spaces $M_{p,\varphi}(w)$

REMARK 5.5. ([31])

(1) The John-Nirenberg inequality : There are constants C_1, $C_2 > 0$, such that for all $b \in BMO(\mathbb{R}^n)$ and $\beta > 0$

$$|\{x \in B : |b(x) - b_B| > \beta\}| \leqslant C_1|B|e^{-C_2\beta/\|b\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

(2) The John-Nirenberg inequality implies that

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |b(y) - b_{B(x,r)}|^p dy\right)^{\frac{1}{p}} \tag{5.1}$$

for $1 < p < \infty$.

(3) Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \leqslant C\|b\|_* \ln \frac{t}{r} \quad \text{for} \quad 0 < 2r < t, \tag{5.2}$$

where C is independent of b, x, r and t.

In the following lemma we get local estimate (see, for example, [23]) for the commutator operator $T_{\Omega,b}$.

LEMMA 5.2. *Suppose that* Ω *be satisfies the conditions* (1.1), (1.2) *and* $\Omega \in L_q(S^{n-1})$, $1 < q \leqslant \infty$. *Let* $b \in BMO(\mathbb{R}^n)$, $T_{\Omega,b}$ *be a commutator sublinear operator satisfying condition* (3.7), *and bounded on* $L_p(\mathbb{R}^n)$ *for* $1 < p < \infty$.

If $q' < p < \infty$ *and* $w \in A_{p/q'}$, *then the inequality*

$$\|T_{\Omega,b}f\|_{L_{p,w}(B(x_0,r))} \lesssim \|b\|_* w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, *and for all* $f \in L_{p,w}^{\text{loc}}(\mathbb{R}^n)$.

If $1 < p < q$ *and* $w^{1-p'} \in A_{p'/q'}$, *then the inequality*

$$\|T_{\Omega,b}f\|_{L_{p,w}(B(x_0,r))} \lesssim \|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x_0,r))} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} \|w\|^{-1/p}_{L_{\frac{q}{q-p}}(B(x_0,t))} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, *and for all* $f \in L_{p,w}^{\text{loc}}(\mathbb{R}^n)$.

Proof. Let $p \in (1,\infty)$ and $b \in BMO(\mathbb{R}^n)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r, $2B = B(x_0,2r)$. We represent f as (4.2) and have

$$\|T_{\Omega,b}f\|_{L_{p,w}(B)} \leqslant \|T_{\Omega,b}f_1\|_{L_{p,w}(B)} + \|T_{\Omega,b}f_2\|_{L_{p,w}(B)}.$$

Since $f_1 \in L_{p,w}(\mathbb{R}^n)$, $T_{\Omega,b}f_1 \in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of $T_{\Omega,b}$ in $L_{p,w}(\mathbb{R}^n)$ for $w \in A_{p/q'}$ and $q' < p < \infty$ (see Theorem 2.4) it follows that

$$\begin{aligned}
\|T_{\Omega,b}f_1\|_{L_{p,w}(B)} &\leqslant \|T_{\Omega,b}f_1\|_{L_{p,w}(\mathbb{R}^n)} \\
&\lesssim \|\Omega\|_{L_q(S^{n-1})} [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \|b\|_* \|f_1\|_{L_{p,w}(\mathbb{R}^n)} \\
&\approx \|\Omega\|_{L_q(S^{n-1})} [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \|b\|_* \|f\|_{L_{p,w}(2B)}.
\end{aligned}$$

For $x \in B$ we have

$$T_{\Omega,b}f_2(x) \lesssim \int_{\complement_{(2B)}} |b(y) - b(x)||\Omega(x-y)| \frac{|f(y)|}{|x_0 - y|^n} dy.$$

Then

$$\begin{aligned}
&\|T_{\Omega,b}f_2\|_{L_{p,w}(B)}\\
&\lesssim \left(\int_B \left(\int_{\complement_{(2B)}} |b(y)-b(x)||\Omega(x-y)|\frac{|f(y)|}{|x_0-y|^n}\,dy\right)^p w(x)dx\right)^{\frac{1}{p}}\\
&\lesssim \left(\int_B \left(\int_{\complement_{(2B)}} |b(y)-b_{B,w}||\Omega(x-y)|\frac{|f(y)|}{|x_0-y|^n}\,dy\right)^p w(x)dx\right)^{\frac{1}{p}}\\
&\quad+\left(\int_B \left(\int_{\complement_{(2B)}} |b(x)-b_{B,w}||\Omega(x-y)|\frac{|f(y)|}{|x_0-y|^n}\,dy\right)^p w(x)dx\right)^{\frac{1}{p}}\\
&=I_1+I_2.
\end{aligned}$$

Let us estimate I_1.

$$\begin{aligned}
I_1 &= w(B)^{\frac{1}{p}}\int_{\complement_{(2B)}} |b(y)-b_{B,w}||\Omega(x-y)|\frac{|f(y)|}{|x_0-y|^n}\,dy\\
&\approx w(B)^{\frac{1}{p}}\int_{\complement_{(2B)}} |b(y)-b_{B,w}||\Omega(x-y)||f(y)|\int_{|x_0-y|}^{\infty}\frac{dt}{t^{n+1}}\,dy\\
&\approx w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\int_{2r\leqslant|x_0-y|\leqslant t} |b(y)-b_{B,w}||\Omega(x-y)||f(y)|\,dy\frac{dt}{t^{n+1}}\\
&\lesssim w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\int_{B(x_0,t)} |b(y)-b_{B,w}||\Omega(x-y)||f(y)|dy\frac{dt}{t^{n+1}}.
\end{aligned}$$

Applying Hölder's inequality and by (5.2), we get

$$\begin{aligned}
I_1 &\lesssim \|b\|_* w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|\Omega(x-\cdot)\|_{L_q(B(x_0,t))}\,\|f\|_{L_{q'}(B(x_0,t))}\,\frac{dt}{t^{n+1}}\\
&\lesssim \|\Omega\|_{L_q(S^{n-1})}\|b\|_* w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L_{p,w}(B(x_0,t))}\\
&\quad\times \|w^{-q'/p}\|_{L_{(p/q')'}(B(x_0,t))}^{\frac{1}{q'}}|B(x_0,t+r)|^{\frac{1}{q}}\,\frac{dt}{t^{n+1}}\\
&\lesssim \|\Omega\|_{L_q(S^{n-1})}[w]_{A_{\frac{p}{q'}}}^{\frac{1}{p}}\,\|b\|_* w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L_{p,w}(B(x_0,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}
\end{aligned}$$

In order to estimate I_2 note that

$$I_2=\left(\int_B |b(x)-b_{B,w}|^p w(x)dx\right)^{\frac{1}{p}}\int_{\complement_{(2B)}}\frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n}\,dy.$$

By (4.3) and (5.2), we get

$$\begin{aligned}
I_2 &\lesssim \|b\|_* w(B)^{\frac{1}{p}}\int_{\complement_{(2B)}}\frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n}\,dy\\
&\lesssim \|\Omega\|_{L_q(S^{n-1})}[w]_{A_{\frac{p}{q'}}}^{\frac{1}{p}}\,\|b\|_* w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}.
\end{aligned}$$

Summing up I_1 and I_2, for all $p\in(1,\infty)$ we get

$$\begin{aligned}&\|T_{\Omega,b}f_2\|_{L_{p,w}(B)}\\ &\lesssim \|\Omega\|_{L_q(S^{n-1})}[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\|b\|_*w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))}w(B(x_0,t))^{-\frac{1}{p}}\frac{dt}{t}.\end{aligned}$$

Thus

$$\begin{aligned}\|T_{\Omega,b}f\|_{L_{p,w}(B)}\lesssim \|\Omega\|_{L_q(S^{n-1})}[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\|b\|_*\Big(&\|f\|_{L_{p,w}(2B)}\\ &+w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))}w(B(x_0,t))^{-\frac{1}{p}}\frac{dt}{t}\Big).\end{aligned}$$

On the other hand, by (4.4) we get

$$\begin{aligned}&\|T_{\Omega,b}f\|_{L_{p,w}(B)}\\ &\lesssim \|\Omega\|_{L_q(S^{n-1})}[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\|b\|_*w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))}w(B(x_0,t))^{-\frac{1}{p}}\frac{dt}{t}\\ &\lesssim \|b\|_*w(B(x_0,r))^{\frac{1}{p}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))}w(B(x_0,t))^{-\frac{1}{p}}\frac{dt}{t}.\end{aligned}$$

With similar techniques for $1<p<q$, $w^{1-p'}\in A_{p'/q'}$ can be achieved and the proof is finished. □

THEOREM 5.10. *Suppose that Ω be satisfies the conditions (1.1), (1.2) and $\Omega\in L_q(S^{n-1})$, $1<q\leqslant\infty$. Let $b\in BMO(\mathbb{R}^n)$, $T_{\Omega,b}$ be a commutator sublinear operator satisfying condition (3.7), and bounded on $L_p(\mathbb{R}^n)$ for $1<p<\infty$. Let also, for $q'<p<\infty$, $w\in A_{p/q'}$ the pair (φ_1,φ_2) satisfies the condition (3.9) and for $1<p<q$, $w^{1-p'}\in A_{p'/q'}$ the pair (φ_1,φ_2) satisfies the condition*

$$\int_r^{\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\operatorname*{ess\,inf}_{t<\tau<\infty}\varphi_1(x,\tau)\|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x,\tau))}}{\|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x,t))}}\frac{dt}{t}\leqslant C\,\varphi_2(x,r)\,\frac{w(B(x,r))^{\frac{1}{p}}}{\|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))}},\tag{5.3}$$

where C does not depend on x and r.

Then the operator $T_{\Omega,b}$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$.

$$\|T_{\Omega,b}f\|_{M_{p,\varphi_2}(w)}\lesssim\|f\|_{M_{p,\varphi_1}(w)}.$$

Proof. When $q'<p<\infty$, $w\in A_{p/q'}$, by Lemma 5.2 and Theorem 2.6 with $v_2(r)=\varphi_2(x,r)^{-1}$, $v_1(r)=\varphi_1(x,r)^{-1}w(B(x,r))^{-\frac{1}{p}}$, $g(r)=\|f\|_{L_{p,w}(B(x,r))}$ and $w(r)=$

$w(B(x,r))^{-\frac{1}{p}} r^{-1}$ we have

$$\begin{aligned}\|T_{\Omega,b}f\|_{M_{p,\varphi_2}(w)} &= \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|\mu_{\Omega,b}f\|_{L_{p,w}(B(x,r))} \\ &\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x,t))} w(B(x,t))^{-\frac{1}{p}} \frac{dt}{t} \\ &\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_{p,w}(B(x,r))} \\ &= \|b\|_* \|f\|_{M_{p,\varphi_1}(w)}.\end{aligned}$$

For the case of $1 < p < q$, $w^{1-p'} \in A_{p'/q'}$, by Lemma 4.1 and Theorem 2.6 with $v_2(r) = \varphi_2(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|w\|_{L_{\frac{q}{q-p}}(B(x,r))}^{\frac{1}{p}}$, $v_1(r) = \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}}$, $g(r) = \|f\|_{L_{p,w}(B(x,r))}$ and $w(r) = \|w\|_{L_{\frac{q}{q-p}}(B(x,r))}^{-\frac{1}{p}} r^{-1}$ we have

$$\begin{aligned}\|T_{\Omega,b}f\|_{M_{p,\varphi_2}(w)} &= \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|\mu_\Omega f\|_{L_{p,w}(B(x,r))} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}} \\ &\quad \times \int_r^\infty \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))} \|w\|_{L_{\frac{q}{q-p}}(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_{p,w}(B(x,r))} \\ &= \|f\|_{M_{p,\varphi_1}(w)}. \quad \square\end{aligned}$$

6. Marcinkiewicz operator with rough kernels $\mu_{j,\Omega}^L$ and its commutator $\mu_{j,\Omega,b}^L$ in the spaces $M_{p,\varphi}(w)$

In this section, we prove the boundedness of the Marcinkiewicz operator μ_j^L on $M_{p,\varphi}(w)$ spaces. For $x \in \mathbb{R}^n$, the function $\rho(x)$ is defined by

$$\rho(x) = \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V\ (y)dy\ \leqslant 1 \right\}.$$

LEMMA 6.3. ([39]) *Let $V \in B_q$ with $q \geqslant n/2$. Then there exists $l_0 > 0$ such that*

$$\frac{l}{C}\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-l_0} \leqslant \frac{\rho(y)}{\rho(x)} \leqslant C\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{l_0/(l_0+1)}.$$

In particular, $\rho(x) \approx \rho(y)$, if $|x-y| < C\rho(x)$.

LEMMA 6.4. ([39]) *Let $V \in B_q$ with $q \geqslant n/2$. For any $l > 0$, there exists $C_l > 0$ such that*

$$\left|K_j^L(x,y)\right| \leqslant \frac{C_l}{\left(1+\frac{|x-y|}{\rho(x)}\right)^l} \frac{1}{|x-y|^{n-1}},$$

and

$$\left|K_j^L(x,y) - K_j(x-y)\right| \leqslant C\frac{\rho(x)^{-1}}{|x-y|^{n-2}}.$$

The following theorem in the case $w = 1$ was proved in [2].

THEOREM 6.11. *Suppose that $\Omega \in L_q(S^{n-1})$, $1 < q \leqslant \infty$ satisfies the conditions (1.1), (1.2) and $V \in B_n$. Then for every $q' < p < \infty$ and $w \in A_{p/q'}$ or $1 < p < q$ and $w^{1-p'} \in A_{p'/q'}$ there is a constant C independent of f such that*

$$\|\mu_{j,\Omega}^L f\|_{L_{p,w}} \leqslant C\|f\|_{L_{p,w}}.$$

Proof. The proof follows from the boundedness of the operators M_Ω and $\mu_{j,\Omega}$ on $L_{p,w}(\mathbb{R}^n)$ for $p > 1$ and the validity of the following inequality

$$\mu_{j,\Omega}^L f(x) \lesssim \mu_{j,\Omega} f(x) + M_\Omega f(x), \;\; a.e. \;\; x \in \mathbb{R}^n,$$

which was proved in the proof of [2, Theorem 5]. □

Note that the operators M_Ω and $\mu_{j,\Omega}$ which are sublinear operators satisfies the condition (3.6) and bounded on $L_{p,w}(\mathbb{R}^n)$ for $p > 1$. Statements of the Lemma 4.1 for the operators M_Ω and $\mu_{j,\Omega}$ is provided. Then we get that the statements of the Lemma 4.1 also true for the operators $\mu_{j,\Omega}^L$, $j = 1,\dots,n$. Therefore, by Lemma 4.1 and Theorem 6.11 the following corollaries are obtained.

COROLLARY 6.1. *Suppose that $\Omega \in L_q(S^{n-1})$, $1 < q \leqslant \infty$ satisfies the conditions (1.1), (1.2) and $V \in B_n$.*

If $q' < p < \infty$ and $w \in A_{p/q'}$, then the inequality

$$\|\mu_{j,\Omega}^L f\|_{L_{p,w}(B(x_0,r))} \lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, and for all $f \in L_{p,w}^{\rm loc}(\mathbb{R}^n)$.

If $1 < p < q$ and $w^{1-p'} \in A_{p'/q'}$, then the inequality

$$\|\mu_{j,\Omega}^L f\|_{L_{p,w}(B(x_0,r))} \lesssim \|w\|_{L_{\frac{q}{q-p}}(B(x_0,r))}^{1/p} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, \|w\|_{L_{\frac{q}{q-p}}(B(x_0,t))}^{-1/p} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, and for all $f \in L_{p,w}^{\rm loc}(\mathbb{R}^n)$.

COROLLARY 6.2. *Suppose that $\Omega \in L_q(S^{n-1})$, $1<q\leqslant\infty$ satisfies the conditions* (1.1), (1.2) *and $V\in B_n$. Let also, for $q'<p<\infty$, $w\in A_{p/q'}$ the pair (φ_1,φ_2) satisfies the condition* (3.8) *and for $1<p<q$, $w^{1-p'}\in A_{p'/q'}$ the pair (φ_1,φ_2) satisfies the condition* (5.3)*. Then the operator $\mu^L_{j,\Omega}$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$. Moreover*

$$\|\mu^L_{j,\Omega}f\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)}.$$

The following theorem in the case $w=1$ was proved in [2].

THEOREM 6.12. *Suppose that $\Omega\in L_q(S^{n-1})$, $1<q\leqslant\infty$ satisfies the conditions* (1.1), (1.2) *and $V\in B_n$. Let also $b\in BMO(\mathbb{R}^n)$. Then for every $q'<p<\infty$, $w\in A_{p/q'}$ or $1<p<q$, $w^{1-p'}\in A_{p'/q'}$ there is a constant C independent of f such that*

$$\|\mu^L_{j,\Omega,b}f\|_{L_{p,w}} \leqslant C\|f\|_{L_{p,w}}.$$

Proof. In the proof we used the idea in [19]. It suffices to show that

$$\mu^L_{j,\Omega,b}f(x) \leqslant \mu_{j,\Omega,b}f(x) + CM_{\Omega,b}f(x), \quad a.e.\ x\in\mathbb{R}^n, \tag{6.1}$$

where $M_{\Omega,b}$ denotes the commutator of the commutator of Hardy-Littlewood operator with rough kernel.

Fixing $x\in\mathbb{R}^n$ and let $r=\rho(x)$. Then

$$\begin{aligned}
\mu^L_{j,\Omega,b}f(x) \leqslant & \left(\int_0^r \left|\int_{|x-y|\leqslant t} |\Omega(x-y)|K^L_j(x,y)[b(x)-b(y)]f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac12}\\
&+\left(\int_r^\infty \left|\int_{|x-y|\leqslant r} |\Omega(x-y)|K^L_j(x,y)[b(x)-b(y)]f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac12}\\
&+\left(\int_r^\infty \left|\int_{r<|x-y|\leqslant t} |\Omega(x-y)|K^L_j(x,y)[b(x)-b(y)]f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac12}\\
\leqslant & \left(\int_0^r \left|\int_{|x-y|\leqslant t} |\Omega(x-y)|[K^L_j(x,y)-K_j(x,y)][b(x)-b(y)]f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac12}\\
&+\left(\int_0^r \left|\int_{|x-y|\leqslant t} |\Omega(x-y)|K_j(x,y)[b(x)-b(y)]f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac12}\\
&+\left(\int_r^\infty \left|\int_{|x-y|\leqslant r} |\Omega(x-y)|K^L_j(x,y)[b(x)-b(y)]f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac12}\\
&+\left(\int_r^\infty \left|\int_{r<|x-y|\leqslant t} |\Omega(x-y)|K^L_j(x,y)[b(x)-b(y)]f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac12}\\
:= & E_1+E_2+E_3+E_4.
\end{aligned}$$

For E_1, by Lemma 6.4, we have

$$E_1 \leqslant C\left(\int_0^r \left|\frac{1}{r}\int_{|x-y|\leqslant t} |\Omega(x-y)||[b(x)-b(y)]\frac{|f(y)|}{|x-y|^{n-2}}dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}$$
$$\leqslant CM_{\Omega,b}f(x).$$

Obviously,

$$E_2 \leqslant \mu_{j,\Omega,b}f(x).$$

For E_3, using Lemma 6.4 again, we get

$$E_3 \leqslant \left(\int_r^\infty \left|\frac{1}{r}\int_{|x-y|\leqslant r} |\Omega(x-y)||[b(x)-b(y)]\frac{|f(y)|}{|x-y|^{n-2}}dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}$$
$$\leqslant CM_{\Omega,b}f(x).$$

It remains to estimate E_4. From Lemma 6.4, we obtain

$$E_4 \leqslant C\left(\int_r^\infty \left|r\int_{r<|x-y|\leqslant t} |\Omega(x-y)||[b(x)-b(y)]\frac{|f(y)|}{|x-y|^{n}}dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}$$
$$\leqslant Cr\left(\int_r^\infty \left|\sum_{k=0}^{[\log_2 t/r]+1}(2^k r)^n\int_{|x-y|\leqslant 2^k r} |\Omega(x-y)||[b(x)-b(y)]||f(y)|dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}$$
$$\leqslant Cr\left(\int_r^\infty \left|([\log_2 t/r]+1)M_{\Omega,b}f(x)\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}$$
$$\leqslant Cr\left(\int_r^\infty \frac{t}{r}M_{\Omega,b}f(x)^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \leqslant CM_{\Omega,b}f(x).$$

Thus, Theorem 6.12 is proved. □

Note that the operators $M_{\Omega,b}$ and $\mu_{j,\Omega,b}$ which are sublinear operators satisfies the condition (3.6) and bounded on $L_{p,w}(\mathbb{R}^n)$ for $p>1$. Statements of the Lemma 5.2 for the operators $M_{\Omega,b}$, $\mu_{j,\Omega,b}$ and also for the operators $\mu^L_{j,\Omega,b}$, $j=1,\dots,n$ are provided. Therefore, by Lemma 5.2 and Theorem 6.12 the following corollaries are obtained.

COROLLARY 6.3. *Suppose that $\Omega \in L_q(S^{n-1})$, $1<q\leqslant\infty$ satisfies the conditions (1.1), (1.2) and $V\in B_n$. Let also $b\in BMO(\mathbb{R}^n)$.*
If $q'<p<\infty$ and $w\in A_{p/q'}$, then the inequality

$$\|\mu^L_{j,\Omega,b}f\|_{L_{p,w}(B(x_0,r))}$$
$$\lesssim w(B(x_0,r))^{\frac{1}{p}}\int_{2r}^\infty \left(1+\ln\frac{t}{r}\right)\|f\|_{L_{p,w}(B(x_0,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\frac{dt}{t}$$

holds for any ball $B(x_0,r)$, *and for all* $f \in L^{\mathrm{loc}}_{p,w}(\mathbb{R}^n)$.

If $1 < p < q$ *and* $w^{1-p'} \in A_{p'/q'}$, *then the inequality*

$$\|\mu^L_{j,\Omega,b}f\|_{L_{p,w}(B(x_0,r))} \lesssim \|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x_0,r))} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))} \, \|w\|^{-1/p}_{L_{\frac{q}{q-p}}(B(x_0,t))} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, *and for all* $f \in L^{\mathrm{loc}}_{p,w}(\mathbb{R}^n)$.

COROLLARY 6.4. *Suppose that* $\Omega \in L_q(S^{n-1})$, $1 < q \leqslant \infty$ *satisfies the conditions* (1.1), (1.2) *and* $b \in BMO(\mathbb{R}^n)$, $V \in B_n$. *Let also, for* $q' < p < \infty$, $w \in A_{p/q'}$ *the pair* (φ_1,φ_2) *satisfies the condition* (3.8) *and for* $1 < p < q$, $w^{1-p'} \in A_{p'/q'}$ *the pair* (φ_1,φ_2) *satisfies the condition* (5.3). *Then the operator* $\mu^L_{j,\Omega,b}$ *is bounded from* $M_{p,\varphi_1}(w)$ *to* $M_{p,\varphi_2}(w)$. *Moreover*

$$\|\mu^L_{j,\Omega,b}f\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)}.$$

Acknowledgements. The authors would like to express their gratitude to the referees for his (her) very valuable comments and suggestions.

REFERENCES

[1] A. AKBULUT, V. S. GULIYEV AND R. MUSTAFAYEV, *On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces*, Math. Bohem. **137** (1) (2012), 27–43.

[2] A. AKBULUT, O. KUZU, *Marcinkiewicz integrals with rough kernel associated with Schrödinger operator on vanishing generalized Morrey spaces*, Azerb. J. Math. **4** (1) (2014), 40–54.

[3] J. ALVAREZ, R. J. BAGBY, D. S. KURTZ AND C. PÉREZ, *Weighted estimates for commutators of linear operators*, Studia Math. **104** (1993), 195–209.

[4] A. BENEDEK, A. P. CALDERON, R. PANZONE, *Convolution operators on Banach value functions*, Proc. Natl. Acad. Sci. USA **48** (1962), 256–265.

[5] A. P. CALDERON, *Commutators of singular integral operators*, Proc. Natl. Acad. Sci. USA **53** (1965), 1092–1099.

[6] A. P. CALDERON, *Cauchy integrals on Lipschitz curves and related operators*, Proc. Natl. Acad. Sci. USA **74** (4) (1977), 1324–1327.

[7] F. CHIARENZA, M. FRASCA, *Morrey spaces and Hardy-Littlewood maximal function*, Rend Mat. **7** (1987), 273–279.

[8] F. CHIARENZA, M. FRASCA, P. LONGO, $W^{2,p}$*-solvability of Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc. **336** (1993), 841–853.

[9] R. COIFMAN, R. ROCHBERG, G. WEISS, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (2) (1976), 611–635.

[10] Y. CHEN, *Regularity of solutions to elliptic equations with VMO coefficients*, Acta Math. Sin. (Engl. Ser.) **20** (2004), 1103–1118.

[11] Y. DING, *On Marcinkiewicz integral*, in: Proc. of the Conference "Singular Integrals and Related Topics, III," Osaka, Japan, 2001, pp. 28–38.

[12] Y. DING, D. FAN, Y. PAN, *Weighted boundedness for a class of rough Marcinkiewicz integrals*, Indiana Univ. Math. J. **48** (1999), 1037–1055.

[13] Y. DING, D. FAN, Y. PAN, L_p*-boundedness of Marcinkiewicz integrals with Hardy space function kernels*, Acta Math. Sinica, English Series, **16** (2000), 593–600.

[14] Y. DING, S. LU, K. YABUTA, *On commutators of Marcinkiewicz integrals with rough kernel*, J. Math. Anal. Appl, **275** (2002), 60–68.

[15] J. DUOANDIKOETXEA, *Weighted norm inequalities for homogeneous singular integrals*, Trans. Amer. Math. Soc. **336** (1993), 869–880.

[16] J. DZIUBAŃSKI, J. ZIENKIEWICZ, *Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality*, Rev. Mat. Iber. **15** (1999), 279–296.

[17] G. DI FAZIO, M. A. RAGUSA, *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Funct. Anal. **112** (1993), 241–256.

[18] D. FAN, S. LU AND D. YANG, *Boundedness of operators in Morrey spaces on homogeneous spaces and its applications*, Acta Math. Sinica (N. S.) **14** (1998), suppl., 625–634.

[19] W. GAO, L. TANG, *Boundedness for Marcinkiewicz integrals associated with Schrödinger operators*, Proc. Indian Acad. Sci. Math. Sci. **124** (2) (2014), 193–203.

[20] M. GIAQUINTA, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Princeton Univ. Press, Princeton, NJ, 1983.

[21] V. S. GULIYEV, *Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$*, Doctoral dissertation, Moscow, Mat. Inst. Steklov, 1994, 329 pp. (in Russian).

[22] V. S. GULIYEV, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. 2009, Art. ID 503948, 20 pp.

[23] V. S. GULIYEV, *Generalized weighted Morrey spaces and higher order commutators of sublinear operators*, Eurasian Math. J. **3** (3) (2012), 33–61.

[24] V. S. GULIYEV, S. S. ALIYEV, T. KARAMAN, *Boundedness of a class of sublinear operators and their commutators on generalized Morrey spaces*, Abstr. Appl. Anal. vol. 2011, Art. ID 356041, 18 pp. doi:10.1155/2011/356041

[25] V. S. GULIYEV, S. S. ALIYEV, *Boundedness of parametric Marcinkiewicz integral operator and their commutators on generalized Morrey spaces*, Georgian Math. J. **19** (2012), 195–208.

[26] V. S. GULIYEV, *Local generalized Morrey spaces and singular integrals with rough kernel*, Azerb. J. Math. **3** (2) (2013), 79–94.

[27] V. S. GULIYEV, *Generalized local Morrey spaces and fractional integral operators with rough kernel*, J. Math. Sci. (N. Y.) **193** (2) (2013), 211–227.

[28] V. S. GULIYEV, T. KARAMAN, R. CH. MUSTAFAYEV AND A. SERBETCI, *Commutators of sublinear operators generated by Calderón-Zygmund operator on generalized weighted Morrey spaces*, Czechoslovak Math. J. **64** (139) (2) (2014), 365–386.

[29] V. S. GULIYEV, M. N. OMAROVA, *Higher order commutators of vector-valued intrinsic square functions on vector-valued generalized weighted Morrey spaces*, Azerb. J. Math. **4** (2) (2014), 64–85.

[30] L. HÖRMANDER, *Translation invariant operators*, Acta Math. **104** (1960), 93–139.

[31] S. JANSON, *On functions with conditions on the mean oscillation*, Ark. Mat. **14** (2) (1976), 189–196.

[32] T. KARAMAN, V. S. GULIYEV, A. SERBETCI, *Boundedness of sublinear operators generated by Calderón-Zygmund operators on generalized weighted Morrey spaces*, Scientic Annals of "Al. I. Cuza" University of Iasi, **60** (1) 2014, 227–244.

[33] Y. KOMORI AND S. SHIRAI, *Weighted Morrey spaces and a singular integral operator*, Math. Nachr. **282** (2) (2009), 219–231.

[34] A. KUFNER, O. JOHN AND S. FUÇIK, *Function Spaces*, Noordhoff International Publishing: Leyden, Publishing House Czechoslovak Academy of Sciences: Prague, 1977.

[35] G. LU, S. LU, D. YANG, *Singular integrals and commutators on homogeneous groups*, Anal. Math. **28** (2002), 103–134.

[36] T. MIZUHARA, *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer-Verlag, Tokyo (1991), 183–189.

[37] C. B. MORREY, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.

[38] B. MUCKENHOUPT, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.

[39] Z. SHEN, *L^p estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier (Grenoble) **45** (1995), 513–546.

[40] F. SORIA, G. WEISS, *A remark on singular integrals and power weights*, Indiana Univ. Math. J. **43** (1994), 187–204.

[41] E. M. STEIN, *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), 430–466.

[42] T. WALSH, *On the function of Marcinkiewicz*, Studia Math. **44** (1972), 203–217.

[43] H. XU, J. C. CHEN AND Y. M. YING, *A note on Marcinkiewicz integrals with H_1 kernels*, Acta. Math. Scientia, Ser. B., **23** (1) (2003) 133–138.

(Received November 28, 2014)

Vagif S. Guliyev
Ahi Evran University, Department of Mathematics
40100, Kirsehir, Turkey
and
Institute of Mathematics and Mechanics of NASA
AZ 1141 Baku, Azerbaijan

Ali Akbulut
Ahi Evran University, Department of Mathematics
40100, Kirsehir, Turkey
e-mail: aakbulut@ahievran.edu.tr

Vugar H. Hamzayev
nstitute of Mathematics and Mechanics of NASA
AZ 1141 Baku, Azerbaijan
and
Nakhchivan Teacher-Training Institute, Nakhchivan, Azerbaijan
e-mail: vugarhamzayev@yahoo.com

Okan Kuzu
Ahi Evran University, Department of Mathematics
40100, Kirsehir, Turkey
e-mail: okan.kuzu@ahievran.edu.tr