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A PRECISE INEQUALITY OF DIFFERENTIAL POLYNOMIALS RELATED TO SMALL FUNCTIONS

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Abstract. In this paper, we consider the value distribution of the differential polynomials $\varphi f^2 f' - 1$ where f is a transcendental meromorphic function and φ is a small function, and obtain a precise inequality by the reduced counting function.

1. Introduction and results

Let $f(z)$ be a meromorphic function in the complex plane, we say $a(z)$ is a small function if $a(z)$ is a non-vanishing meromorphic function such that $T(r,a) = S(r,f)$ and $S(r,f)$ denotes $o(T(r,f))$ $(r \to \infty)$, possibly outside a set of r of finite linear measure. We assumed that the reader is familiar with the notations of Nevanlinna theory (see, e.g., [1, 6, 7]).

DEFINITION 1. Let k be a positive integer, for any constant a in the complex plane. We denote by $N_{k)}(r,1/(f-a))$ the counting function of a-points of f with multiplicity $\leqslant k$, by $N_{(k}(r,1/(f-a))$ the counting function of a-points of f with multiplicity $\geqslant k$, by $N_k(r,1/(f-a))$ the counting function of a-points of f with multiplicity of k. and denote the reduced counting function by $\overline{N}_{k)}(r,1/(f-a))$, $\overline{N}_{(k}(r,1/(f-a))$ and $\overline{N}_k(r,1/(f-a))$, respectively.

In 1979, E. Mues [1] proved that for a transcendental meromorphic function f in the open plane, $f^2 f' - 1$ has infinitely many zeros. This is a qualitative result. In 1992, Q. Zhang [8] has obtained a quantitative result and proved the following theorem.

THEOREM A. *Let f be transcendental meromorphic in the complex plane, then*

$$T(r,f) \leqslant 6N\left(r, \frac{1}{f^2 f' - 1}\right) + S(r,f). \tag{1}$$

In [4], Xu, Yi and Zhang improved Theorem A by the reduced counting function and proved the following.

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THEOREM B. *Let f be a transcendental meromorphic function. Then*

$$T(r,f) \leqslant 6\overline{N}\left(r,\frac{1}{f^2f'-1}\right)+S(r,f). \tag{2}$$

In the value distribution theory, it is a very important problem whether we can use the small function to instead of the constant in the counting function (or the reduced counting function)? For example, K. Yamanoi proved the second Nevanlinna main theorem for small functions in [5]. It's one of the most important work in the value distribution theory in the recent years.

In 1993, Q. Zhang [9] studied the zeros of $f^2(z)f'(z)-a(z)$, where $a(z)\not\equiv 0$ is a small function, and improved Theorem A.

THEOREM C. *Let $f(z)$ be transcendental meromorphic in the complex plane and $\varphi(z)(\not\equiv 0)$ be a small function, then*

$$T(r,f) \leqslant 6N\left(r,\frac{1}{\varphi f^2f'-1}\right)+S(r,f). \tag{3}$$

Corresponding Theorem B, it is naturally to consider the value distribution of $\varphi f^2f'-$ by the reduced counting function. In fact, we proved the following result.

THEOREM 1. *Let $f(z)$ be a transcendental meromorphic function and $\varphi(z)(\not\equiv 0)$ be a small function. Then*

$$T(r,f) \leqslant 6\overline{N}\left(r,\frac{1}{\varphi f^2f'-1}\right)+S(r,f). \tag{4}$$

Obviously, our result improves the conclusion of Q. D. Zhang in [8, 9] and Xu, Yi and Zhang in [4] greatly.

2. Proof of the theorem 1

In order to prove our result, we need the following lemma.

LEMMA 1. *Let f be a transcendental meromorphic function, and let $\varphi(z)(\not\equiv 0)$ be a small function. Then*

$$\begin{aligned}3T(r,f) \leqslant{}& \overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{\varphi f^2f'-1}\right)\\&-N_0\left(r,\frac{1}{(\varphi f^2f')'}\right)+S(r,f),\end{aligned} \tag{5}$$

where $N_0\left(r,\frac{1}{(\varphi f^2f')'}\right)$ denotes the counting function of the zeros of $(\varphi f^2f')'$, not of $f(\varphi f^2f'-1)$.

Proof. We first claim that $\varphi f^2 f' \not\equiv$ constant. If $\varphi f^2 f' \equiv C$, where C is a constant. Obviously, $C \neq 0$. Then

$$\frac{1}{f^3} \equiv \frac{\varphi}{C}\frac{f'}{f}, \quad \frac{1}{f^2 f'} \equiv C\varphi.$$

Therefore,

$$\begin{aligned} m\left(r, \frac{1}{f}\right) &\leqslant \frac{1}{3} m\left(r, \frac{1}{C}\varphi\frac{f'}{f}\right) \\ &\leqslant \frac{1}{3} m(r,\varphi) + \frac{1}{3} m\left(r, \frac{f'}{f}\right) + O(1) = S(r,f), \\ N\left(r, \frac{1}{f}\right) &\leqslant N\left(r, \frac{1}{f^2 f'}\right) \\ &= N\left(r, \frac{1}{C}\varphi\right) = S(r,f). \end{aligned}$$

From the above, we have $T(r,f) = S(r,f)$. It is a contradiction. Hence $\varphi f^2 f'$ is not equivalent to a constant.

Let

$$\frac{1}{f^3} \equiv \frac{\varphi f^2 f'}{f^3} - \frac{(\varphi f^2 f')'}{f^3}\frac{\varphi f^2 f' - 1}{(\varphi f^2 f')'},$$

we have

$$\begin{aligned} 3m\left(r, \tfrac{1}{f}\right) &= m\left(r, \frac{1}{f^3}\right) \\ &\leqslant m\left(r, \frac{\varphi f^2 f' - 1}{(\varphi f^2 f')'}\right) + m\left(r, \varphi\frac{f'}{f}\right) + m\left(r, \frac{(\varphi f^2 f')'}{f^3}\right) + O(1) \\ &\leqslant N\left(r, \frac{(\varphi f^2 f')'}{\varphi f^2 f' - 1}\right) - N\left(r, \frac{\varphi f^2 f'}{(\varphi f^2 f')'}\right) + S(r,f) \\ &= N(r, (\varphi f^2 f')') + N\left(r, \frac{1}{\varphi f^2 f' - 1}\right) - N\left(r, \frac{1}{(\varphi f^2 f')'}\right) \\ &\quad - N(r, \varphi f^2 f') + S(r,f) \\ &= \overline{N}(r,f) + N\left(r, \frac{1}{\varphi f^2 f' - 1}\right) - N\left(r, \frac{1}{(\varphi f^2 f')'}\right) + S(r,f). \end{aligned}$$

Hence

$$\begin{aligned} 3T(r,f) &= 3m\left(r, \frac{1}{f}\right) + 3N\left(r, \frac{1}{f}\right) + O(1) \\ &= \overline{N}(r,f) + 3N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{\varphi f^2 f' - 1}\right) - N\left(r, \frac{1}{(\varphi f^2 f')'}\right) + S(r,f). \end{aligned} \tag{6}$$

Let

$$N\left(r,\frac{1}{(\varphi f^2 f')'}\right)=N_{000}\left(r,\frac{1}{(\varphi f^2 f')'}\right)+N_{00}\left(r,\frac{1}{(\varphi f^2 f')'}\right)+N_0\left(r,\frac{1}{(\varphi f^2 f')'}\right), \tag{7}$$

where $N_{000}\left(r,\frac{1}{(\varphi f^2 f')'}\right)$ denotes the counting function of the zeros of $(\varphi f^2 f')'$, which come from the zeros of $\varphi f^2 f'-1$, $N_{00}\left(r,\frac{1}{(\varphi f^2 f')'}\right)$ denotes the counting function of the zeros of $(\varphi f^2 f')'$, which come from the zeros of f. Hence we have

$$N\left(r,\frac{1}{\varphi f^2 f'-1}\right)-N_{000}\left(r,\frac{1}{(\varphi f^2 f')'}\right)=\overline{N}\left(r,\frac{1}{\varphi f^2 f'-1}\right). \tag{8}$$

Supposed that z_0 is a zero of f with multiplicity q and the pole of φ with multiplicity of t.

Case I. Supposed that $t\leqslant 2q-1$. If $q=1$, then z_0 is a zero of $(\varphi f^2 f')'$ with multiplicity at least $2q-1-t$; if $q\geqslant 2$, then z_0 is a zero of $(\varphi f^2 f')'$ with multiplicity at least $3q-2-t$.

Case II. Supposed that $t\geqslant 2q$, z_0 is at most the pole of φ^2.

Hence we have

$$\begin{aligned}3N\left(r,\frac{1}{f}\right)-N_{00}\left(r,\frac{1}{(\varphi f^2 f')'}\right)&\leqslant N_{1)}\left(r,\frac{1}{f}\right)+\overline{N}_{1)}\left(r,\frac{1}{f}\right)+2\overline{N}_{(2}\left(r,\frac{1}{f}\right)+N(r,\varphi^2)\\&=N_{1)}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}_{(2}\left(r,\frac{1}{f}\right)+S(r,f).\end{aligned} \tag{9}$$

Combining (6)–(9), we have

$$\begin{aligned}3T(r,f)\leqslant{}&\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+N_{1)}\left(r,\frac{1}{f}\right)+\overline{N}_{(2}\left(r,\frac{1}{f}\right)\\&+\overline{N}\left(r,\frac{1}{\varphi f^2 f'-1}\right)-N_0\left(r,\frac{1}{(\varphi f^2 f')'}\right)+S(r,f).\end{aligned}$$

This completes the proof of the lemma. □

Now we begin to prove Theorem 1.

Let $F(z)=\varphi(z)f^2(z)f'(z)-1$ and

$$h(z)=\frac{F'(z)}{f(z)}=\varphi(z)\{2f'^2(z)+f(z)f''(z)\}+\varphi'(z)f(z)f'(z).$$

Obviously, $h(z)\not\equiv 0$. Also let

$$\begin{aligned}G(z)={}&13\left(\frac{F'(z)}{F(z)}\right)^2+20\left(\frac{F'(z)}{F(z)}\right)'-24\frac{F'(z)}{F(z)}\frac{h'(z)}{h(z)}+8\left(\frac{h'(z)}{h(z)}\right)^2-8\left(\frac{h'(z)}{h(z)}\right)'\\&-4\frac{\varphi'(z)}{\varphi(z)}\frac{F'(z)}{F(z)}+8\frac{\varphi'(z)}{\varphi(z)}\frac{h'(z)}{h(z)}-8\left(\frac{\varphi'(z)}{\varphi(z)}\right)'.\end{aligned}$$

By Lemmas 4 and 7 in [9], we know $G(z)\not\equiv 0$ and the simple poles of $f(z)$ are the zeros of $G(z)$.

By differentiating the equation $F=\varphi f^2f'-1$, we get

$$f\beta=-\frac{F'}{F}, \tag{10}$$

where

$$\beta=\varphi' ff'+2\varphi(f')^2+\varphi ff''-\varphi ff'\frac{F'}{F}, \quad h=-\beta F. \tag{11}$$

Note that the poles of $G(z)$ whose multiplicity are at most two come from the multiple poles of $f(z)$, $F(z)$ or the zeros of $h(z)$.

We consider the poles of β^2G. We can see the zeros of h either are the zeros of F, or the zeros of β. From the above we know that the multiple poles of f with the multiplicity $q(\geqslant 2)$ are the zeros of β with the multiplicity of $q-1$. Hence the poles of β^2G only come from the zeros of F, and the multiplicity is at most 4. Hence,

$$N(r,\beta^2G)\leqslant 4\overline{N}(r,1/F).$$

Note that $m(r,G)=S(r,f)$, therefore $m(r,\beta^2G)=S(r,f)$. Hence

$$T(r,\beta^2G)\leqslant 4\overline{N}(r,1/F).$$

Since the multiple zeros of f with the multiplicity $p(\geqslant 2)$ are the multiple zeros of β with multiplicity at least $2p-2$, therefore, are at least the zeros of β^2G with the multiplicity $2(2p-2)-2=4p-6$. Also note that the simple poles of f are the zeros of β^2G. Hence we have

$$\overline{N}_{1)}(r,f)+2N\left(r,\frac{1}{f}\right)-2\overline{N}\left(r,\frac{1}{f}\right)\leqslant N\left(r,\frac{1}{\beta^2G}\right)\leqslant T(r,\beta^2G)\leqslant 4\overline{N}\left(r,\frac{1}{F}\right). \tag{12}$$

From (5), we have

$$m(r,f)+N(r,f)-\overline{N}(r,f)+2m\left(r,\frac{1}{f}\right)+2N\left(r,\frac{1}{f}\right)-2\overline{N}\left(r,\frac{1}{f}\right)\leqslant\overline{N}\left(r,\frac{1}{F}\right)+S(r,f). \tag{13}$$

Combining the twice times of (13) and (12), we have

$$\begin{aligned}&T(r,f)+N_{(2}(r,f)-2\overline{N}_{(2}(r,f)+m(r,f)+4m\left(r,\frac{1}{f}\right)+6N\left(r,\frac{1}{f}\right)-6\overline{N}\left(r,\frac{1}{f}\right)\\&\leqslant 6\overline{N}\left(r,\frac{1}{F}\right)+S(r,f),\end{aligned}$$

Hence we have

$$T(r,f)<6\overline{N}\left(r,\frac{1}{\varphi f^2f'-1}\right)+S(r,f).$$

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