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THE MOLECULAR DECOMPOSITION OF HERZ–MORREY–HARDY SPACES WITH VARIABLE EXPONENTS AND ITS APPLICATION

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Abstract. The molecular decomposition of Herz-Morrey-Hardy spaces with variable exponents is given. As its application, the boundedness of a convolution type singular integral on Herz-Morrey-Hardy spaces with variable exponents is obtained.

1. Introduction

Recent three decades, there is a increasing interest in variable exponent spaces, see [15] and the monographs [5, 6] and the references therein. Moreover, variable Hardy spaces, variable Herz spaces, variable Besov and Triebel-Lizorkin spaces, variable Morrey spaces and Banach space valued variable Lebesgue space and Sobolev spaces have been introduced; see [1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 19, 20, 21, 22, 29, 30, 31, 34]. Meanwhile, Morrey type Besov and Triebel spaces and their generalizations have been developed; see [17, 18, 23, 24, 25, 27, 28, 35]. Motivated by these references, we introduced Herz-Morrey-Hardy spaces with variable exponents and gave their atomic decomposition in [33]. As a continuation of [33], we establish their molecular decomposition in next section. For the molecular decompositions of the classical Hardy spaces, the variable Hardy spaces, the Herz type Hardy spaces and the non-homogeneous Herz type Besov and Triebel-Lizorkin spaces, we refer the reader to [26], [19], [16] and [32], respectively. In Section 3, by using their atomic and molecular decompositions, we shall give the boundedness of a convolution type singular integral on Herz-Morrey-Hardy spaces with variable exponents.

2. Molecular decomposition

Let $L^1_{\rm loc}(\mathbb{R}^n)$ be the collection of all locally integrable functions on $\mathbb{R}^n$. Given a function $f \in L^1_{\rm loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator $\mathscr{M}$ is defined by

$$\mathscr{M} f(x) := \sup_{r>0} r^{-n} \int_{B(x,r)} |f(y)| \mathrm{d}y, \ \forall x \in \mathbb{R}^n,$$

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where and follows $B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$. We also use the following notation: $p_- := \operatorname{ess\,inf}\{p(x) : x \in \mathbb{R}^n\}$ and $p_+ := \operatorname{ess\,sup}\{p(x) : x \in \mathbb{R}^n\}$. The set $\mathscr{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$. $\mathscr{B}(\mathbb{R}^n)$ is the set of $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfying the condition that $\mathscr{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (for its definition see [33]).

In this paper we use the symbol $A \lesssim B$ to denote that there exists a positive exponent c such that $A \leqslant cB$.

LEMMA 2.1. (see [12]) *Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then there exist $0 < \delta_1, \delta_2 < 1$ and a positive constant C depending only on $p(\cdot)$ and n such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}}}{\|\chi_S\|_{L^{p(\cdot)}}} \leqslant C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{p(\cdot)}}}{\|\chi_B\|_{L^{p(\cdot)}}} \leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_1} \quad \textit{and} \quad \frac{\|\chi_S\|_{L^{p'(\cdot)}}}{\|\chi_B\|_{L^{p'(\cdot)}}} \leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_2}.$$

LEMMA 2.2. (see [12]) *Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for any ball B in $\mathbb{R}^n$,*

$$\|\chi_B\|_{L^{p(\cdot)}} \|\chi_B\|_{L^{p'(\cdot)}} \leqslant C|B|.$$

Let $k \in \mathbb{Z}$, $B_k := \{x \in \mathbb{R}^n : |x| \leqslant 2^k\}$, $D_k := B_k \backslash B_{k-1}$, $\chi_k := \chi_{D_k}$. The symbol $\mathbb{N}_0$ denotes the set of all non-negative integers. For any $m \in \mathbb{N}_0$, we denote $\tilde{\chi}_m := \chi_{D_m}, m \geqslant 1$ and $\tilde{\chi}_0 := \chi_{B_0}$ respectively.

DEFINITION 2.1. Let $0 < q \leqslant \infty, p(\cdot) \in \mathscr{P}(\mathbb{R}^n), 0 \leqslant \lambda < \infty$. Let $\alpha(\cdot)$ be a bounded real-valued measurable function on $\mathbb{R}^n$. The homogeneous Herz-Morrey space $M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ and non-homogeneous Herz-Morrey space $MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ are defined respectively by

$$M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda} := \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \backslash 0) : \|f\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} < \infty \right\},$$

and

$$MK^{\alpha(\cdot),q}_{p(\cdot),\lambda} := \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}} < \infty \right\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} := \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} \|2^{\alpha(\cdot)k} f \chi_k\|^q_{L^{p(\cdot)}} \right)^{\frac{1}{q}},$$

and

$$\|f\|_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}} := \sup_{L \in \mathbb{N}_0} 2^{-L\lambda} \left(\sum_{k=0}^{L} \|2^{\alpha(\cdot)k} f \tilde{\chi}_k\|^q_{L^{p(\cdot)}} \right)^{\frac{1}{q}}.$$

Here there is the usual modification when $q = \infty$.

LEMMA 2.3. (see [33]) *Let* $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$, $q\in(0,\infty]$ *and* $\lambda\in[0,\infty)$. *If* $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathscr{P}_0^{\log}(\mathbb{R}^n)\cap\mathscr{P}_\infty^{\log}(\mathbb{R}^n)$, *then*

$$\|f\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}\approx\max\left\{\sup_{L\leqslant 0,L\in\mathbb{Z}}2^{-L\lambda}\left(\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\|f\chi_k\|^q_{L^{p(\cdot)}}\right)^{\frac{1}{q}},\right.$$
$$\left.\sup_{L>0,L\in\mathbb{Z}}\left[2^{-L\lambda}\left(\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\|f\chi_k\|^q_{L^{p(\cdot)}}\right)^{\frac{1}{q}}+2^{-L\lambda}\left(\sum_{k=0}^{L}2^{k\alpha(\infty)q}\|f\chi_k\|^q_{L^{p(\cdot)}}\right)^{\frac{1}{q}}\right]\right\}.$$

Let G_Nf be the grand maximal function of f defined by

$$G_Nf(x):=\sup_{\phi\in\mathscr{A}_N}|\phi^*_\triangledown(f)(x)|,\ x\in\mathbb{R}^n,$$

where $\mathscr{A}_N:=\{\phi\in\mathscr{S}(\mathbb{R}^n):\sup_{|\alpha|,|\beta|\leqslant N,\ \forall x\in\mathbb{R}^n}|x^\alpha D^\beta\phi(x)|\leqslant 1\}$ and $N>n+1$, $\phi^*_\triangledown$ is the nontangential maximal operator defined by

$$\phi^*_\triangledown(f)(x):=\sup_{|y-x|<t}|\phi_t*f(y)|,\ \forall x\in\mathbb{R}^n\text{ with }\phi_t(\cdot)=t^{-n}\phi(\cdot/t).$$

DEFINITION 2.2. Let $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$, $0<q\leqslant\infty$, $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$, $0\leqslant\lambda<\infty$ and $N>n+1$. The homogeneous Herz-Morrey-Hardy space with variable exponents $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ and non-homogeneous Herz-Morrey-Hardy space with variable exponents $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ are defined respectively by

$$HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}:=\left\{f\in\mathscr{S}'(\mathbb{R}^n):\ \|f\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}:=\|G_Nf\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}<\infty\right\},$$

and

$$HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}:=\left\{f\in\mathscr{S}'(\mathbb{R}^n):\ \|f\|_{HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}}:=\|G_Nf\|_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}}<\infty\right\},$$

Let $\mathscr{P}_0^{\log}(\mathbb{R}^n)$ and $\mathscr{P}_\infty^{\log}(\mathbb{R}^n)$ be the set of log-Hölder continuous functions at origin and the set of log-Hölder continuous functions at infinity, respectively; for their definitions see [33].

Now, we give the notion of molecules and atoms.

DEFINITION 2.3. Let $0<q<\infty$, $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$, $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathscr{P}_0^{\log}(\mathbb{R}^n)\cap\mathscr{P}_\infty^{\log}(\mathbb{R}^n)$ such that $n\delta_1\leqslant\alpha(0)$, $\alpha_\infty<\infty$ where δ_1 as in Lemma 2.1. Let non-negative integer $s\geqslant\max\{[\alpha(0)-n\delta_1],[\alpha_\infty-n\delta_1]\}$ and $\varepsilon>\max\{s/n,\alpha(0)/n+\delta_1-1,\alpha_\infty/n+\delta_1-1\}$. Let $b=1-\delta_1+\varepsilon$. If l is a negative integer, let $\alpha_l=\alpha(0)$ and $a=1-\delta_1-\alpha(0)/n+\varepsilon$; if l is a non-negative integer, let $\alpha_l=\alpha_\infty$, and $a=1-\delta_1-\alpha_\infty/n+\varepsilon$.

(i) A function $M_l\in L^{p(\cdot)}$ with $l\in\mathbb{Z}$ is called a dyadic central $(\alpha(\cdot),p(\cdot);s,\varepsilon)_l$-molecule if it satisfies (i$_1$) $\|M_l\|_{L^{p(\cdot)}} \leqslant 2^{-l\alpha_l}$;

(i$_2$) $\mathscr{R}_{p(\cdot)}(M_l) := \|M_l\|_{L^{p(\cdot)}}^{a/b} \||\cdot|^{nb}M_l\|_{L^{p(\cdot)}}^{1-a/b} < \infty$;

(i$_3$) $\int_{\mathbb{R}^n} M_l(x)x^\beta \mathrm{d}x = 0$ for any multi index β with $|\beta| \leqslant s$.

(ii) When $l \in \mathbb{N}_0$, a dyadic central $(\alpha(\cdot), p(\cdot); s, \varepsilon)_l$-molecule M_l is also called of restricted type.

DEFINITION 2.4. Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathscr{P}_0^{\log}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{\log}(\mathbb{R}^n)$, and non-negative integer $s \geqslant [\alpha_r - n\delta_2]$, here $\alpha_r = \alpha(0)$, if $r < 1$, $\alpha_r = \alpha_\infty$, if $r \geqslant 1$, $n\delta_2 \leqslant \alpha_r < \infty$ and δ_2 as in Lemma 2.1.

(i) A function a on $\mathbb{R}^n$ is called a central $(\alpha(\cdot), p(\cdot))$-atom, if it satisfies: (1) supp $a \subset B(0, 2^r)$; (2) $\|a\|_{L^{p(\cdot)}} \leqslant |B(0, 2^r)|^{-\alpha_r/n}$; (3) $\int_{\mathbb{R}^n} a(x)x^\beta \mathrm{d}x = 0$, $|\beta| \leqslant s$.

(ii) A function a on $\mathbb{R}^n$ is called a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type, if it satisfies: (1) supp $a \subset B(0, 2^r)$, $r \geqslant 1$; (2) $\|a\|_{L^{p(\cdot)}} \leqslant |B(0, 2^r)|^{-\alpha_r/n}$; (3) $\int_{\mathbb{R}^n} a(x)x^\beta \mathrm{d}x = 0$, $|\beta| \leqslant s$.

Next lemma shows that an atom is a molecule.

LEMMA 2.4. *Let* $0<q<\infty$, $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathscr{P}_0^{\log}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{\log}(\mathbb{R}^n)$ *such that* $\max\{n\delta_1, n\delta_2\} \leqslant \alpha(0), \alpha_\infty < \infty$, *where* δ_1, δ_2 *as in Lemma* 2.1. *Define a non-negative integer* $s \geqslant [\max\{\alpha(0), \alpha_\infty\} - \min\{n\delta_1, n\delta_2\}]$, $\varepsilon > \max\{s/n, \alpha(0)/n + \delta_1 - 1, \alpha_\infty/n + \delta_1 - 1\}$, $a = 1 - \delta_1 - \alpha(0)/n + \varepsilon$ *or* $a = 1 - \delta_1 - \alpha_\infty/n + \varepsilon$ *and* $b = 1 - \delta_1 + \varepsilon$.

(i) *If* M *is a central* $(\alpha(\cdot), p(\cdot))$*-atom, then* M *is a central* $(\alpha(\cdot), p(\cdot); s, \varepsilon)$*-molecule such that* $\mathscr{R}_{p(\cdot)}(M) \leqslant C$ *with* C *independent of* M.

(ii) *If* M *is a central* $(\alpha(\cdot), p(\cdot))$*-atom of restricted type, then* M *is a central* $(\alpha(\cdot), p(\cdot); s, \varepsilon)$*-molecule of restricted type such that* $\mathscr{R}_{p(\cdot)}(M) \leqslant C$ *with* C *independent of* M.

Proof. Let M is a $(\alpha(\cdot), p(\cdot))$-atom with support on a ball $B(0, r)$, then we get

$$\|M\|_{L^{p(\cdot)}}^{a/b} \||\cdot|^{nb}M\|_{L^{p(\cdot)}}^{1-a/b} \leqslant r^{nb(1-a/b)} \|M\|_{L^{p(\cdot)}} \lesssim r^{\alpha_r} r^{-\alpha_r} \lesssim 1.$$

Then the results follow from the definition of atoms. $\square$

Now there is a position to state the molecular decompositions of Herz-Morrey-Hardy spaces with variable exponents.

THEOREM 2.1. *Let* $0 < q < \infty$, $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathscr{P}_0^{\log}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{\log}(\mathbb{R}^n)$ *such that* $\max\{n\delta_1, n\delta_2\} \leqslant \alpha(0), \alpha_\infty < \infty$, *where* δ_1, δ_2 *as in Lemma* 2.1. $0 \leqslant \lambda \leqslant \frac{1}{2}\min\{\alpha(0), \alpha_\infty\}$. *Define a non-negative integer* $s \geqslant [\max\{\alpha(0), \alpha_\infty\} - \min\{n\delta_1, n\delta_2\}]$. *Suppose* $\varepsilon > \max\{s/n, \alpha(0)/n + \delta_1 - 1, \alpha_\infty/n + \delta_1 - 1\}$.

(i) $f \in HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ *if and only if* f *can be represented as* $f = \sum_{k=-\infty}^{\infty} \lambda_k M_k$ *in the sense of* $\mathscr{S}'(\mathbb{R}^n)$, *where each* M_k *is a dyadic central* $(\alpha(\cdot), p(\cdot); s, \varepsilon)_k$*-molecule and*

$\mathscr{R}_{p(\cdot)}(M_k)$ are uniformly bounded, and $\sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum\limits_{k=-\infty}^{L} |\lambda_k|^q < \infty$. Moreover

$$\|f\|_{H M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \approx \inf \left(\sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} |\lambda_k|^q \right)^{1/q},$$

where the infimum is taken over all above decompositions of f.

(ii) *$f \in HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ if and only if $f = \sum_{k=0}^{\infty} \lambda_k M_k$ in the sense of $\mathscr{S}'(\mathbb{R}^n)$, where each M_k is a dyadic central $(\alpha(\cdot), p(\cdot); s, \varepsilon)_k$-molecule of restricted type and $\mathscr{R}_{p(\cdot)}(M_k)$ are uniformly bounded, and $\sup_{L\in\mathbb{N}_0} 2^{-L\lambda q} \sum\limits_{k=0}^{L} |\lambda_k|^q < \infty$. Moreover*

$$\|f\|_{HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \approx \inf \left(\sup_{L\in\mathbb{N}_0} 2^{-L\lambda q} \sum_{k=0}^{L} |\lambda_k|^q \right)^{1/q},$$

where the infimum is taken over all above decompositions of f.

To prove the theorem, we need the following decomposition of molecules, which is a adjustment from classical setting in [16].

LEMMA 2.5. *For any dyadic central $(\alpha(\cdot), p(\cdot); s, \varepsilon)_l$-molecule $M_l \in L^{p(\cdot)}$, $l \in \mathbb{Z}$, there is a decomposition*

$$M_l = \sum_{i=l}^{\infty} g_{li} + \sum_{i=l}^{\infty} Q_{li}$$

such that (a) *for $i \geqslant l$, $\int_{\mathbb{R}^n} g_{li}(x) x^\beta \mathrm{d}x = 0$ for each $|\beta| \leqslant s$ and $\|g_{li}\|_{L^{p(\cdot)}} \leqslant C 2^{-inb} 2^{lna}$ where the constant C independent of M_l;*

(b) *$\sum_{i=l}^{\infty} Q_{li} = C \sum_{|j|\leqslant s} \sum_{i=l}^{\infty} a^l_{ji}$, where each a^l_{ji} is a central $(\alpha(\cdot), p(\cdot); s, \varepsilon)_k$-atom.*

Proof. For $l \in \mathbb{Z}$, without loss of generality, let

$$\|M_l\|_{L^{p(\cdot)}} = 2^{-l\alpha_l}.$$

Let $E_{ll} := \{x : |x| \leqslant 2^l\}$. For $i > l$, let $E_{li} := \{x : 2^{i-1} < |x| \leqslant 2^i\}$. We denote by χ_{li} the characteristic function of E_{li}. So we have

$$M_l(x) = \sum_{i=l}^{\infty} M_l(x) \chi_{li}(x).$$

We denote by $\mathscr{P}_s$ the class of all real polynomials of degree no more than s. Let $M_{li} := M_l \chi_{li}$ and let $P_{E_{li}} M_{li} \in \mathscr{P}_s$ be the unique polynomial satisfying

$$\int_{E_{li}} (M_{li}(x) - P_{E_{li}} M_{li}(x)) x^\beta \mathrm{d}x = 0, \ |\beta| \leqslant s. \tag{1}$$

Let $Q_{li} := (P_{E_{li}} M_{li})\chi_{li}$. Denote $g_{li} := M_{li} - Q_{li}$, then we have

$$M_l = \sum_{i=l}^{\infty} g_{li} + \sum_{i=l}^{\infty} Q_{li}.$$

Now we first show (a). Equation (1) means $\int_{\mathbb{R}^n} g_{li}(x) x^{\beta} \mathrm{d}x = 0$ for $i \geqslant l$. We still to show the size estimates for g_{li}. Without loss of generality, we can suppose that $\mathscr{R}_{p(\cdot)}(M_l) = 1$, which implies that

$$\left\| |\cdot|^{nb} M_l \right\|_{L^{p(\cdot)}} = \|M_l\|_{L^{p(\cdot)}}^{-a/(b-a)} = 2^{lna},$$

where $a = 1 - \delta_1 - \alpha_l/n + \varepsilon$ and $b = 1 - \delta_1 + \varepsilon$. Then there exists a positive constant C independent of M_l such that for $i \geqslant l$,

$$\|M_{li}\|_{L^{p(\cdot)}} \leqslant C 2^{-inb} 2^{lna}$$

Since if $i > l$,

$$\|M_{li}\|_{L^{p(\cdot)}} \lesssim \| |\cdot|^{nb} M_l \|_{L^{p(\cdot)}} |2^i|^{-nb} \lesssim 2^{-inb} 2^{lna};$$

if $i = l$,

$$\|M_{ll}\|_{L^{p(\cdot)}} \leqslant \|M_l\|_{L^{p(\cdot)}} = 2^{-l\alpha_l} = 2^{-lnb} 2^{lna}.$$

Choose $\{\varphi_j^{li} : |j| \leqslant s\} \subset \mathscr{P}_s$ such that

$$\langle \varphi_\mu^{li}, \varphi_\nu^{li} \rangle_{E_{li}} = \frac{1}{|E_{li}|} \int_{E_{li}} \varphi_\mu^{li}(x) \varphi_\nu^{li}(x) \mathrm{d}x = \delta_{\mu\nu}.$$

Then

$$Q_{li}(x) = \sum_{|\beta| \leqslant s} \langle M_{li}, \varphi_\beta^{li} \rangle_{E_{li}} \varphi_\beta^{li}(x), \text{ if } x \in E_{li}. \tag{2}$$

By scaling we have

$$|Q_{li}(x)| \lesssim \frac{1}{|E_{li}|} \int_{E_{li}} |M_{li}(x)| \mathrm{d}x.$$

Thus for any $i \geqslant l$, by Lemmas 2.1 and 2.2 we have

$$\begin{aligned} \|g_{li}\|_{L^{p(\cdot)}} &\leqslant \|M_{li}\|_{L^{p(\cdot)}} + \|Q_{li}\|_{L^{p(\cdot)}} \\ &\lesssim \|M_{li}\|_{L^{p(\cdot)}} + \frac{1}{|E_{li}|} \|M_{li}\|_{L^{p(\cdot)}} \|\chi_{E_{li}}\|_{L^{p'(\cdot)}} \|\chi_{E_{li}}\|_{L^{p(\cdot)}} \\ &\lesssim \|M_{li}\|_{L^{p(\cdot)}} \\ &\lesssim 2^{-inb} 2^{lna}. \end{aligned}$$

Next we turnto show (b). Let $\{\psi_j^{li} : |j| \leqslant s\} \subset \mathscr{P}_s$ be the dual basis of $\{x^\gamma : |\gamma| \leqslant s\}$ with respect to the weight $\frac{1}{|E_{li}|}$ on E_{li}, that is

$$\langle \psi_j^{li}, x^\gamma \rangle = \frac{1}{|E_{li}|} \int_{E_{li}} \psi_j^{li}(x) x^\gamma \mathrm{d}x = \delta_{j\gamma}.$$

If set $\varphi_j^{li}(x) := \sum_{|\nu|\leqslant s} \beta_{\nu j}^{li} x^\nu$ and $\psi_j^{li}(x) := \sum_{|\nu|\leqslant s} \tau_{\nu j}^{li} \varphi_\nu^{li}(x),$ then we have

$$\tau_{\nu j}^{li} = \langle \psi_j^{li}, \varphi_\nu^{li} \rangle = \sum_{|\gamma|\leqslant s} \beta_{\nu\gamma}^{li} \langle \psi_j^{li}, x^\gamma \rangle = \sum_{|\gamma|\leqslant s} \beta_{\nu\gamma}^{li} \delta_{j\gamma} = \beta_{\nu j}^{li}.$$

So $\psi_j^{li}(x) = \sum_{|\nu|\leqslant s} \beta_{\nu j}^{li} \varphi_\nu^{li}(x).$ For any $x \in E_{li},$ we have

$$\langle M_{li}, \varphi_j^{li} \rangle_{E_{li}} \varphi_j^{li}(x) = \langle M_{li}, \sum_{|\nu|\leqslant s} \beta_{\nu j}^{li} x^\nu \rangle_{E_{li}} \varphi_j^{li}(x) = \sum_{|\nu|\leqslant s} \langle M_{li}, x^\nu \rangle_{E_{li}} \beta_{\nu j}^{li} \varphi_j^{li}(x),$$

which together with (2) implies that

$$Q_{li}(x) = \sum_{|j|\leqslant s} \langle M_{li}, x^j \rangle_{E_{li}} \psi_j^{li}(x), \text{ if } x \in E_{li}. \tag{3}$$

We denote $E := \{x \in \mathbb{R}^n : 1 \leqslant |x| \leqslant 2\},$ $F := \{x \in \mathbb{R}^n : |x| \leqslant 1\},$ $\{e_j : |j| \leqslant s\} \subset \mathscr{P}_s(\mathbb{R}^n)$ satisfying $\frac{1}{|E|}\int_E e_j(x) x^\gamma \mathrm{d}x = \delta_{j\gamma}$, and $\{\tilde{e}_j : |j| \leqslant s\} \subset \mathscr{P}_s(\mathbb{R}^n)$ satisfying $\frac{1}{|F|}\int_F \tilde{e}_j(x) x^\gamma \mathrm{d}x = \delta_{j\gamma}$. Noting that if $i > l,$

$$\delta_{j\gamma} = \frac{1}{|E_{li}|}\int_{E_{li}} \psi_j^{li}(x) x^\gamma \mathrm{d}x = \frac{1}{|E|}\int_E (2^{i-1})^{|\gamma|} \psi_j^{li}(2^{i-1}y) y^\gamma \mathrm{d}y,$$

we get $e_j(y) = (2^{i-1})^{|j|} \psi_j^{li}(2^{i-1}y).$ This in turn leads to that for $i > l,$

$$\psi_j^{li}(x) = (2^{i-1})^{-|j|} e_j\left(\frac{x}{2^{i-1}}\right), \ \ x \in E_{li}.$$

By the similar way, we have

$$\psi_j^{ll}(x) = (2^{l-1})^{-|j|} \tilde{e}_j(x), \ \ x \in F.$$

By taking $C := \sup_{j:|j|\leqslant s} \{\|e_j\|_{L^\infty(\mathbb{R}^n)}, \|\tilde{e}_j\|_{L^\infty(\mathbb{R}^n)}\}$ we have

$$|\psi_j^{li}(x)| \leqslant C(2^{i-1})^{-|j|}, \ \text{ for } \ i \geqslant l. \tag{4}$$

Now we can conclude (b). Let

$$N_j^{li} := \sum_{d=i}^{\infty} |E_{ld}| \langle M_{ld}, x^j \rangle_{E_{ld}}, \ \ i \geqslant l.$$

It is easy to see that

$$N_j^{ll} = \sum_{d=l}^{\infty} |E_{ld}| \langle M_{ld}, x^j \rangle_{E_{ld}} = \sum_{d=l}^{\infty} \int_{E_{ld}} M_l(x) x^j \mathrm{d}x = \int_{\mathbb{R}^n} M_l(x) x^j \mathrm{d}x = 0.$$

For any $l \in \mathbb{Z}$, and measurable subset E_{ld} we have

$$\min\{|E_{ld}|^{\frac{1}{p^+}}, |E_{ld}|^{\frac{1}{p^-}}\} \leqslant \|\chi_{E_{ld}}\|_{L^{p(\cdot)}} \leqslant \max\{|E_{ld}|^{\frac{1}{p^+}}, |E_{ld}|^{\frac{1}{p^-}}\}.$$

By Lemmas 2.1 and 2.2, we have for $i > l$

$$
\begin{aligned}
|N_j^{li}| &\leqslant \sum_{d=i}^{\infty} \int_{E_{ld}} |M_{ld}(x)x^j| \mathrm{d}x \\
&\lesssim \sum_{d=i}^{\infty} \| M_{ld} | \cdot |^j \|_{L^{p(\cdot)}} \| \chi_{E_{ld}} \|_{L^{p'(\cdot)}} \\
&\lesssim \sum_{d=i}^{\infty} \| M_{ld} | \cdot |^j \|_{L^{p(\cdot)}} |E_{ld}| \frac{1}{\| \chi_{E_{ld}} \|_{L^{p(\cdot)}}} \\
&\lesssim \sum_{d=i}^{\infty} (2^d)^{|j|-nb} \| | \cdot |^{nb} M_l \|_{L^{p(\cdot)}} |E_{ld}| \max\{2^{-nd\frac{1}{p^+}}, 2^{-nd\frac{1}{p^-}}\} \\
&\lesssim \sum_{d=i}^{\infty} 2^{lan} \max\left\{2^{d(|j|-nb+n-\frac{n}{p^-})}, 2^{d(|j|-nb+n-\frac{n}{p^+})}\right\} \\
&= \sum_{d=i}^{\infty} 2^{lan} \max\left\{2^{d(|j|-n\varepsilon+n\delta_1-\frac{n}{p^-})}, 2^{d(|j|-n\varepsilon+n\delta_1-\frac{n}{p^+})}\right\}.
\end{aligned}
$$

By (4) we have

$$
\begin{aligned}
|E_{li}|^{-1} |N_j^{li} \psi_j^{li}(x) \chi_{li}(x)| &\lesssim \sum_{d=i}^{\infty} 2^{lan} 2^{-dn(\varepsilon+1+\frac{1}{p^+}-\delta_1)} \\
&\leqslant \sum_{d=i}^{\infty} 2^{dn(a-\varepsilon-1-\frac{1}{p^+}+\delta_1)} \\
&= \sum_{d=i}^{\infty} 2^{dn(-\alpha_l/n-\frac{1}{p^+})} \\
&\lesssim 2^{in(-\alpha_l/n-\frac{1}{p^+})}.
\end{aligned}
$$

When $i > 0$ is sufficiently large

$$
2^{in(-\alpha_l/n-\frac{1}{p^+})} \to 0, \quad i \to \infty. \tag{5}
$$

Using Abel's transform, from (3) and (5) we get

$$
\begin{aligned}
\sum_{i=l}^{\infty} Q_{li}(x) &= \sum_{i=l}^{\infty} \sum_{|j| \leqslant s} \langle M_i, x^j \rangle_{E_{li}} \psi_j^{li}(x) \\
&= \sum_{|j| \leqslant s} \sum_{i=l}^{\infty} \left(\sum_{d=i}^{\infty} |E_{ld}| \langle M_d, x^j \rangle_{E_{li}} \right) \\
&\quad \times \left\{ |E_{li}|^{-1} \psi_j^{li}(x) \chi_{li}(x) - |E_{l(i+1)}|^{-1} \psi_j^{l(i+1)}(x) \chi_{l(i+1)}(x) \right\} \\
&= \sum_{|j| \leqslant s} \sum_{i=l}^{\infty} (-N_j^{l(i+1)}) \left\{ |E_{li}|^{-1} \psi_j^{li}(x) \chi_{li}(x) - |E_{l(i+1)}|^{-1} \psi_j^{l(i+1)}(x) \chi_{l(i+1)}(x) \right\}.
\end{aligned}
$$

Let $a_{ji}^l = (-N_j^{l(i+1)})\left\{|E_{li}|^{-1}\psi_j^{li}(x)\chi_{li}(x) - |E_{l(i+1)}|^{-1}\psi_j^{l(i+1)}(x)\chi_{l(i+1)}(x)\right\}$ and by (5)

$$\begin{aligned}\|a_{ji}^l\|_{L^{p(\cdot)}} &= \|(-N_j^{l(i+1)})\left\{|E_{li}|^{-1}\psi_j^{li}\chi_{li} - |E_{l(i+1)}|^{-1}\psi_j^{l(i+1)}\chi_{l(i+1)}\right\}\|_{L^{p(\cdot)}} \\ &\lesssim |N_j^{l(i+1)}|\left(|E_{li}|^{-1}\|\psi_j^{li}\|_\infty\|\chi_{li}\|_{L^{p(\cdot)}} + |E_{l(i+1)}|^{-1}\|\psi_j^{l(i+1)}\|_\infty\left\|\chi_{l(i+1)}\right\|_{L^{p(\cdot)}}\right) \\ &\lesssim 2^{-i|j|}2^{-in}|N_j^{l(i+1)}|\|\chi_{B_{(i+1)}}\|_{L^{p(\cdot)}}.\end{aligned}$$

But by Lemmas 2.1 and 2.2, we have

$$\begin{aligned}|N_j^{l(i+1)}|\|\chi_{B_{(i+1)}}\|_{L^{p(\cdot)}} &\leqslant \sum_{d=i+1}^{\infty}\int_{E_{ld}}|M_{ld}(x)x^j|\mathrm{d}x\|\chi_{B_{(i+1)}}\|_{L^{p(\cdot)}} \\ &\lesssim \sum_{d=i+1}^{\infty}\|M_{ld}|\cdot|^j\|_{L^{p(\cdot)}}\|\chi_{E_{ld}}\|_{L^{p'(\cdot)}}\|\chi_{B_{(i+1)}}\|_{L^{p(\cdot)}} \\ &\lesssim \sum_{d=i+1}^{\infty}\|M_{ld}|\cdot|^j\|_{L^{p(\cdot)}}\|\chi_{B_d}\|_{L^{p'(\cdot)}}\|\chi_{B_{(i+1)}}\|_{L^{p(\cdot)}} \\ &\lesssim \sum_{d=i+1}^{\infty}(2^d)^{|j|-nb}\||\cdot|^{nb}M_l\|_{L^{p(\cdot)}}|B_d|\left(\frac{|B_{i+1}|}{|B_d|}\right)^{\delta_1} \\ &\lesssim \sum_{d=i+1}^{\infty}2^{d(|j|-n\varepsilon)}2^{lna}2^{n(i+1)\delta_1} \\ &\lesssim 2^{i(|j|-n\varepsilon)}2^{in\delta_1}2^{lna}.\end{aligned}$$

Thus we obtain

$$\|a_{ji}^l\|_{L^{p(\cdot)}} \lesssim 2^{-i|j|}2^{-in}2^{i(|j|-n\varepsilon)}2^{in\delta_1}2^{lna} \lesssim 2^{-inb}2^{lna}. \tag{6}$$

Also we have

$$\|a_{ji}^l\|_{L^{p(\cdot)}} \lesssim 2^{-inb}2^{lna} \lesssim 2^{-inb+ina} = 2^{-i\alpha_l}.$$

By the preceding definition we get

$$|E_{l(i+1)}|^{-1}\int_{\mathbb{R}^n}\psi_j^{i+1}(x)\chi_{l(i+1)}(x)x^j\mathrm{d}x = |E_{li}|^{-1}\int_{\mathbb{R}^n}\psi_j^i(x)\chi_{li}(x)x^j\mathrm{d}x.$$

Hence,

$$\begin{aligned}&\int_{\mathbb{R}^n}a_{ji}^l(x)x^j\mathrm{d}x \\ &= \int_{\mathbb{R}^n}(-N_j^{i+1})\left\{|E_{li}|^{-1}\psi_j^i(x)\chi_{li}(x) - |E_{l(i+1)}|^{-1}\psi_j^{i+1}(x)\chi_{l(i+1)}(x)\right\}x^j\mathrm{d}x \\ &= (-N_j^{i+1})\left(|E_{li}|^{-1}\int_{\mathbb{R}^n}\psi_j^i(x)\chi_{li}(x)x^j\mathrm{d}x - |E_{l(i+1)}|^{-1}\int_{\mathbb{R}^n}\psi_j^{i+1}(x)\chi_{l(i+1)}(x)x^j\mathrm{d}x\right) \\ &= 0.\end{aligned}$$

This finishes the proof. □

Proof of Theorem 2.1. We shall show (i). The proof of (ii) is similar and simpler. The necessity follows from Lemma 2.4 and Theorem 13 in [33]. Thus we only need to prove the sufficiency. Suppose $f=\sum_{k=-\infty}^{\infty}\lambda_k M_k$ in the sense of $\mathscr{S}'(\mathbb{R}^n)$, where each M_k is a dyadic central $(\alpha(\cdot),p(\cdot);s,\varepsilon)_k$-molecule and $\mathscr{R}_{p(\cdot)}(M_k)$ are uniformly bounded, and $\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}|\lambda_k|^q<\infty$. We need to estimate the norm of $G_N f$ in space $M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$. For convenience, we denote $\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{i=-\infty}^{L}|\lambda_i|^q=\Lambda$. By Lemma 2.3 we have

$$
\begin{aligned}
\|G_N f\|^q_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \approx \max\Bigg\{&\sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\|(G_N f)\chi_k\|^q_{L^{p(\cdot)}},\\
&\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\left(\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\|(G_N f)\chi_k\|^q_{L^{p(\cdot)}}\sum_{k=0}^{L}2^{kq\alpha(\infty)}\|(G_N f)\chi_k\|^q_{L^{p(\cdot)}}\right)\Bigg\}\\
&\lesssim \max\{I,II+III\},
\end{aligned}
$$

where

$$
\begin{aligned}
I &:= \sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\|(G_N f)\chi_k\|^q_{L^{p(\cdot)}}\\
II &:= \sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\|(G_N f)\chi_k\|^q_{L^{p(\cdot)}}\\
III &:= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha(\infty)}\|(G_N f)\chi_k\|^q_{L^{p(\cdot)}}.
\end{aligned}
$$

To complete the proof, it suffices to show that $I,II,III\lesssim\Lambda$. To do so, we estimate I,II,III step by step. In the summation of $f=\sum_{l=-\infty}^{\infty}\lambda_l M_l$, we consider it into two parts. For the smaller indices l, we shall use the boundedness of the grand maximal operator G_N acting on M_l in the variable exponent Lebesgue space, for the larger indices l, we shall use Lemma 2.5, the decomposition of M_l. Therefore, we have

$$
\begin{aligned}
I &= \sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\|(G_N f)\chi_k\|^q_{L^{p(\cdot)}}\\
&\lesssim \sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{l=k-1}^{\infty}|\lambda_l|\|G_N M_l\chi_k\|_{L^{p(\cdot)}}\right)^q\\
&\quad+\sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-2}|\lambda_l|\|G_N M_l\chi_k\|_{L^{p(\cdot)}}\right)^q\\
&\lesssim \sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{l=k-1}^{\infty}|\lambda_l|\|M_l\|_{L^{p(\cdot)}}\right)^q\\
&\quad+\sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-2}|\lambda_l|\sum_{i=l}^{\infty}\|G_N g_{li}\chi_k\|_{L^{p(\cdot)}}\right)^q
\end{aligned}
$$

$$+\sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-2}|\lambda_l|\|G_N\left(\sum_{i=l}^{\infty}Q_{li}\right)\chi_k\|_{L^{p(\cdot)}}\right)^q$$

$$:= I_1+I_2+I_3.$$

$$I_1 := \sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{kq\alpha(0)}\left(\sum_{l=k-1}^{\infty}|\lambda_l|\|M_l\|_{L^{p(\cdot)}}\right)^q.$$

$$I_2 := \sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-2}|\lambda_l|\sum_{i=l}^{\infty}\|G_N g_{li}\chi_k\|_{L^{p(\cdot)}}\right)^q$$

$$\lesssim \sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-2}\sum_{i=l}^{k-2}|\lambda_l|\|G_N g_{li}\chi_k\|_{L^{p(\cdot)}}\right)^q$$

$$+\sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-2}\sum_{i=k-1}^{\infty}|\lambda_l|\|g_{li}\|_{L^{p(\cdot)}}\right)^q$$

$$:= I_{21}+I_{22}.$$

$$I_3 \lesssim \sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-2}|\lambda_l|\|G_N\left(\sum_{|j|\leqslant s}\sum_{i=l}^{\infty}a_{ji}^l\right)\chi_k\|_{L^{p(\cdot)}}\right)^q$$

$$\lesssim \sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-2}|\lambda_l|\sum_{|j|\leqslant s}\sum_{i=k-1}^{\infty}\|a_{ji}^l\|_{L^{p(\cdot)}}\right)^q$$

$$+\sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-2}|\lambda_l|\sum_{|j|\leqslant s}\sum_{i=l}^{k-2}\|(G_N a_{ji}^l)\chi_k\|_{L^{p(\cdot)}}\right)^q$$

$$:= I_{31}+I_{32}.$$

$$II = \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)}\|(G_N f)\chi_k\|_{L^{p(\cdot)}}^q$$

$$\lesssim \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)}\left(\sum_{l=k-1}^{\infty}|\lambda_l|\|M_l\|_{L^{p(\cdot)}}\right)^q$$

$$+\sum_{k=-\infty}^{-1} 2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-2}|\lambda_l|\|G_N M_l\chi_k\|_{L^{p(\cdot)}}\right)^q$$

$$\lesssim \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)}\left(\sum_{l=k-1}^{\infty}|\lambda_l|\|M_l\|_{L^{p(\cdot)}}\right)^q$$

$$+\sum_{k=-\infty}^{-1} 2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-2}|\lambda_l|\sum_{i=l}^{\infty}\|G_N g_{li}\chi_k\|_{L^{p(\cdot)}}\right)^q$$

$$+\sum_{k=-\infty}^{-1} 2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-2}|\lambda_l|\|G_N\left(\sum_{i=l}^{\infty}Q_{li}\right)\chi_k\|_{L^{p(\cdot)}}\right)^q$$

$$:= II_1+II_2+II_3.$$

$$II_1 = \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=k-1}^{\infty} |\lambda_l| \|M_l\|_{L^{p(\cdot)}} \right)^q .$$

$$\begin{aligned}
II_2 &= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-2} |\lambda_l| \sum_{i=l}^{\infty} \|G_N g_{li} \chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\leqslant \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-2} \sum_{i=l}^{k-2} |\lambda_l| \|G_N g_{li} \chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\quad + \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-2} \sum_{i=k-1}^{\infty} |\lambda_l| \|g_{li}\|_{L^{p(\cdot)}} \right)^q \\
&:= II_{21} + II_{22}.
\end{aligned}$$

$$\begin{aligned}
II_3 &\lesssim \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-2} |\lambda_l| \|G_N \left(\sum_{|j|\leqslant s} \sum_{i=l}^{\infty} a_{ji}^l \right) \chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\lesssim \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-2} |\lambda_l| \sum_{|j|\leqslant s} \sum_{i=k-1}^{\infty} \|a_{ji}^l\|_{L^{p(\cdot)}} \right)^q \\
&\quad + \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-2} |\lambda_l| \sum_{|j|\leqslant s} \sum_{i=l}^{k-2} \|(G_N a_{ji}^l) \chi_k\|_{L^{p(\cdot)}} \right)^q \\
&:= II_{31} + II_{32}.
\end{aligned}$$

$$\begin{aligned}
III &= \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \|(G_N f)\chi_k\|_{L^{p(\cdot)}}^q \\
&\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{l=k-1}^{\infty} |\lambda_l| \|G_N M_l \chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\quad + \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{l=-\infty}^{k-2} |\lambda_l| \|G_N M_l \chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{l=k-1}^{\infty} |\lambda_l| \|M_l \chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\quad + \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{l=-\infty}^{k-2} |\lambda_l| \sum_{i=l}^{\infty} \|G_N M_{li} \chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\quad + \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{l=-\infty}^{k-2} |\lambda_l| \|G_N \left(\sum_{i=l}^{\infty} Q_{li} \right) \chi_k\|_{L^{p(\cdot)}} \right)^q \\
&:= III_1 + III_2 + III_3.
\end{aligned}$$

$$III_1 := \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{l=k-1}^{\infty} |\lambda_l| \|M_l\|_{L^{p(\cdot)}} \right)^q .$$

$$
\begin{aligned}
III_2 &:= \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{l=-\infty}^{k-2} |\lambda_l| \sum_{i=l}^{\infty} \|G_N g_{li}\chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\leqslant \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{l=-\infty}^{k-2} \sum_{i=l}^{k-2} |\lambda_l| \|G_N g_{li}\chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\quad + \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{l=-\infty}^{k-2} \sum_{i=k-1}^{\infty} |\lambda_l| \|g_{li}\|_{L^{p(\cdot)}} \right)^q \\
&:= III_{21} + III_{22} \\
III_3 &\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{l=-\infty}^{k-2} |\lambda_l| \|G_N \left(\sum_{|j|\leqslant s} \sum_{i=l}^{\infty} a_{ji}^l \right) \chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{l=-\infty}^{k-2} |\lambda_l| \sum_{|j|\leqslant s} \sum_{i=k-1}^{\infty} \|a_{ji}^l\|_{L^{p(\cdot)}} \right)^q \\
&\quad + \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{l=-\infty}^{k-2} |\lambda_l| \sum_{|j|\leqslant s} \sum_{i=l}^{k-2} \|(G_N a_{ji}^l)\chi_k\|_{L^{p(\cdot)}} \right)^q \\
&:= III_{31} + III_{32}.
\end{aligned}
$$

To proceed, we need a pointwise estimate for $G_N g_{li}(x)$ on D_k. Let $\phi \in \mathscr{A}_N, s \in \mathbb{N}$ such that $s+1+n\delta_2 > nb$. Denote by P_s the s-th order Taylor series expansion of ϕ at y/t. If $|x-y| < t$, then from the vanishing moment condition of g_{li} we have

$$
\begin{aligned}
|g_{li}| * \phi_t(y)| &= t^{-n} \left| \int_{\mathbb{R}^n} g_{li}(z) \left(\phi\left(\frac{y-z}{t} \right) - P_s\left(-\frac{z}{t} \right) \right) \mathrm{d}z \right| \\
&\lesssim t^{-n} \int_{\mathbb{R}^n} |g_{li}(z)| \left| \frac{z}{t} \right|^{s+1} (1+|(y-\theta z)/t|)^{-(n+s+1)} \mathrm{d}z \\
&\lesssim \int_{\mathbb{R}^n} |g_{li}(z)||z|^{s+1} (t+|y-\theta z|)^{-(n+s+1)} \mathrm{d}z,
\end{aligned}
$$

where $0 < \theta < 1$. Since $x \in D_k$ for $k \in \mathbb{Z}$, we have $|x| \geqslant 2^{k-1}$. From $|x-y| < t$, $|z| < 2^i$, and $i \leqslant k-2$ we have

$$
t + |y-\theta z| \geqslant |x-y| + |y-\theta z| \geqslant |x| - |z| \geqslant \frac{1}{2}|x|.
$$

Thus,

$$
\begin{aligned}
|g_{li} * \phi_t(y)| &\lesssim \int_{\mathbb{R}^n} |g_{li}(z)||z|^{s+1} (|x-y| + |y-\theta z|)^{-(n+s+1)} \mathrm{d}z \\
&\lesssim 2^{i(s+1)} |x|^{-(n+s+1)} \int_{\mathbb{R}^n} |(g_{li})(z)| \mathrm{d}z \\
&\lesssim 2^{i(s+1)} |x|^{-(n+s+1)} 2^{-inb} 2^{lna} \|\chi_{B_i}\|_{L^{p'(\cdot)}}.
\end{aligned}
$$

Therefore, for $i \leqslant k-2$

$$
G_N(g_{li})(x) \lesssim 2^{i(s+1)-k(s+n+1)} 2^{-inb} 2^{lna} \|\chi_{B_i}\|_{L^{p'(\cdot)}}, \ x \in D_k, \text{ and } k \in \mathbb{Z}. \tag{7}
$$

We also need a pointwise estimate for $G_N a^l_{ji}(x)$ on D_k. As the argument as above, we have for $i \leqslant k-2$,

$$G_N a^l_{ji}(x) \lesssim 2^{i(s+1)-k(s+n+1)} 2^{-inb} 2^{lna} \|\chi_{B_i}\|_{L^{p'(\cdot)}}, \ x \in D_k, \text{ and } k \in \mathbb{Z}. \tag{8}$$

Since a^l_{ji} is similar to g_{li}, then $G_N a^l_{ji}$ is similar to $G_N g_{li}$. Therefore the estimation of I_3, II_3, and III_3 are similar to that of I_2, II_2 and III_2 respectively. So we omit the details for the estimation of I_3, II_3, and III_3. To complete the proof, we consider them into two cases $0 < q \leqslant 1$ and $1 < q < \infty$.

Case 1: $0 < q \leqslant 1$. In this case, we always use the inequality:

$$\left(\sum_{i=1}^{\infty} a_i\right)^q \leqslant \sum_{i=1}^{\infty} a_i^q \text{ for } a_i \geqslant 0, \ i \in \mathbb{N}, \tag{9}$$

and the convergence of a geometric series and exchange order of summation and the convergence of geometric power series. Using the condition (i_1) in Definition 2.3, we have

$$\begin{aligned}
I_1 &= \sup_{L \leqslant 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \left(\sum_{l=k-1}^{\infty} |\lambda_l| \|M_l\|_{L^{p(\cdot)}} \right)^q \\
&\lesssim \sup_{L \leqslant 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \left(\sum_{l=k-1}^{\infty} |\lambda_l| 2^{-\alpha_l l} \right)^q \\
&\lesssim \sup_{L \leqslant 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \left(\sum_{l=k-1}^{-1} |\lambda_l|^q 2^{-\alpha(0)lq} + \sum_{l=0}^{\infty} |\lambda_l|^q 2^{-\alpha_\infty lq} \right) \\
&\lesssim \sup_{L \leqslant 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \sum_{l=k-1}^{-1} |\lambda_l|^q 2^{\alpha(0)(k-l)q} \\
&\quad + \sup_{L \leqslant 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \sum_{l=0}^{\infty} |\lambda_l|^q 2^{-\alpha_\infty lq} \\
&\lesssim \sup_{L \leqslant 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \sum_{l=k-1}^{-1} 2^{-l\lambda q} |\lambda_l|^q 2^{l\lambda q} 2^{\alpha(0)(k-l)q} \\
&\quad + \sup_{L \leqslant 0, L \in \mathbb{Z}} \sum_{l=0}^{\infty} 2^{-l\lambda q} |\lambda_l|^q 2^{(\lambda-\alpha_\infty)lq} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \\
&\lesssim \Lambda \sup_{L \leqslant 0, L \in \mathbb{Z}} \sum_{k=-\infty}^{L} \sum_{l=k-1}^{-1} 2^{(l-L)\lambda q} 2^{\alpha(0)(k-l)q} \\
&\quad + \Lambda \sup_{L \leqslant 0, L \in \mathbb{Z}} \sum_{l=0}^{\infty} 2^{(\lambda-\alpha_\infty)lq} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq - L\lambda q} \\
&\lesssim \Lambda.
\end{aligned}$$

In the last inequality, we used the condition $\lambda < \alpha_\infty$.

By (8), Lemma 2.1 and the assumpton $s+1+n\delta_2>nb$ we have

$$
\begin{aligned}
I_{21} &\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{kq\alpha(0)}\sum_{l=-\infty}^{k-2}|\lambda_l|^q\sum_{i=l}^{k-2}\|(G_N g_{li})\chi_k\|_{L^{p(\cdot)}}^q\\
&\lesssim \sum_{|j|\leqslant s}\sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{l=-\infty}^{L}|\lambda_l|^q\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\sum_{i=l}^{k-2}2^{q(i-k)(s+1+n\delta_2)-iqnb+lqna}\\
&\lesssim \sum_{|j|\leqslant s}\sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q}\Lambda\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\sum_{i=l}^{k-2}2^{q(i-k)(s+1+n\delta_2)-iq\alpha(0)}\\
&\lesssim \Lambda\sum_{|j|\leqslant s}\sup_{L\leqslant 0, L\in\mathbb{Z}}\sum_{k=-\infty}^{L}\sum_{i=l}^{k-2}2^{q(i-k)(s+1+n\delta_2-\alpha(0))}\\
&\lesssim \Lambda.
\end{aligned}
$$

By (a) in Lemma 2.5, we obtain that

$$
\begin{aligned}
I_{22} &\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{kq\alpha(0)}\sum_{l=-\infty}^{k-2}|\lambda_l|^q\sum_{|j|\leqslant s}\sum_{i=k-1}^{\infty}\|g_{li}\|_{L^{p(\cdot)}}^q\\
&\lesssim \sum_{|j|\leqslant s}\sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{l=-\infty}^{L}|\lambda_l|^q\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\sum_{i=k-1}^{\infty}2^{-inbq}2^{lnaq}\\
&\lesssim \sum_{|j|\leqslant s}\sup_{L\leqslant 0, L\in\mathbb{Z}}\Lambda\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\sum_{i=k-1}^{\infty}2^{-iq\alpha(0)}\\
&\lesssim \Lambda\sum_{|j|\leqslant s}\sup_{L\leqslant 0, L\in\mathbb{Z}}\sum_{k=-\infty}^{L}\sum_{i=k-1}^{\infty}2^{(k-i)q\alpha(0)}\\
&\lesssim \Lambda.
\end{aligned}
$$

Then we estimate II. Using the condition (i_1) in Definition 2.3, we have

$$
\begin{aligned}
II_1 &= \sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left(\sum_{l=k}^{\infty}|\lambda_l|\|M_l\|_{L^{p(\cdot)}}\right)^q\\
&\lesssim \sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left(\sum_{l=k}^{\infty}|\lambda_l|2^{-l\alpha_l}\right)^q\\
&\lesssim \sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\left(\sum_{l=k}^{-1}|\lambda_l|^q 2^{-\alpha(0)lq}+\sum_{l=0}^{\infty}|\lambda_l|^q 2^{-\alpha_\infty lq}\right)\\
&\lesssim \sum_{k=-\infty}^{-1}\sum_{l=k}^{-1}|\lambda_l|^q 2^{\alpha(0)(k-l)q}+\sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\sum_{l=0}^{\infty}|\lambda_l|^q 2^{-\alpha_\infty lq}\\
&\lesssim \sum_{l=-\infty}^{-1}|\lambda_l|^q\sum_{k=-\infty}^{l}2^{\alpha(0)(k-l)q}+\sum_{l=0}^{\infty}|\lambda_l|^q 2^{-\alpha_\infty lq}\sum_{k=-\infty}^{-1}2^{\alpha(0)kq}
\end{aligned}
$$

$$\begin{aligned}
&\lesssim \sum_{l=-\infty}^{-1}|\lambda_l|^q+\sum_{l=0}^{\infty}2^{-l\lambda q}|\lambda_l|^q 2^{(\lambda-\alpha_\infty)lq}\\
&\lesssim \Lambda+\Lambda\sum_{l=0}^{\infty}2^{(\lambda-\alpha_\infty)lq}\\
&\lesssim \Lambda.
\end{aligned}$$

In the last inequality, we used the condition $\lambda<\alpha_\infty$.

By (8), Lemma 2.1 and the assumption $s+1+n\delta_2>nb$,

$$\begin{aligned}
II_{21}&\lesssim \sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-2}\sum_{i=l}^{k-2}|\lambda_l|2^{i(s+1)-k(s+n+1)}2^{-inb}2^{lna}\|\chi_{B_i}\|_{L^{p'(\cdot)}}\|\chi_k\|_{L^{p(\cdot)}}\right)^q\\
&\lesssim \sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\sum_{l=-\infty}^{k-2}|\lambda_l|^q\sum_{|j|\leqslant s}\sum_{i=l}^{k-2}2^{q(i-k)(s+1+n\delta_2)}2^{-iqnb}2^{lqna}\\
&\lesssim \sum_{|j|\leqslant s}\sum_{l=-\infty}^{-3}|\lambda_l|^q\sum_{k=-\infty}^{-1}\sum_{i=l}^{k-2}2^{q(i-k)(s+1+n\delta_2)-iq\alpha(0))}\\
&\lesssim \sum_{|j|\leqslant s}\sum_{l=-\infty}^{-3}|\lambda_l|^q\sum_{k=-\infty}^{-1}\sum_{i=l}^{k-2}2^{q(i-k)(s+1+n\delta_2-\alpha(0))}\\
&\lesssim \Lambda.
\end{aligned}$$

By (a) of Lemma 2.5,

$$\begin{aligned}
II_{22}&\lesssim \sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-2}\sum_{i=k-1}^{\infty}|\lambda_l|2^{-inb}2^{lna}\right)^q\\
&\lesssim \sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\sum_{l=-\infty}^{k-2}|\lambda_l|^q\sum_{|j|\leqslant s}\sum_{i=k-1}^{\infty}2^{-iqnb}2^{lqna}\\
&\lesssim \sum_{|j|\leqslant s}\sum_{l=-\infty}^{-3}|\lambda_l|^q\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\sum_{i=k-1}^{\infty}2^{-iq\alpha(0)}\\
&\lesssim \Lambda\sum_{|j|\leqslant s}\sum_{k=-\infty}^{-1}\sum_{i=k-1}^{0}2^{(k-i)q\alpha(0)}\\
&\lesssim \Lambda.
\end{aligned}$$

Now we turn to estimate III. Similar to I_1, we use the condition (i_1) in Definition 2.3 and obtain that

$$\begin{aligned}
III_1&=\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{l=k-1}^{\infty}|\lambda_l|\|M_l\|_{L^{p(\cdot)}}\right)^q\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\sum_{l=k-1}^{\infty}|\lambda_l|2^{-lq\alpha_\infty}q
\end{aligned}$$

$$\begin{aligned}
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}\sum_{l=k-1}^{\infty}|\lambda_l|^q 2^{(k-l)\alpha_\infty q}\\
&= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\left[\sum_{l=-1}^{L-1}|\lambda_l|^q\sum_{k=0}^{l+1}2^{(k-l)\alpha_\infty q}+\sum_{l=L-1}^{\infty}|\lambda_l|^q\sum_{k=0}^{L}2^{(k-l)\alpha_\infty q}\right]\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{l=-1}^{L}|\lambda_l|^q+\sup_{L>0,L\in\mathbb{Z}}\sum_{l=L-1}^{\infty}2^{(l\lambda q-L\lambda q)}2^{-l\lambda q}|\lambda_l|^q\sum_{k=0}^{L}2^{(k-l)\alpha_\infty q}\\
&\lesssim \Lambda+\Lambda\sup_{L>0,L\in\mathbb{Z}}\sum_{l=L-1}^{\infty}2^{(l-L)\lambda q}2^{(L-l)\alpha_\infty q}\\
&\lesssim \Lambda+\Lambda\sup_{L>0,L\in\mathbb{Z}}\sum_{l=L-1}^{\infty}2^{(l-L)q(\lambda-\alpha_\infty)}\\
&\lesssim \Lambda.
\end{aligned}$$

In the last inequality, we used the condition $\lambda<\alpha_\infty$.

By Lemma 1.1, (8) and the hypothesis $s+1+n\delta_2>nb$,

$$\begin{aligned}
III_{21}&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{l=-\infty}^{k-2}\sum_{i=l}^{k-2}|\lambda_l|2^{i(s+1)-k(s+n+1)}2^{-inb}2^{lna}\|\chi_{B_i}\|_{L^{p'(\cdot)}}\|\chi_k\|_{L^{p(\cdot)}}\right)^q\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{l=-\infty}^{k-2}\sum_{i=l}^{k-2}|\lambda_l|2^{(i-k)(s+1+n\delta_2)}2^{-inb}2^{lna}\right)^q\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\sum_{l=-\infty}^{k-2}\sum_{i=l}^{k-2}|\lambda_l|^q2^{(i-k)(s+1+n\delta_2)q}2^{-inbq}2^{lnaq}\\
&= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{l=-\infty}^{-2}|\lambda_l|^q2^{lnaq}\sum_{k=0}^{L}2^{-k(s+1+n\delta_2-\alpha_\infty)q}\sum_{i=l}^{k-2}2^{i(s+1+n\delta_2-nb)q}\\
&\quad+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{l=-1}^{L-2}|\lambda_l|^q2^{lnaq}\sum_{k=l+2}^{L}2^{-k(s+1+n\delta_2-\alpha_\infty)q}\sum_{i=l}^{k-2}2^{i(s+1+n\delta_2-nb)q}\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{l=-\infty}^{-2}|\lambda_l|^q2^{lnaq}\sum_{k=0}^{L}2^{-k(s+1+n\delta_2-\alpha_\infty)q}2^{k(s+1+n\delta_2-nb)q}\\
&\quad+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{l=-1}^{L-2}|\lambda_l|^q2^{lnaq}\sum_{k=l+2}^{L}2^{-k(s+1+n\delta_2-\alpha_\infty)q}2^{k(s+1+n\delta_2-nb)q}\\
&= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{l=-\infty}^{-2}|\lambda_l|^q2^{lnaq}\sum_{k=0}^{L}2^{k(\alpha_\infty-nb)q}\\
&\quad+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{l=-1}^{L-2}|\lambda_l|^q2^{lnaq}\sum_{k=l+2}^{L}2^{k(\alpha_\infty-nb)q}\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{l=-\infty}^{-2}|\lambda_l|^q2^{lnaq}+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{l=-1}^{L-2}|\lambda_l|^q2^{lnaq}2^{l(\alpha_\infty-nb)q}
\end{aligned}$$

$$\lesssim \sum_{l=-\infty}^{-2} |\lambda_l|^q + \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{l=-1}^{L-2} |\lambda_l|^q$$
$$\lesssim \Lambda.$$

By (a) of Lemma 2.5, we have

$$\begin{aligned}
III_{22} &\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \sum_{l=-\infty}^{k-2} \sum_{i=k-1}^{\infty} |\lambda_l|^q 2^{-inbq} 2^{lnaq} \\
&\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \sum_{l=-\infty}^{k-2} |\lambda_l|^q 2^{lnaq} 2^{-knbq} \\
&= \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{l=-\infty}^{-2} |\lambda_l|^q 2^{lnaq} \sum_{k=0}^{L} 2^{k(\alpha_\infty - nb)q} \\
&\quad + \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{l=-1}^{L-2} |\lambda_l|^q 2^{lnaq} \sum_{k=l+2}^{L} 2^{k(\alpha_\infty - nb)q} \\
&\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{l=-\infty}^{-2} |\lambda_l|^q 2^{lnaq} \\
&\quad + \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{l=-1}^{L-2} |\lambda_l|^q 2^{lnaq} 2^{l(\alpha_\infty - nb)q} \\
&\lesssim \sum_{l=-\infty}^{-2} |\lambda_l|^q + \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{l=-1}^{L-2} |\lambda_l|^q \\
&\lesssim \Lambda.
\end{aligned}$$

Case 2: $q > 1$. In this case, similar to Case 1, we always exchange order of summation and use the convergence of a geometric series, but use Hölder's inequality instead of the inequality (9). Denote by q' the conjugate exponent of q. Using the condition (i_1) in Definition 2.3 and Hölder's inequality, we get

$$\begin{aligned}
I_1 &= \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{l=k-1}^{\infty} |\lambda_l| \|M_l\|_{L^{p(\cdot)}} \right)^q \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{l=k-1}^{\infty} |\lambda_l| |B_l|^{-\alpha_l/n} \right)^q \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left(\sum_{l=k-1}^{-1} |\lambda_l| 2^{\alpha(0)(k-l)} \right)^q \\
&\quad + \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \left(\sum_{l=0}^{\infty} |\lambda_l| 2^{-\alpha_\infty l} \right)^q
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left(\sum_{l=k-1}^{-1} |\lambda_l|^q 2^{\alpha(0)(k-l)q/2}\right) \left(\sum_{l=k-1}^{-1} 2^{\alpha(0)(k-l)q'/2}\right)^{q/q'} \\
&\quad + \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \left(\sum_{l=0}^{\infty} |\lambda_l|^q 2^{-\alpha_\infty lq/2}\right) \left(\sum_{l=0}^{\infty} 2^{-\alpha_\infty lq'/2}\right)^{q/q'} \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \sum_{l=k-1}^{-1} |\lambda_l|^q 2^{\alpha(0)(k-l)q/2} \\
&\quad + \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \sum_{l=0}^{\infty} |\lambda_l|^q 2^{-\alpha_\infty lq/2} \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \left(\sum_{l=-\infty}^{L-1} |\lambda_l|^q \sum_{k=-\infty}^{l+1} 2^{\alpha(0)(k-l)q/2} + \sum_{l=L-1}^{-1} |\lambda_l|^q \sum_{k=-\infty}^{L} 2^{\alpha(0)(k-l)q/2}\right) \\
&\quad + \sup_{L\leqslant 0, L\in\mathbb{Z}} \sum_{l=0}^{\infty} 2^{-l\lambda q} |\lambda_l|^q 2^{(\lambda-\alpha_\infty/2)lq} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \\
&\leqslant \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{l=-\infty}^{L} |\lambda_l|^q + \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{l=L-1}^{-1} |\lambda_l|^q \sum_{k=-\infty}^{l} 2^{\alpha(0)(k-l)q/2} \\
&\quad + \Lambda \sup_{L\leqslant 0, L\in\mathbb{Z}} \sum_{l=0}^{\infty} 2^{(\lambda-\alpha_\infty/2)lq} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq-L\lambda q} \\
&\lesssim \Lambda + \sup_{L\leqslant 0, L\in\mathbb{Z}} \sum_{l=L-1}^{-1} 2^{-l\lambda q} |\lambda_l|^q 2^{(l-L)\lambda q} \sum_{k=-\infty}^{l} 2^{\alpha(0)(k-l)q/2} + \Lambda \\
&\lesssim \Lambda + \Lambda \sup_{L\leqslant 0, L\in\mathbb{Z}} \sum_{l=L-1}^{-1} 2^{(l-L)\lambda q} \sum_{k=-\infty}^{j} 2^{\alpha(0)(k-l)q/2} \\
&\lesssim \Lambda.
\end{aligned}
$$

In the last three inequality, we used the condition $2\lambda < \alpha_\infty$.

Then by (a) of Lemma 2.5, Hölder's inequality, (8) and the assumption $s+1+n\delta_2 > \alpha_k$, we obtain that

$$
\begin{aligned}
I_{21} &\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-2} \sum_{i=l}^{k-2} |\lambda_l| 2^{(i-k)(s+1+n\delta_2)} 2^{-inb} 2^{lna}\right)^q \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{i=-\infty}^{k-2} \sum_{l=-\infty}^{i} |\lambda_l| 2^{(i-k)(s+1+n\delta_2)} 2^{-inb} 2^{lna}\right)^q \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \sum_{i=-\infty}^{k-2} \sum_{l=-\infty}^{i} |\lambda_l|^q 2^{\frac{q}{2}(i-k)(s+1+n\delta_2)} 2^{-\frac{q}{2}inb} 2^{\frac{q}{2}lna} \\
&\quad \times \left(\sum_{i=-\infty}^{k-2} \sum_{l=-\infty}^{i} 2^{\frac{q'}{2}(i-k)(s+1+n\delta_2)} 2^{-\frac{q'}{2}inb} 2^{\frac{q'}{2}lna}\right)^{q/q'}
\end{aligned}
$$

$$\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} \sum_{l=-\infty}^{L} |\lambda_l|^q 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \sum_{i=-\infty}^{k-2} 2^{\frac{q}{2}(i-k)(s+1+n\delta_2)} 2^{-\frac{q}{2}i\alpha(0)}$$
$$\times \left(\sum_{i=-\infty}^{k-2} 2^{\frac{q'}{2}(i-k)(s+1+n\delta_2)} 2^{-\frac{q'}{2}i\alpha(0)} \right)^{q/q'}$$
$$\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} \Lambda \sum_{k=-\infty}^{L} \sum_{i=-\infty}^{k-2} 2^{\frac{q}{2}(i-k)(s+1+n\delta_2-\alpha(0))} \left(\sum_{i=-\infty}^{k-2} 2^{\frac{q'}{2}(i-k)(s+1+n\delta_2-\alpha(0))} \right)^{q/q'}$$
$$\lesssim \Lambda.$$

By by (a) of Lemma 2.5 and Hölder's inequality again, we have

$$I_{22} \lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{i=k-2}^{+\infty} \sum_{l=-\infty}^{k-2} |\lambda_l| 2^{-inb} 2^{lna} \right)^q$$
$$\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \sum_{i=k-2}^{+\infty} \sum_{l=-\infty}^{k-2} |\lambda_l|^q 2^{-\frac{q}{2}inb} 2^{\frac{q}{2}lna}$$
$$\times \left(\sum_{i=k-2}^{+\infty} \sum_{l=-\infty}^{k-2} 2^{-\frac{q'}{2}inb} 2^{\frac{q'}{2}lna} \right)^{q/q'}$$
$$\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{l=-\infty}^{L} |\lambda_l|^q \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \sum_{i=k-2}^{+\infty} 2^{-\frac{q}{2}i\alpha(0)} \left(\sum_{i=k-2}^{+\infty} 2^{-\frac{q'}{2}i\alpha(0)} \right)^{q/q'}$$
$$\lesssim \Lambda \sum_{k=-\infty}^{L} \sum_{i=k-2}^{+\infty} 2^{(k-i)\frac{q}{2}\alpha(0)} \left(\sum_{i=k-2}^{+\infty} 2^{(k-i)\frac{q'}{2}\alpha(0)} \right)^{q/q'}$$
$$\lesssim \Lambda.$$

Next we turn to estimate II. Using the condition (i_1) in Definition 2.3, we have

$$II_1 = \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=k}^{\infty} |\lambda_l| \|M_l\|_{L^{p(\cdot)}} \right)^q$$
$$\lesssim \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=k}^{\infty} |\lambda_l| 2^{-l\alpha_l} \right)^q$$
$$\lesssim \sum_{k=-\infty}^{-1} \left(\sum_{l=k}^{-1} |\lambda_l| 2^{\alpha(0)(k-l)} \right)^q + \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{l=0}^{\infty} |\lambda_l| 2^{-\alpha_\infty l} \right)^q$$
$$\lesssim \sum_{k=-\infty}^{-1} \left(\sum_{l=k}^{-1} |\lambda_l|^q 2^{\alpha(0)(k-l)q/2} \right) \left(\sum_{l=k}^{-1} 2^{\alpha(0)(k-l)q'/2} \right)^{q/q'}$$
$$+ \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{l=0}^{\infty} |\lambda_l|^q 2^{-\alpha_\infty lq/2} \right) \left(\sum_{l=0}^{\infty} 2^{-\alpha_\infty lq'/2} \right)^{q/q'}$$

$$\begin{aligned}
&\lesssim \sum_{k=-\infty}^{-1} |\lambda_l|^q \sum_{k=-\infty}^{l} 2^{\alpha(0)(k-l)q/2} + \sum_{l=0}^{\infty} |\lambda_l|^q 2^{-\alpha_\infty lq/2} \\
&\lesssim \sum_{l=-\infty}^{-1} |\lambda_l|^q + \sum_{l=0}^{\infty} 2^{(\lambda-\alpha_\infty/2)lq} 2^{-l\lambda q} \sum_{i=-\infty}^{l} |\lambda_l|^q \\
&\leqslant \Lambda + \Lambda \sum_{l=0}^{\infty} 2^{(\lambda-\alpha_\infty/2)lq} \\
&\lesssim \Lambda.
\end{aligned}$$

In the last inequality, we used the condition $2\lambda < \alpha_\infty$.

By (a) of Lemma 2.5 and the hypothesis that $s+1+n\delta_2 > \alpha_k$, we get

$$\begin{aligned}
II_{21} &\lesssim \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-2} \sum_{i=l}^{k-2} |\lambda_l| 2^{(i-k)(s+1+n\delta_2)} 2^{-inb} 2^{lna} \right)^q \\
&\lesssim \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{i=-\infty}^{k-2} \sum_{l=-\infty}^{i} |\lambda_l| 2^{(i-k)(s+1+n\delta_2)} 2^{-inb} 2^{lna} \right)^q \\
&\lesssim \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \sum_{i=-\infty}^{k-2} \sum_{l=-\infty}^{i} |\lambda_l|^q 2^{\frac{q}{2}(i-k)(s+1+n\delta_2)} 2^{-\frac{q}{2}inb} 2^{\frac{q}{2}lna} \\
&\quad \times \left(\sum_{i=-\infty}^{k-2} \sum_{l=-\infty}^{i} 2^{\frac{q'}{2}(i-k)(s+1+n\delta_2)} 2^{-\frac{q'}{2}inb} 2^{\frac{q'}{2}lna} \right)^{q/q'} \\
&\lesssim \sum_{l=-\infty}^{-3} |\lambda_l|^q \sum_{k=-\infty}^{-1} \sum_{i=-\infty}^{k-2} 2^{\frac{q}{2}(i-k)(s+1+n\delta_2-\alpha(0))} \\
&\quad \times \left(\sum_{i=-\infty}^{k-2} 2^{\frac{q'}{2}(i-k)(s+1+n\delta_2-\alpha(0))} \right)^{q/q'} \\
&\lesssim \Lambda.
\end{aligned}$$

By (a) of Lemma 2.5, we obtain

$$\begin{aligned}
II_{22} &\lesssim \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{i=k-1}^{+\infty} \sum_{l=-\infty}^{k-2} |\lambda_l| 2^{-inb} 2^{lna} \right)^q \\
&\lesssim \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \sum_{i=k-2}^{+\infty} \sum_{l=-\infty}^{k-2} |\lambda_l|^q 2^{-\frac{q}{2}inb} 2^{\frac{q}{2}lna} \left(\sum_{i=k-2}^{+\infty} \sum_{l=-\infty}^{k-2} 2^{-\frac{q'}{2}inb} 2^{\frac{q'}{2}lna} \right)^{q/q'} \\
&\lesssim \sum_{l=-\infty}^{-3} |\lambda_l|^q \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \sum_{i=k-2}^{\infty} 2^{-\frac{q}{2}i\alpha(0)} \left(\sum_{i=k-2}^{+\infty} 2^{-\frac{q'}{2}i\alpha(0)} \right)^{q/q'} \\
&\lesssim \Lambda \sum_{k=-\infty}^{-1} \sum_{i=k-2}^{\infty} 2^{(k-i)\frac{q}{2}\alpha(0)} \left(\sum_{i=k-2}^{+\infty} 2^{(k-i)\frac{q'}{2}\alpha(0)} \right)^{q/q'} \\
&\lesssim \Lambda.
\end{aligned}$$

Finally, we estimate III. Using the condition (i_1) in Definition 2.3, we have

$$
\begin{aligned}
III_1 &= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L} 2^{kq\alpha_\infty}\left(\sum_{l=k}^{\infty}|\lambda_l|\|M_l\|_{L^{p(\cdot)}}\right)^q \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L} 2^{kq\alpha_\infty}\left(\sum_{l=k}^{\infty}|\lambda_l|2^{-l\alpha_l}\right)^q \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L} 2^{kq\alpha_\infty}\left(\sum_{l=k}^{\infty}|\lambda_l|^q 2^{-l\alpha_\infty q/2}\right)\left(\sum_{l=k}^{\infty}2^{-l\alpha_\infty q'/2}\right)^{q/q'} \\
&= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}\left(\sum_{l=k}^{\infty}|\lambda_l|^q 2^{(k-l)\alpha_\infty q/2}\right)\left(\sum_{l=k}^{\infty}2^{(k-l)\alpha_\infty q'/2}\right)^{q/q'} \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}\sum_{l=k}^{\infty}|\lambda_l|^q 2^{(k-l)\alpha_\infty q/2} \\
&= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\left[\sum_{l=0}^{L}|\lambda_l|^q\sum_{k=0}^{l}2^{(k-l)\alpha_\infty q/2}+\sum_{l=L}^{\infty}|\lambda_l|^q\sum_{k=0}^{L}2^{(k-l)\alpha_\infty q/2}\right] \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{l=0}^{L}|\lambda_l|^q+\sup_{L>0,L\in\mathbb{Z}}\sum_{l=L}^{\infty}2^{(l\lambda q-L\lambda q)}2^{-l\lambda q}\sum_{i=-\infty}^{l}|\lambda_i|^q\sum_{k=0}^{L}2^{(k-l)\alpha_\infty q/2} \\
&\lesssim \Lambda+\Lambda\sup_{L>0,L\in\mathbb{Z}}\sum_{l=L}^{\infty}2^{(l-L)\lambda q}2^{(L-l)\alpha_\infty q/2} \\
&\lesssim \Lambda+\Lambda\sup_{L>0,L\in\mathbb{Z}}\sum_{l=L}^{\infty}2^{(l-L)q(\lambda-\alpha_\infty/2)} \\
&\lesssim \Lambda.
\end{aligned}
$$

In the last inequality, we used the condition $2\lambda<\alpha_\infty$.

By (a) of Lemma 2.5, (8) and the hypothesis that $s+1+n\delta_2>\alpha_k$,

$$
\begin{aligned}
III_{21} &\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{l=-\infty}^{k-2}\sum_{i=l}^{k-2}|\lambda_l|2^{i(s+1)-k(s+n+1)}2^{-inb}2^{lna}\|\chi_{B_i}\|_{L^{p'(\cdot)}}\|\chi_k\|_{L^{p(\cdot)}}\right)^q \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{i=-\infty}^{k-2}\sum_{l=-\infty}^{i}|\lambda_l|2^{i(s+1)-k(s+n+1)}2^{-inb}2^{lna}\|\chi_{B_i}\|_{L^{p'(\cdot)}}\|\chi_k\|_{L^{p(\cdot)}}\right)^q \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{i=-\infty}^{k-2}\sum_{l=-\infty}^{i}|\lambda_l|2^{(i-k)(s+1+n\delta_2)}2^{-inb}2^{lna}\right)^q \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\sum_{i=-\infty}^{k-2}\sum_{l=-\infty}^{i}|\lambda_l|^q 2^{\frac{q}{2}(i-k)(s+1+n\delta_2)}2^{-\frac{q}{2}inb}2^{\frac{q}{2}lna} \\
&\quad\times\left(\sum_{i=-\infty}^{k-2}\sum_{l=-\infty}^{i}2^{\frac{q'}{2}(i-k)(s+1+n\delta_2)}2^{-\frac{q'}{2}inb}2^{\frac{q'}{2}lna}\right)^{q/q'}
\end{aligned}
$$

$$\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\sum_{i=-\infty}^{k-2}\left(\sum_{l=-\infty}^{0}|\lambda_l|^q 2^{\frac{q}{2}(i-k)(s+1+n\delta_2)}2^{-\frac{q}{2}inb}\right.$$

$$\left.+\sum_{l=1}^{i}|\lambda_l|^q 2^{\frac{q}{2}(i-k)(s+1+n\delta_2)}2^{-\frac{q}{2}i\alpha_\infty}\right)$$

$$\times\left[\sum_{i=-\infty}^{k-2}\left(2^{\frac{q'}{2}(i-k)(s+1+n\delta_2)}2^{-\frac{q'}{2}inb}+2^{\frac{q'}{2}(i-k)(s+1+n\delta_2)}2^{-\frac{q'}{2}i\alpha_\infty}\right)\right]^{q/q'}$$

$$\lesssim \Lambda\sup_{L>0,L\in\mathbb{Z}}\sum_{k=0}^{L}2^{kq\alpha_\infty}\sum_{i=-\infty}^{k-2}\left(2^{\frac{q}{2}(i-k)(s+1+n\delta_2)}2^{-\frac{q}{2}inb}+2^{\frac{q}{2}(i-k)(s+1+n\delta_2)}2^{-\frac{q}{2}i\alpha_\infty}\right)$$

$$\times\left[\sum_{i=-\infty}^{k-2}\left(2^{\frac{q'}{2}(i-k)(s+1+n\delta_2)}2^{-\frac{q'}{2}inb}+2^{\frac{q'}{2}(i-k)(s+1+n\delta_2)}2^{-\frac{q'}{2}i\alpha_\infty}\right)\right]^{q/q'}$$

$$\lesssim \Lambda\sup_{L>0,L\in\mathbb{Z}}\sum_{k=0}^{L}\left(\sum_{i=-\infty}^{k-2}2^{\frac{q}{2}k(\alpha_\infty-nb)}2^{\frac{q}{2}(i-k)(s+1+n\delta_2-nb)}+\sum_{i=-\infty}^{k-2}2^{\frac{q}{2}(i-k)(s+1+n\delta_2-\alpha_\infty)}\right)$$

$$\times\left(\sum_{i=-\infty}^{k-2}2^{\frac{q}{2}k(\alpha_\infty-nb)}2^{\frac{q'}{2}(i-k)(s+1+n\delta_2-nb)}+\sum_{i=-\infty}^{k-2}2^{\frac{q'}{2}(i-k)(s+1+n\delta_2-\alpha_\infty)}\right)^{q/q'}$$

$$\lesssim \Lambda\sup_{L>0,L\in\mathbb{Z}}\sum_{k=0}^{L}\left(\sum_{i=-\infty}^{k-2}2^{\frac{q}{2}(i-k)(s+1+n\delta_2-nb)}+\sum_{i=-\infty}^{k-2}2^{\frac{q}{2}(i-k)(s+1+n\delta_2-\alpha_\infty)}\right)$$

$$\times\left(\sum_{i=-\infty}^{k-2}2^{\frac{q'}{2}(i-k)(s+1+n\delta_2-nb)}+\sum_{i=-\infty}^{k-2}2^{\frac{q'}{2}(i-k)(s+1+n\delta_2-\alpha_\infty)}\right)^{q/q'}$$

$$\lesssim \Lambda.$$

By (a) of Lemma 2.5,

$$III_{22}\lesssim \sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{i=k-2}^{+\infty}\sum_{l=-\infty}^{k-2}|\lambda_l|2^{-inb}2^{lna}\right)^q$$

$$\lesssim \sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\sum_{i=k-2}^{+\infty}\sum_{l=-\infty}^{k-2}|\lambda_l|^q 2^{-\frac{q}{2}inb}2^{\frac{q}{2}lna}\left(\sum_{i=k-2}^{+\infty}\sum_{l=-\infty}^{k-2}2^{-\frac{q'}{2}inb}2^{\frac{q'}{2}lna}\right)^{q/q'}$$

$$\lesssim \sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{i=k-2}^{+\infty}\sum_{l=-\infty}^{0}|\lambda_l|^q 2^{-\frac{q}{2}inb}2^{\frac{q}{2}lna}+\sum_{i=k-2}^{+\infty}\sum_{l=1}^{k-2}|\lambda_l|^q 2^{-\frac{q}{2}inb}2^{\frac{q}{2}lna}\right)$$

$$\times\left(\sum_{i=k-2}^{+\infty}\sum_{l=-\infty}^{0}2^{-\frac{q'}{2}inb}2^{\frac{q'}{2}lna}+\sum_{i=k-2}^{+\infty}\sum_{l=1}^{k-2}2^{-\frac{q'}{2}inb}2^{\frac{q'}{2}lna}\right)^{q/q'}$$

$$\lesssim \Lambda\sup_{L>0,L\in\mathbb{Z}}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{i=k-2}^{+\infty}2^{-\frac{q}{2}inb}+\sum_{i=k-2}^{+\infty}2^{-\frac{q}{2}i\alpha_\infty}\right)\left(\sum_{i=k-2}^{+\infty}2^{-\frac{q'}{2}inb}+\sum_{i=k-2}^{+\infty}2^{-\frac{q'}{2}i\alpha_\infty}\right)^{q/q'}$$

$$\begin{aligned}
&\lesssim \Lambda \sup_{L>0, L\in\mathbb{Z}} \sum_{k=0}^{L} \left(\sum_{i=k-2}^{+\infty} 2^{\frac{q}{2}k\alpha_\infty} 2^{-\frac{q}{2}inb} + \sum_{i=k-2}^{+\infty} 2^{(k-i)\frac{q}{2}\alpha_\infty} \right) \\
&\quad \times \left(\sum_{i=k-2}^{+\infty} 2^{\frac{q}{2}k\alpha_\infty} 2^{-\frac{q'}{2}inb} + \sum_{i=k-2}^{+\infty} 2^{(k-i)\frac{q'}{2}\alpha_\infty} \right)^{q/q'} \\
&\lesssim \Lambda \sup_{L>0, L\in\mathbb{Z}} \sum_{k=0}^{L} \left(2^{\frac{q}{2}k(\alpha_\infty - nb)} + \sum_{i=k-2}^{+\infty} 2^{(k-i)\frac{q}{2}\alpha_\infty} \right) \left(2^{\frac{q}{2}k(\alpha_\infty - nb)} + \sum_{i=k-2}^{+\infty} 2^{(k-i)\frac{q'}{2}\alpha_\infty} \right)^{q/q'} \\
&\lesssim \Lambda. \quad \square
\end{aligned}$$

3. An application

By using atomic and molecular decompositions, we have the following result.

THEOREM 3.1. *Let* $0 < q < \infty$, $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. *Let*

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y) f(y) \mathrm{d}y$$

be a bounded operator on $L^{p(\cdot)}$ *with the kernel* K *satisfies*

$$|K(x-y) - K(x)| \leqslant C \frac{|y|^\delta}{|x|^{n+\delta}}, \; |x| \geqslant |y|,$$

where C *is a positive constant and* $0 < \delta \leqslant 1$, *and* $T^*(1) = 0$. *If* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathscr{P}_0^{\log}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{\log}(\mathbb{R}^n)$ *such that* $\max\{n\delta_1, n\delta_2\} \leqslant \alpha(0), \alpha_\infty < \min\{n\delta_1 + \delta, n\delta_2 + \delta\}$, *where* δ_1, δ_2 *as in Lemma 2.1*, $0 \leqslant \lambda \leqslant \frac{1}{2}\min\{\alpha(0), \alpha_\infty\}$, *then the operator* T *is a bounded operator on* $HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ *and* $HMK_{p(\cdot),\lambda}^{\alpha(\cdot),q}$.

The idea of the proof of Theorem 3.1 comes from Theorem 6.2.3 in [16] for Herz Type Hardy spaces. To complete the proof of Theorem 3.1 we need the following definition and lemmas firstly.

DEFINITION 3.1. For $s \in \mathbb{Z}_+$, let $\mathscr{D}(\mathbb{R}^n)$ be the space of infinitely differentiable complex-valued functions with compact supported in $\mathbb{R}^n$.

$$\mathscr{D}_s(\mathbb{R}^n) = \left\{ f \in \mathscr{D}(\mathbb{R}^n) : \int f(x) x^\beta \mathrm{dx} = 0, \; \text{for all } |\beta| \leqslant \mathrm{s} \right\}$$

and

$$\dot{\mathscr{D}}_s(\mathbb{R}^n) = \{ f \in \mathscr{D}_s(\mathbb{R}^n), \; 0 \notin \text{ supp} f \}.$$

LEMMA 3.1. *Let* $0<q<\infty$, $p(\cdot)\in\mathscr{B}(\mathbb{R}^n)$, $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathscr{P}_0^{\log}(\mathbb{R}^n)\cap\mathscr{P}_\infty^{\log}(\mathbb{R}^n)$ *such that* $\max\{n\delta_1,n\delta_2\}\leqslant\alpha(0),\alpha_\infty<\infty$, *where* δ_1,δ_2 *as in Lemma 2.1.* $0\leqslant\lambda\leqslant\frac{1}{2}\min\{\alpha(0),\alpha_\infty\}$. *Let* s *be a non-negative integer such that* $s\geqslant[\max\{\alpha(0),\alpha_\infty\}-\min\{n\delta_1,n\delta_2\}]$. *Suppose that* $f\in\mathscr{D}_s(\mathbb{R}^n)$ *and* $\operatorname{supp} f\in B_{k_0-1}$ *for some* $k_0\in\mathbb{N}$.

(i) *There exist a sequence of numbers* $\{\lambda_k\}_{k\in\mathbb{Z}}$ *and a sequence of cental* $(\alpha(\cdot),p(\cdot))$*-atoms* $\{a_k\}_{k\in\mathbb{Z}}\subset\mathscr{D}_s(\mathbb{R}^n)$ *with* $\operatorname{supp} a_k\in B_{k+2}\backslash B_{k-1}$ *such that*

$$f(x)=\sum_{k\in\mathbb{Z}}\lambda_k a_k(x)$$

for all $x\in\mathbb{R}^n\backslash\{0\}$ *and in the sense of* $\mathscr{S}'(\mathbb{R}^n)$, *and*

$$\left(\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}|\lambda_k|^q\right)^{1/q}\leqslant C\|f\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}.$$

Furthermore, if $\operatorname{supp} f\in B_{k_0-1}\backslash B_{k_1+1}$ *for some* $k_1\in\mathbb{Z}$, *then* $\lambda_k=0$ *for all* $k>k_0$ *and* $k<k_1$;

(ii) *There exist a sequences of numbers* $\{\lambda_k\}_{k=0}^{k_0}$ *and a sequence of cental* $(\alpha(\cdot),p(\cdot))$*-atoms of restrict type* $\{a_k\}_{k\in\mathbb{Z}}\subset\mathscr{D}_s(\mathbb{R}^n)$ *with* $\operatorname{supp} a_k\in B_{k+1}$ *such that*

$$f(x)=\sum_{k=0}^{k_0}\lambda_k a_k(x)$$

for all $x\in\mathbb{R}^n$, *and*

$$\left(\sup_{L\in\mathbb{N}_0}2^{-L\lambda q}\sum_{k=0}^{L}|\lambda_k|^q\right)^{1/q}\leqslant C\|f\|_{HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}}.$$

Proof. We only prove the conclusion (i), the proof of conclusion (ii) is analogous and omitted. Let $\phi\in\mathscr{S}(\mathbb{R}^n)$ such that

$$0\leqslant\phi(x)\leqslant1,\ x\in\mathbb{R}^n;\ \phi(x)=1,\ |x|\leqslant\frac{1}{2}+\frac{1}{10};\ \phi(x)=0,\ \text{if}\ |x|>1-\frac{1}{10}.$$

Let

$$\varphi(x)=\phi(x)-\phi(\frac{x}{2})\ \text{ and }\ \Phi_k(x)=\varphi(2^{-k}x).$$

Then $\operatorname{supp}\varphi\subset\{\frac{1}{2}\leqslant|x|\leqslant2\}$. $\operatorname{supp}\Phi_k\subset D_k\cup D_{k+1}$ and $\sum_k\Phi(x)=1$ for any $x\in\mathbb{R}^n\backslash\{0\}$. Let $R_0=\{\frac{1}{2}\leqslant|x|\leqslant2\}$, $R_k=D_k\cup D_{k+1}$ for $k\geqslant1$. Let $\{\widetilde{\psi}_\beta:\beta\leqslant s\}\in\mathscr{S}(\mathbb{R}^n)$ be a dual basis of $\{x^\beta:|\beta|\leqslant s\}$ with respect to the weight $|R_0|^{-1}\varphi$, namely

$$\frac{1}{|R_0|}\int_{\mathbb{R}^n}x^\beta\widetilde{\psi}_\gamma(x)\varphi(x)\mathrm{d}x=\delta_{\beta\gamma},$$

which implies that

$$2^{-k(n+|\beta|)}\frac{1}{|R_0|}\int_{\mathbb{R}^n}x^\beta\widetilde{\psi}_\gamma(2^{-k}x)\varphi(2^{-k}x)\mathrm{d}x=\delta_{\beta\gamma}.$$

For all $x \in \mathbb{R}^n,\ k \in \mathbb{Z}$ and $|\beta| \leqslant s,$ set

$$\psi_{k,\beta}(x) = 2^{-k(n+|\beta|)} \frac{1}{|R_0|} \widetilde{\psi}_\beta(2^{-k}x)\Phi_k(x).$$

Then $\psi_{k,\beta}(x) \in \mathscr{S}(\mathbb{R}^n),$ $\operatorname{supp} \psi_{k,\beta} \subset R_k,$

$$\|\psi_{k,\beta}\|_{L^\infty(\mathbb{R}^n)} \leqslant C2^{-k(n+|\beta|)} \tag{10}$$

and

$$\int_{\mathbb{R}^n} \psi_{k,\beta}(x)x^\gamma \mathrm{d}x = \delta_{\beta\gamma}. \tag{11}$$

For each fixed $f \in \mathscr{D}_s(\mathbb{R}^n)$ and $k \in \mathbb{Z},$ set $f_k(x) = f(x)\Phi_k(x)$ and

$$P_k(x) = \sum_{|\beta|\leqslant s} \psi_{k,\beta}(x) \int_{\mathbb{R}^n} f_k(y)y^\beta \mathrm{d}y.$$

It is easy to see that $f_k - P_k \in \mathscr{D}_s(\mathbb{R}^n)$ and $\operatorname{supp}(f_k - P_k) \subset R_k$ for $k \in \mathbb{Z}.$ Decompose f as

$$f(x) = \sum_{k\in\mathbb{Z}} [f_k(x) - P_k(x)] + \sum_{k\in\mathbb{Z}} P_k(x) \tag{12}$$

$$= \sum_{k\in\mathbb{Z}} [f_k(x) - P_k(x)] + \sum_{k\in\mathbb{Z}} \sum_{|\beta|\leqslant s} [\psi_{k,\beta}(x) - \psi_{k+1,\beta}(x)] \int_{\mathbb{R}^n} \sum_{l=-\infty}^{k} f_l(y)y^\beta \mathrm{d}y \tag{13}$$

where the equality holds for $x \in \mathbb{R}^n \backslash \{0\}.$

Notice that $|f(x)| \leqslant G_N(f)(x)$ for any $x \in \mathbb{R}^n.$ Then by (10), the Hölder inequality and Lemma 2.2 we have

$$\begin{aligned}\|P_k\|_{L^{p(\cdot)}} &\lesssim \sum_{|\beta|\leqslant s} 2^{-kn} \|\chi_{B_k}\|_{L^{p(\cdot)}} \int_{\mathbb{R}^n} G_N(f)(x)\chi_{R_k}(x)\mathrm{d}x \\ &\lesssim 2^{-kn} \|\chi_{B_k}\|_{L^{p(\cdot)}} \|\chi_{B_k}\|_{L^{p'(\cdot)}} \|G_N(f)\chi_{R_k}\|_{L^{p(\cdot)}} \\ &\lesssim \|G_N(f)\chi_{R_k}\|_{L^{p(\cdot)}}.\end{aligned}$$

This implies that

$$\|f_k - P_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|\chi_{R_k} G_N(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Let $\lambda_k = |B_{k+1}|^{\alpha_k/n} \|\chi_{R_k} G_N(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ and $a_k \equiv \lambda_k^{-1}(f_k - P_k).$ Then for any $k \in \mathbb{Z},$ Ca_k is a central of $(\alpha(\cdot), p(\cdot))$-atom supported in $B_{k+1} \backslash B_{k-1}.$ Moreover'

$$\left(\sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} |\lambda_k|^q\right)^{1/q} \leqslant C\|f\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}. \tag{14}$$

To estimate the second summation, for $|\beta| \leqslant s,$ let ζ^β be the function satisfies that

$$\zeta^\beta(y) = \sum_{l=-\infty}^{0} \varphi(2^{-l}y)y^\beta,\ y \neq 0$$

and $\zeta^\beta(0)=0$ if $|\beta|>0$, $\zeta^\beta(0)=1$ if $|\beta|=0$. Then there exists a constant $C>0$ such that $C\zeta^\beta\in\mathscr{A}_N(\mathbb{R}^n)$ for all $|\beta|\leqslant s$. Moreover, it is easy to see that

$$\int_{\mathbb{R}^n}\sum_{l=-\infty}^{k} f_l(y)y^\beta\,\mathrm{d}y = 2^{k(n+|\beta|)}|\zeta^\beta_{2^k}*f(0)|\leqslant C2^{k(n+|\beta|)}\chi_{B_{k+1}}(x)G_N(f)(x). \tag{15}$$

By the definition of $\psi_{k,\beta}$ and (11) we have that $\psi_{k,\beta}-\psi_{k-1,\beta}\in\mathscr{D}_s(\mathbb{R}^n)$ and

$$|\psi_{k,\beta}-\psi_{k+1,\beta}|\leqslant C2^{-k(n+|\beta|)}\chi_{R_k\cup R_{k+1}} \tag{16}$$

Let $\mu_k=C|B_{k+2}|^{\alpha_k/n}\|\chi_{R_k}G_N(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ and

$$b_k=(\mu_k^{-1})\sum_{|\beta|\leqslant s}(\psi_{k,\beta}-\psi_{k-1,\beta})\int_{\mathbb{R}^n}\sum_{l=-\infty}^{k} f_l(y)y^\beta\,\mathrm{d}y.$$

Then by (15) and (16), we have $b_k\in\mathscr{D}_s(\mathbb{R}^n)$ supported in $B_{k+2}\backslash B_{k-1}$ and $\|b_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leqslant C|B_{k+2}|^{\alpha(\cdot)/n}$, which implies that $Cb_k\in\mathscr{D}_s(\mathbb{R}^n)$ is a central $(\alpha(\cdot),p(\cdot))$-atom supported in $B_{k+2}\backslash B_{k-1}$ for all $k\in\mathbb{Z}$. Again by (15),

$$\left(\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}|\mu_k|^q\right)^{1/q}\leqslant C\|f\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}. \tag{17}$$

Since for $|k-l|>2$, $\operatorname{supp}a_k\cap\operatorname{supp}a_l=\emptyset$ and $\operatorname{supp}b_k\cap\operatorname{supp}b_l=\emptyset$, it follows that

$$f(x)=\sum_{k\in\mathbb{Z}}(\lambda_k a_k(x)+\mu_k b_k(x))$$

for all $x\in\mathbb{R}^n\backslash\{0\}$ and in $\mathscr{S}'(\mathbb{R}^n)$, which together with (14), (17) and the facts that $\{Ca_k,Cb_k\}_{k\in\mathbb{Z}}$ are central $(\alpha(\cdot),p(\cdot))$-atoms gives the central atomic decomposition of f.

For $f\in\mathscr{D}_s(\mathbb{R}^n)$ with $\operatorname{supp}f\subset B_{k_0-1}\backslash B_{k_1+1}$, observing that when $k>k_0$ or $k<k_1$, $f_k=0$ and

$$\int_{\mathbb{R}^n}\sum_{l=-\infty}^{k} f_l(y)y^\beta\,\mathrm{d}y=\int_{\mathbb{R}^n}f(y)y^\beta\,\mathrm{d}y=0,$$

and so $a_k=0$ and $b_k=0$, we have

$$f(x)=\sum_{|k|\leqslant k_0}(\lambda_k a_k(x)+\mu_k b_k(x))$$

for all $x\in\mathbb{R}^n$. This concludes the proof of the conclusion (i). $\square$

LEMMA 3.2. *Let* $0<q<\infty$, $p(\cdot)\in\mathscr{B}(\mathbb{R}^n)$, $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathscr{P}_0^{\log}(\mathbb{R}^n)\cap\mathscr{P}_\infty^{\log}(\mathbb{R}^n)$ *such that* $\max\{n\delta_1,n\delta_2\}\leqslant\alpha(0),\alpha_\infty<\infty$, *where* δ_1,δ_2 *as in Lemma 2.1.* $0\leqslant\lambda\leqslant\frac{1}{2}\min\{\alpha(0),\alpha_\infty\}$. *Let s be a non-negative integer such that* $s\geqslant[\max\{\alpha(0),\alpha_\infty\}-\min\{n\delta_1,n\delta_2\}]$. *Then*

(i) $\dot{\mathscr{D}}_s(\mathbb{R}^n)$ *is dense in* $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$;

(ii) $\mathscr{D}_s(\mathbb{R}^n)$ *is dense in* $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}$;

Proof. Let $f \in HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$. Theorem 13 in [33] shows that there exist a sequence of numbers $\{\lambda_k\}_{k\in\mathbb{Z}}$ and a sequences of central $(\alpha(\cdot),p(\cdot))$ atoms $\{a_k\}_{k\in\mathbb{Z}}$ such that $f=\sum_{k=-\infty}^{\infty}\lambda_k a_k$ in the sense of $\mathscr{S}'(\mathbb{R}^n)$ and

$$\left(\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}|\lambda_k|^q\right)^{1/q} \leqslant C\|f\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}.$$

For $T\in\mathbb{N}$, let

$$f_T=\sum_{|k|<T}\lambda_k a_k.$$

It is easy to see that $f_T\in HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ and

$$\|f-f_T\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \leqslant \left(\sup_{L\in\mathbb{N}_0} 2^{-L\lambda q}\sum_{k=T}^{L}|\lambda_k|^q\right)^{1/q} \to 0.$$

Take $\psi\in\mathscr{D}(\mathbb{R}^n)$ with integral 1. Since

$$f_T\in L^{p(\cdot)}(\mathbb{R}^n)\cap HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}, \quad \operatorname{supp} f_T\in B_{T-1},$$

it follows that $\psi_t * f_T\in\mathscr{D}_s$, where $\psi_t(\cdot)=t^{-n}\psi(\frac{\cdot}{t})$. Moreover, note that for any $t\in(0,2^{-T})$ and k with $|k|<T$, $\|a_k\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}\leqslant 1$ and

$$|B_{k+1}|^{-(k+1)\alpha_k/n}\|\psi_t * a_k - a_k\|^{-1}_{L^{p(\cdot)}}(\psi * a_k - a_k)$$

is a central $(\alpha(\cdot),p(\cdot))$-atom supported in B_{k+1}. This in turn implies that

$$\begin{aligned}\|\psi_t * f_T - f_T\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} &= \left\|\sum_{|k|<T}\lambda_k(\psi * a_k - a_k)\right\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}\\ &\leqslant \left\{\sup_{|L|\leqslant T+1} 2^{-L\lambda q}\sum_{|k|<T}|\lambda_k|^q|B_{k+1}|^{q(k+1)\alpha_k/n}\|\psi_t * a_k - a_k\|^q_{L^{p(\cdot)}}\right\}^{1/q}\\ &\to 0, \quad \text{as } t\to 0.\end{aligned}$$

For any $f_T\in\mathscr{D}_s(\mathbb{R}^n)$, by Lemma 3.1 (i), we know that f_T has an atom decomposition

$$f_T=\sum_{k\in\mathbb{Z}}\lambda_{T,k}a_{T,k}$$

almost everywhere and in the sense of $\mathscr{S}'(\mathbb{R}^n)$, where $a_{T,k}\in\mathscr{D}_s(\mathbb{R}^n)$ supported in $B_{k+2}\backslash B_{k-1}$ and $\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}|\lambda_{T,k}|^q\leqslant C\|f_T\|^q_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}$. Let

$$f_{T,J}=\sum_{|k|\leqslant J}\lambda_{T,k}a_{T,k}.$$

Then $f_{T,J} \in \dot{\mathscr{D}}_s(\mathbb{R}^n)$ and

$$\lim_{J\to\infty} \|f_{T,J} - f_T\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} = 0.$$

Therefore $\dot{\mathscr{D}}_s(\mathbb{R}^n)$ is dense in $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$.

Using Theorem 13 in [33] and Lemma 3.1 (ii) and the same argument above we can prove (ii). $\square$

LEMMA 3.3. *Let* $0<q<\infty$, $p(\cdot)\in\mathscr{B}(\mathbb{R}^n)$, $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathscr{P}_0^{\log}(\mathbb{R}^n)\cap\mathscr{P}_\infty^{\log}(\mathbb{R}^n)$ *such that* $\max\{n\delta_1,n\delta_2\}\leqslant\alpha(0),\alpha_\infty<\infty$, *where* δ_1,δ_2 *as in Lemma* 2.1. $0\leqslant\lambda\leqslant\frac{1}{2}\min\{\alpha(0),\alpha_\infty\}$. *Let* s *be a non-negative integer such that* $s\geqslant[\max\{\alpha(0),\alpha_\infty\}-\min\{n\delta_1,n\delta_2\}]$. *Suppose* $\varepsilon>\max\{s/n,\alpha(0)/n+\delta_1-1,\alpha_\infty/n+\delta_1-1\}$. *Let* T *be a linear operator on* $\mathscr{D}(\mathbb{R}^n)$.

(i) *If there is a constant* $C>0$ *such that for any central* $(\alpha(\cdot),p(\cdot))$ *atom* b, CTb *is a central* $(\alpha(\cdot),p(\cdot),s,\varepsilon)$*-molecule. Then* T *can be boundedly extended to* $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$.

(ii) *If there is a constant* $C>0$ *such that for any central* $(\alpha(\cdot),p(\cdot))$ *atom* b *of restrict type,* CTb *is a central* $(\alpha(\cdot),p(\cdot),s,\varepsilon)$*-molecule of restrict type. Then* T *can be boundedly extended to* $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}$.

Proof. We only prove (i). The proof of (ii) is similar to (i). Let $f\in\dot{\mathscr{D}}_s(\mathbb{R}^n)\cap HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ supported in $B_{k_0-1}\backslash B_{-k_0+1}$. It follows from (i) of Lemma 3.1 that there exists a sequence of numbers $\{\lambda_k\}_{|k|\leqslant k_0}$ and a sequence of central $(\alpha(\cdot),q(\cdot))$ atoms $\{a_k\}_{|k|\leqslant k_0}$ such that $f=\sum_k\lambda_k a_k$ in the sense of $\mathscr{S}'(\mathbb{R}^n)$ and

$$\left(\sup_{L\leqslant k_0} 2^{-L\lambda q}\sum_{k=-\infty}^{L}|\lambda_k|^q\right)^{1/q} \lesssim \|f\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}.$$

We then have that

$$Tf=\sum_{|k|\leqslant k_0}\lambda_k Ta_k$$

and

$$\|Tf\|^r_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \lesssim \left\{\sup_{L\leqslant k_0} 2^{-L\lambda q}\sum_{k=-\infty}^{L}|\lambda_k|^q\right\}^{r/q} \lesssim \|f\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}.$$

Since the density of $\dot{\mathscr{D}}_s(\mathbb{R}^n)$ in $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$, T can be extended to be a bounded operator on $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$. $\square$

Proof of Theorem 3.1. We only prove the result for homogeneous case, the non-homogeneous can be handled by the similar way. Let $f\in HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$, by Theorem

13 in [33] then $f=\sum_{k=-\infty}^{\infty}\lambda_k a_k$ in the sense of $\mathscr{S}'(\mathbb{R}^n)$, where each a_k is a central $(\alpha(\cdot),p(\cdot))$-atom with support contained in B_k and

$$\|f\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \approx \inf\sup_{L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L}|\lambda_k|^q\right)^{1/q}.$$

By Lema 3.1 to 3.3, we only need to show that Ta_k is a central $(\alpha(\cdot),p(\cdot);0,\varepsilon)$-molecule for any central $(\alpha(\cdot),p(\cdot))$-atom with support contained in B_k in $\mathscr{D}(\mathbb{R}^n)$. The moment condition is guaranteed by $T^*(1)=0$. So we only need to verify the size condition for molecules. To do this, let $t=1-\delta_1-\alpha_k/n+\varepsilon$, $b=1-\delta_1+\varepsilon$.

$$\mathscr{R}_{p(\cdot)}(Ta_k)=\|Ta_k\|_{L^{p(\cdot)}}^{t/b}\||\cdot|^{nb}(Ta_k)\|_{L^{p(\cdot)}}^{1-t/b}<\infty,$$

First we estimate $\||\cdot|^{nb}(Ta_k)\|_{L^{p(\cdot)}}^{1-t/b}$. In fact, we have

$$\||\cdot|^{nb}(Ta_k)\|_{L^{p(\cdot)}(|\cdot|\leqslant 2k)}\lesssim r^{nb}\|Ta_k\|_{L^{p(\cdot)}}\lesssim r^{nb-\alpha_k}.$$

On the other hand, for any x with $|x|>2k$, the vanishing moment of a_k and the regularity of K, we have

$$\begin{aligned}
|Ta_k(x)| &= \left|\int_{\mathbb{R}^n}(K(x-y)-K(x))\,a_k(y)\mathrm{d}y\right|\\
&\lesssim \int_{\mathbb{R}^n}\frac{|y|^{\delta}}{|x-y|^{n+\delta}}|a_k(y)|\mathrm{d}y\\
&\lesssim k^{n+\delta}|x|^{-(n+\delta)}\frac{1}{|B(0,k)|}\int_{B(0,k)}|a_k(y)|\mathrm{d}y\\
&\lesssim k^{n+\delta}|x|^{-(n+\delta)}\mathscr{M}a_k(x)\\
&\leqslant \mathscr{M}a_k(x)
\end{aligned}$$

and

$$\||\cdot|^{nb}(Ta_k)\|_{L^{p(\cdot)}(|\cdot|>2k)}\lesssim \||\cdot|^{nb}\mathscr{M}a_k\|_{L^{p(\cdot)}(|\cdot|>2k)}\lesssim k^{nb}\||a_k\|_{L^{p(\cdot)}}\lesssim k^{nb-\alpha_k}.$$

Thus, we get

$$\begin{aligned}
\mathscr{R}_{p(\cdot)}(Ta_k) &= \|Ta_k\|_{L^{p(\cdot)}}^{t/b}\||\cdot|^{nb}(Ta_k)\|_{L^{p(\cdot)}}^{1-t/b}\\
&\lesssim \|a_k\|_{L^{p(\cdot)}}^{t/b}k^{(nb-\alpha_k)(1-t/b)}\\
&\lesssim k^{-\alpha_k t/b+(nb-\alpha_k)(1-t/b)}\\
&=1.
\end{aligned}$$

Therefore, the proof of Theorem 3.1 is concluded. □

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REFERENCES

[1] A. ALMEIDA AND D. DRIHEM, *Maximal, potential and singular type operators on Herz spaces with variable exponts*, J. Math. Anal. Appl. **394** (2012), no. 2, 781–795.

[2] A. ALMEIDA, J. HASANOV AND S. SAMKO, *Maximal and potential operators in variable exponent Morrey spaces*, Georgian Math. J. **15** (2008), no. 2, 195–208.

[3] A. ALMEIDA AND P. HÄSTÖ, *Besov spaces with variable smoothness and integrability*, J. Funct. Anal. **258** (2010), no. 5, 1628–1655.

[4] C. CHENG AND J. XU, *Geometric properties of Banach space valued Bochner-Lebesgue spaces with variable exponent*, J. Math. Inequal. **7** (2013), no. 3, 461–475.

[5] D. CRUZ-URIBE AND A. FIORENZA, *Variable Lebesgue Spaces*, Springer, Heidelberg, Germany, 2013.

[6] L. DIENING, P. HARJULEHTO, P. HÄSTÖ AND M. RŮŽIČKA, *Lebesgue and Sobolev spaces with variable exponents*, Springer, Berlin, Germany, 2011.

[7] L. DIENING, P. HÄSTÖ AND S. ROUDENKO, *Function spaces of variable smoothness and integrablity*, J. Funct. Anal. **256** (2009), no. 6, 1731–1768.

[8] B. DONG AND J. XU, *New Herz type Besov and Triebel-Lizorkin spaces with variable exponents*, J. Funct. Spaces Appl. **2012** (2012), Article ID 384593, 27 pages.

[9] B. DONG AND J. XU, *Local characterizations of Besov and Triebel-Lizorkin spaces with variable exponent*, J. Funct. Spaces Appl. **2014** (2014), Article ID 162518, 12 pages.

[10] B. DONG AND J. XU, *Herz-Morrey type Besov and Triebel-Lizorkin spaces with variable exponents*, Banach J. Math. Anal. **9** (2015), no. 1, 75–101.

[11] J. FU AND J. XU, *Characterizations of Morrey type Besov and Triebel-Lizorkin spaces with variable exponents*, J. Math. Anal. Appl. **381** (2011), no. 1, 280–298.

[12] M. IZUKI, *Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization*, Anal. Math. **36** (2010), no. 1, 33–50.

[13] H. KEMPKA, *2-Microlocal Besov and Triebel-Lizorkin spaces of variable integrability*, Rev. Mat. Complut. **22** (2009), no. 1, 227–251.

[14] H. KEMPKA, *Atomic, molecular and wavelet decomposition of generalized 2-microlocal Besov spaces*, J. Funct. Spaces Appl. **8** (2010), no. 2, 129–165.

[15] O. KOVÁČIK AND J. RÁKOSNÍK, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41** (1991), no. 4, 592–681.

[16] S. LU, D. YANG AND G. HU, *Herz type spaces and their applications*, Science Press, Beijing, 2008.

[17] A. MAZZUCATO, *Decomposition of Besov-Morrey Spaces, Harmonic Analysis at Mount Holyoke*, AMS Series in Contemporary Mathematics **320** (2003), 279–294.

[18] A. MAZZUCATO, *Besov-Morrey Spaces: Function Space Theory and Applications to Non-Linear PDE*, Trans. Amer. Math. Soc. **355** (2003), 1297–1364.

[19] E. NAKAI AND Y. SAWANO, *Hardy spaces with variable exponents and generalized Campanato spaces*, J. Funct. Anal. **262** (2012), no. 9, 3665–3748.

[20] P. ROCHA AND M. URCIUOLO, *Factional type integral operators on variable Hardy spaces*, Acta Math. Hungar. **143** (2014), no. 2, 502–514.

[21] C. SHI AND J. XU, *Herz type Besov and Triebel-Lizorkin spaces with variable exponents*, Front. Math. China, vol. 8, no. 4, pp. 907–921, 2013.

[22] S. SAMKO, *Variable exponent Herz spaces*, Mediterr. J. Math., vol. 10, no. 4, pp. 2007–2025, 2013.

[23] Y. SAWANO, *Wavelet characterization of Besov-Morrey and Triebel-Lizorkin-Morrey spaces*, Funct. Approx. Comment. Math. **38** (2008), 93–108.

[24] Y. SAWANO, *Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces on domains in $\mathbb{R}^n$*, Math. Nachr. **283** (2010), 1456–1487.

[25] Y. SAWANO AND H. TANAKA, *Decompositions of Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces*, Math. Z. **257** (2007), no. 4, 871–905.

[26] M. H. TAIBLESON AND G. WEISS, *The molecular characterization of certain Hardy spaces*, Astérisque **77** (1980), 67–149.

[27] L. TANG AND J. XU, *Some properties of Morrey type Besov-Triebel spaces*, Math. Nachr. **278** (2005), no. 7–8, 904–917.

[28] H. TRIEBEL, *Local Function Spaces, Heat and Navier-stokes Equations*, European Mathematical Society, 2013.

[29] H. WANG AND Z. LIU, *The Herz-type Hardy spaces with variable exponent and their applications*, Taiwanese J. Math. **16** (2012), no. 4, 1363–1389.
[30] J. XU, *Variable Besov spaces and Triebel-Lizorkin spaces*, Ann. Acad. Sci. Fenn. Math. **33** (2008), 511–522.
[31] J. XU, *An atomic decomposition of variable Besov and Triebel-Lizorkin spaces*, Armenian J. Math. **2** (2009), 1–12.
[32] J. XU, *Decompositions of non-homogeneous Herz type Besov and Triebel-Lizorkin spaces*, Sci. China Math. **57** (2014), no. 2, 315–331.
[33] J. XU AND X. YANG, *Herz-Morrey-Hardy spaces with variable exponents and their applications*, J. Function Spaces **2015** (2015), Article ID 160635, 19 pages.
[34] J. XU AND X. YANG, *Variable exponent Herz type Besov and Triebel-Lizorkin spaces*, Georgian Math. J., accepted.
[35] W. YUAN, W. SICKEL AND D. YANG, *Morrey and Campanato Meet Besov, Lizorkin and Triebel*, Lecture Notes in Mathematics **2005**, Springer, 2010.

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