Journal of
Mathematical
Inequalities
Volume 10, Number 4 (2016), 1013–1018 doi:10.7153/jmi-10-81

QUANTUM OSTROWSKI INEQUALITIES FOR q–DIFFERENTIABLE CONVEX FUNCTIONS

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(*Communicated by J. Pečarić*)

Abstract. In this paper, we establish a new quantum analogue of classical integral identity. Using this quantum integral identity, we derive some quantum analogues of Ostrowski type inequalities for q-differentiable convex functions.

1. Introduction

Quantum calculus or q-calculus is sometimes called as calculus without limits. In this, we obtain q-analogues of mathematical objects that can be recaptured as $q \to 1$. There are two types of q-addition, the Nalli-Ward-Al-Salam q-addition (NWA) and the Jackson-Hahn-Cigler q-addition (JHC). The first one is commutative and associative, while the second one is neither. That is why sometimes more than one q-analogue exists. These two operators form the basis of the method which unities hypergeometric series and q-hypergeometric series and which gives many formulas of q-calculus a natural form. The history of quantum calculus can be traced back to Euler (1707–1783), who first introduced the q in the tracks of Newton's infinite series. In recent years many researchers have shown keen interest in studying and investigating quantum calculus thus it emerges as interdisciplinary subject. This is of course due to the fact that quantum analysis is very helpful numerous fields and has large applications in various areas of pure and applied sciences such as computer science and particle physics, and also acts as an important tool for researchers working with analytic number theory or in theoretical physics. The quantum calculus can be viewed as bridge between Mathematics and Physics. Many scientists who are using quantum calculus are physicists, as quantum calculus has many applications in quantum group theory. For some recent developments in quantum calculus interested readers are referred to [3, 4, 5, 6, 7, 8, 10, 11, 12].

In recent decades theory of convex functions has been extensively studied due to its great importance in various fields of pure and applied sciences. Theory of inequalities and theory of convex functions are closely related to each other, thus various inequalities can be found in the literature which are proved for convex functions and

Mathematics subject classification (2010): 26D15, 26A51.
Keywords and phrases: Convex functions, quantum, q-differentiable, q-integration, Ostrowski.

differentiable convex functions, see [1, 2, 5, 6, 7, 8, 9, 10, 11]. Tariboon et al. [11, 12] introduced the notions of quantum derivative and quantum integral on finite intervals. Tariboon et al. [11] developed various quantum analogues of classical integral inequalities such as Holder's inequality, Hermite-Hadamard's inequality and Ostrowski's inequality etc.

Motivated by this ongoing research, we etsablish some new quantum analogues of Ostrowski inequalities for q-differentiable convex functions. This is the main motivation of this paper. The ideas and techniques of the paper may open new venue of further research in this field.

2. Preliminaries

In this section, we recall some previously known concepts.

Let $J = [a,b] \subseteq \mathbb{R}$ be an interval and $0 < q < 1$ be a constant. The q-derivative of a function $f : J \to \mathbb{R}$ at a point $x \in J$ on $[a,b]$ is defined as follows.

DEFINITION 2.1. ([11]) Let $f : J \to \mathbb{R}$ be a continuous function and let $x \in J$. Then q-derivative of f on J at x is defined as

$$ {}_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a. \tag{2.1}$$

It is obvious that ${}_aD_q f(a) = \lim_{x \to a} {}_aD_q f(x)$.

A function f is q-differentiable on J if ${}_aD_q f(x)$ exists for all $x \in J$. Also if $a = 0$ in (2.1), then ${}_0D_q f = D_q f$, where D_q is the q-derivative of the function f [3].

DEFINITION 2.2. ([11]) Let $f : J \to \mathbb{R}$ is a continuous function. A second-order q-derivative on J, which is denoted as ${}_aD_q^2 f$, provided ${}_aD_q f$ is q-differentiable on J is defined as ${}_aD_q^2 f =_a D_q({}_aD_q f) : J \to \mathbb{R}$. Similarly higher order q-derivative on J is defined by ${}_aD_q^n f =: J \to \mathbb{R}$.

LEMMA 2.1. ([11]) *Let* $\alpha \in \mathbb{R}$, *then*

$$ {}_aD_q (x-a)^{\alpha} = \left(\frac{1-q^{\alpha}}{1-q} \right) (x-a)^{\alpha-1}. $$

Tariboon et al. [11] defined the q-integral as follows

DEFINITION 2.3. ([11]) Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then q-integral on I is defined as

$$ \int_a^x f(t)\, {}_ad_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), \tag{2.2}$$

for $x \in I$.

If $a = 0$ in (2.2), then we have the classical q-integral [3].

LEMMA 2.2. ([11]) *Let* $\alpha \in \mathbb{R} \setminus \{-1\}$, *then*

$$ \int_a^x (t-a)_a^{\alpha} d_q t = \left(\frac{1-q}{1-q^{\alpha+1}} \right) (x-a)^{\alpha+1}. $$

EXAMPLE 2.1. ([11]) Let $f(x) = x$ for $x \in J$, then, we have

$$\int_a^x f(t)_a d_q t = \int_a^x t_a d_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n (q^n x + (1-q^n)a) = \frac{(x-a)(x+qa)}{1+q}.$$

Now we recall the classical Ostrowski inequality.

THEOREM 2.1. (Ostrowski's Inequality) *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I, the interior of the interval I, such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'(x)| \leqslant M$, then, the following inequality,*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \leqslant \frac{M}{b-a}\left[\frac{(x-a)^2+(b-x)^2}{2}\right], \tag{2.3}$$

holds.

3. Main results

In this section, we derive some Ostrowski type inequalities for q-differentiable convex functions. For this we first establish a new quantum integral identity which will serve as an auxiliary result in development of next results.

LEMMA 3.1. *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_aD_q$ be continuous and integrable on I where $0 < q < 1$, then*

$$\begin{aligned} & f(x) - \frac{1}{b-a}\int_a^b f(u)_a d_q u \\ &= \frac{q(x-a)^2}{b-a}\int_0^1 t_a D_q f(tx+(1-t)a)_0 d_q t + \frac{q(b-x)^2}{b-a}\int_0^1 t_a D_q f(tx+(1-t)b)_0 d_q t \end{aligned}$$

Proof. From Definition 2.1, Definition 2.4 and

$$\frac{(x-a)^2}{b-a}\int_0^1 t_a D_q f(tx+(1-t)a)_0 d_q t + \frac{(b-x)^2}{b-a}\int_0^1 t_a D_q f(tx+(1-t)b)_0 d_q t = I_1 + I_2. \tag{3.1}$$

Now

$$\begin{aligned} I_1 &= \frac{(x-a)^2}{b-a}\int_0^1 t_a D_q f(tx+(1-t)a)_0 d_q t \\ &= \frac{(x-a)^2}{b-a}\int_0^1 t \frac{f(tx+(1-t)a) - f(tqx+(1-tq)a)}{(1-q)(x-a)t}\, {}_0 d_q t \\ &= \frac{x-a}{b-a}\left[\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) - \sum_{n=0}^{\infty} q^n f(q^{n+1}x + (1-q^{n+1})a)\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{x-a}{b-a}\sum_{n=0}^{\infty} q^n f(q^n x+(1-q^n)a) - \frac{x-a}{q(b-a)}\sum_{n=1}^{\infty} q^n f(q^n x+(1-q^n)a) \\
&= \frac{x-a}{b-a}\sum_{n=0}^{\infty} q^n f(q^n x+(1-q^n)a) + \frac{x-a}{q(b-a)} f(x) \\
&\quad - \frac{x-a}{q(b-a)}\sum_{n=0}^{\infty} q^n f(q^n x+(1-q^n)a) \\
&= \frac{x-a}{q(b-a)} f(x) + \left[\frac{(q-1)(x-a)}{q(b-a)}\right]\sum_{n=0}^{\infty} q^n f(q^n x+(1-q^n)a) \\
&= \frac{x-a}{q(b-a)} f(x) - \left[\frac{(1-q)(x-a)}{q(b-a)}\right]\frac{1}{(1-q)(x-a)}\int_a^x f(u)\,{}_a d_q u \\
&= \frac{x-a}{q(b-a)} f(x) - \frac{1}{q(b-a)}\int_a^x f(u)\,{}_a d_q u. \qquad (3.2)
\end{aligned}
$$

Similarly

$$
I_2 = \frac{b-x}{q(b-a)} f(x) - \frac{1}{q(b-a)}\int_x^b f(u)\,{}_a d_q u. \qquad (3.3)
$$

Combining (3.1), (3.2) and (3.3) and multiplying by q completes the proof. □

THEOREM 3.1. *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_aD_q$ be continuous and integrable on I where $0 < q < 1$. If $|{}_aD_q f|$ is convex function and $|{}_aD_q f(x)| \leqslant M$, then*

$$
\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,{}_a d_q u \right| \leqslant \frac{qM[(x-a)^2+(b-x)^2]}{(b-a)(1+q)}.
$$

Proof. Using Lemma 3.1 and the fact that $|{}_aD_q f|$ is convex function, we have

$$
\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,{}_a d_q u \right| \\
&= \left| \frac{q(x-a)^2}{b-a}\int_0^1 t\,{}_aD_q f(tx+(1-t)a)\,{}_0 d_q t + \frac{q(b-x)^2}{b-a}\int_0^1 t\,{}_aD_q f(tx+(1-t)b)\,{}_0 d_q t \right| \\
&\leqslant \frac{q(x-a)^2}{b-a}\int_0^1 t|{}_aD_q f(tx+(1-t)a)|\,{}_0 d_q t + \frac{q(b-x)^2}{b-a}\int_0^1 t|{}_aD_q f(tx+(1-t)b)|\,{}_0 d_q t
\end{aligned}
$$

$$\leqslant \frac{q(x-a)^2}{b-a}\int_0^1 t[t|{}_aD_qf(x)|+(1-t)|{}_aD_qf(a)|]_0d_qt$$

$$+\frac{q(b-x)^2}{b-a}\int_0^1 t[t|{}_aD_qf(x)|+(1-t)|{}_aD_qf(b)|]_0d_qt$$

$$\leqslant \frac{qM[(x-a)^2+(b-x)^2]}{(b-a)(1+q)}.$$

This completes the proof. □

THEOREM 3.2. *Let $f: I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_aD_q$ be continuous and integrable on I where $0<q<1$. If $|{}_aD_qf|^r$ is convex function and $|{}_aD_qf(x)|\leqslant M$, then for $p,r>1$, $\frac{1}{p}+\frac{1}{r}=1$, we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)_ad_qu\right|\leqslant \frac{qM[(x-a)^2+(b-x)^2]}{(b-a)}\left(\frac{1-q}{1-q^{p+1}}\right)^{\frac{1}{p}}.$$

Proof. Using Lemma 3.1, Holder's inequality and the fact that $|{}_aD_qf|^r$ is convex function, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)_ad_qu\right|$$

$$=\left|\frac{q(x-a)^2}{b-a}\int_0^1 t_aD_qf(tx+(1-t)a)_0d_qt+\frac{q(b-x)^2}{b-a}\int_0^1 t_aD_qf(tx+(1-t)b)_0d_qt\right|$$

$$\leqslant \frac{q(x-a)^2}{b-a}\left(\int_0^1 t_0^pd_qt\right)^{\frac{1}{p}}\left(\int_0^1 |{}_aD_qf(tx+(1-t)a)|_0^rd_qt\right)^{\frac{1}{r}}$$

$$+\frac{q(b-x)^2}{b-a}\left(\int_0^1 t_0^pd_qt\right)^{\frac{1}{p}}\left(\int_0^1 |{}_aD_qf(tx+(1-t)b)|_0^rd_qt\right)^{\frac{1}{r}}$$

$$\leqslant \frac{q(x-a)^2}{b-a}\left(\frac{1-q}{1-q^{p+1}}\right)^{\frac{1}{p}}\left(\int_0^1 [t|{}_aD_qf(x)|^r+(1-t)|{}_aD_qf(a)|^r]_0d_qt\right)^{\frac{1}{r}}$$

$$+\frac{q(b-x)^2}{b-a}\left(\frac{1-q}{1-q^{p+1}}\right)^{\frac{1}{p}}\left(\int_0^1 [t|{}_aD_qf(x)|^r+(1-t)|{}_aD_qf(b)|^r]_0d_qt\right)^{\frac{1}{r}}$$

$$\leqslant \frac{qM[(x-a)^2+(b-x)^2]}{(b-a)}\left(\frac{1-q}{1-q^{p+1}}\right)^{\frac{1}{p}}.$$

This completes the proof. □

Acknowledgement. The authors are thankful to editor and anonymous referees for their valuable suggestions and comments. The authors are also thankful to Dr. S.

M. Junaid Zaidi, Rector, COMSATS Institute of Information Technology, Pakistan for providing excellent research facilities.

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(Received May 5, 2015)

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