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GENERALIZED OPIAL—TYPE INEQUALITIES FOR DIFFERENTIAL AND INTEGRAL OPERATORS WITH SPECIAL KERNELS IN FRACTIONAL CALCULUS

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Abstract. In this paper we give Opial–type inequalities for two functions and multiple Opial–type inequalities by using generalized fractional differential and integral operators with special kernels. Also, we deduce some results that already have been proved in [9, 10].

1. Introduction

Mathematical inequalities which involve derivatives and integrals of functions are of great interest. Opial's inequality [14] is of great importance in mathematics with respect to applications in theory of differential equations and difference equations. Many researchers have been published its improvements and generalizations, one can see (for instance, [1, 2]) and references there in. In 1960. Opial established the following integral inequality.

Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then

$$\int_0^h |x(t)x'(t)|dt \leqslant \frac{h}{4}\int_0^h \left(x'(t)\right)^2 dt, \tag{1.1}$$

where constant $\frac{h}{4}$ is the best possible.

Agarwal and Pang [1] studied Opial-type inequalities involving ordinary derivatives and their applications in differential equations and difference equations. Iqbal et al. in [9] gave Opial-type inequalities for two functions for general kernels and provided a connection between their results and results in [6]. They presented fractional versions of Opial-type inequalities regarding fractional derivatives of Riemann-Liouville, Caputo and Canavati type.

By $C^m[a,b]$ we denote the space of all functions which have continuous derivatives up to order m, and $AC[a,b]$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^m[a,b]$ we denote the space of all functions $f \in C^{m-1}[a,b]$ with $f^{(m-1)} \in$

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$AC[a,b]$. By $L_p[a,b]$, $1 \leqslant p < \infty$, we denote the space of all Lebesgue measurable functions f for which $|f|^p$ is Lebesgue integrable on $[a,b]$, and by $L_\infty[a,b]$ the set of all functions measurable and essentially bounded on $[a,b]$. Clearly, $L_\infty[a,b] \subset L_p[a,b]$ for all $p \geqslant 1$. We say that a function $g : [a,b] \longrightarrow \mathbb{R}$ belongs to the class $U(f,K)$ if it admits the representation

$$g(x) \leqslant \int_a^x K(x,t) f(t)\, dt,$$

where f is a continuous function and K is an arbitrary non-negative kernel such that $f(x) > 0$ implies $g(x) > 0$ for every $x \in [a,b]$. We also assume that all integrals under consideration exist and that they are finite.

Iqbal et al. in [9] proved following Opial-type inequalities involving two functions for general kernel with related extreme case.

THEOREM 1.1. *Let $g_1 \in U(f_1,K)$, $g_2 \in U(f_2,K)$. Let $\varphi > 0$, $w \geqslant 0$ be measurable functions on $[a,x]$, and K be a non-negative measurable kernel. Let $r > 1$, $r > q > 0$ and $p \geqslant 0$. Let f_1, $f_2 \in L_r[a,b]$. Then the following inequality holds:*

$$\int_a^x w(t)\left(|g_1(t)|^p |f_2(t)|^q + |g_2(t)|^p |f_1(t)|^q\right) dt \tag{1.2}$$

$$\leqslant 2^{1-\frac{q}{r}} \left(\frac{q}{p+q}\right)^{\frac{q}{r}} \left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{r}}$$

$$\times \left(\int_a^x [h(t)]^{\frac{r}{r-q}}\, dt\right)^{\frac{r-q}{r}} \left(\int_a^x \varphi(\tau)\left[|f_1(\tau)|^r + |f_2(\tau)|^r\right] d\tau\right)^{\frac{p+q}{r}},$$

where

$$h(t) = w(t)\left[\int_a^t K(t,\tau)^{\frac{r}{r-1}} \varphi(\tau)^{\frac{1}{1-r}}\, d\tau\right]^{\frac{p(r-1)}{r}} [\varphi(t)]^{\frac{-q}{r}}, \tag{1.3}$$

and

$$d_{\frac{p}{q}} = \begin{cases} 2^{1-\frac{p}{q}}, & 0 \leqslant p \leqslant q \\ 1, & p \geqslant q. \end{cases} \tag{1.4}$$

THEOREM 1.2. *Let $g_i \in U(f_1,K_i)$, $\widetilde{g}_i \in U(f_2,K_i)$, $(i = 1,2)$. Let $w \geqslant 0$ be measurable function on $[a,x]$ and $p, q_1, q_2 \geqslant 0$ and f_1, $f_2 \in L_\infty[a,b]$. Then the following inequality holds:*

$$\int_a^x w(t)\left[|g_1(t)|^{q_1}|\widetilde{g}_2(t)|^{q_2}|f_1(t)|^p+|g_2(t)|^{q_2}|\widetilde{g}_1(t)|^{q_1}|f_2(t)|^p\right]dt \tag{1.5}$$

$$\leqslant \|w\|_\infty \int_a^x \left(\int_a^t K_1(t,\tau)\,d\tau\right)^{q_1}\left(\int_a^t K_2(t,\tau)\,d\tau\right)^{q_2} dt$$

$$\times\frac{1}{2}\left[\|f_1\|_\infty^{2(q_1+p)}+\|f_1\|_\infty^{2q_2}+\|f_2\|_\infty^{2q_2}+\|f_2\|_\infty^{2(q_1+p)}\right].$$

THEOREM 1.3. *Let $g_1 \in U(f_1,K)$, $g_2 \in U(f_2,K)$. Let $\varphi > 0$, $w \geqslant 0$ be measurable functions on $[a,x]$, and K be a non-negative measurable kernel. Let $r < 0$, $q > 0$ and $p \geqslant 0$. Let f_1, $f_2 \in L_r[a,b]$ each of which is of fixed sign a.e. on $[a,b]$, with $\frac{1}{f_1}, \frac{1}{f_2} \in L_r[a,b]$. Then the following inequality holds:*

$$\int_a^x w(t)\left(|g_1(t)|^p|f_2(t)|^q+|g_2(t)|^p|f_1(t)|^q\right)dt \tag{1.6}$$

$$\geqslant 2^{1-\frac{q}{r}}\left(\frac{q}{p+q}\right)^{\frac{q}{r}}\left(c_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}$$

$$\times\left(\int_a^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}}\left(\int_a^x \varphi(\tau)\left[|f_1(\tau)|^r+|f_2(\tau)|^r\right]d\tau\right)^{\frac{p+q}{r}},$$

where h is defined by (1.3) and

$$c_{\frac{p}{q}}=\begin{cases}1, & 0\leqslant p\leqslant q\\ 2^{1-\frac{p}{q}}, & p\geqslant q.\end{cases} \tag{1.7}$$

2. Fractional differential and integral operators

Fractional calculus is a theory of integral and differential operators of non-integer order (real or complex number powers). Several mathematicians contributed to this subject over years. People like Liouville, Riemann, and Weyl made major contributions to the theory of fractional calculus. The story on the fractional calculus continued with contributions from Fourier, Abel, Lacroix, Leibniz, Grunwald and Letnikov. For a historical survey the reader may see [11, 12].

Fractional integral inequalities are useful in establishing the uniqueness of solutions for certain fractional partial differential equations. They also provide upper and lower bounds for solutions of fractional boundary value problems. These considerations have led various researchers in the field of integral inequalities to explore certain extensions and generalizations by involving fractional calculus operators.

Many authors have established Opial–type integral inequalities for different kinds of fractional derivatives and fractional integral operators for example Riemann-Liouville, Caputo, Canvati etc. In this paper we give Opial–type integral inequalities for

Hilfer differential operator and fractional integral operator containing a generalized Mittag–Leffler function in the kernel [10, 11, 12, 15, 16, 17, 18, 19].

For any $f \in L_1[a,b]$ the *Riemann-Liouville fractional integral* of f of order ν is defined by

$$\left(I_{a+}^{\nu} f\right)(s) = \frac{1}{\Gamma(\nu)} \int_a^s (s-t)^{\nu-1} f(t)dt = \left(f * K_{\nu}\right)(s), \quad s \in [a,b],\ \nu > 0, \tag{2.1}$$

where $K_{\nu}(s) = \frac{(s-t)^{\nu-1}}{\Gamma(\nu)}$. The *Riemann-Liouville fractional derivative of* $f \in L_1[a,b]$ of order ν is defined by

$$\left(D_{a+}^{\nu} f\right)(x) = \frac{d^n}{dx^n}\left(I_{a+}^{n-\nu} f\right)(x), \quad (\nu > 0,\ n = [\nu]+1). \tag{2.2}$$

Let $\mu > 0$ and $\mu \notin \{1,2,\ldots\}$, $m = [\mu]+1$, $f \in AC^m[a,b]$. The Caputo derivative of order μ is defined as

$$\left({}^C D_{a+}^{\mu} f\right)(x) = \left(I_{a+}^{m-\mu} \frac{d^m}{dx^m} f\right)(x) = \frac{1}{\Gamma(m-\mu)} \int\limits_a^x (x-s)^{m-\mu-1} \frac{d^m}{ds^m} f(s)\, ds. \tag{2.3}$$

If $\mu = m \in \{1,2,\ldots\}$ and usual derivative $f^{(m)}(x)$ of order m exists, then Caputo derivative $\left({}^C D_{a+}^{\mu} f\right)(x)$ coincides with $f^{(m)}(x)$ (see, [11, p. 92]).

DEFINITION 2.1. [18] Let $f \in L_1[a,b]$, $f * K_{(1-\nu)(1-\mu)} \in AC[a,b]$. The fractional derivative operator $D_{a+}^{\mu,\nu}$ of order $0 < \mu < 1$ and type $0 \leqslant \nu \leqslant 1$ with respect to $x \in [a,b]$ is defined by

$$\left(D_{a+}^{\mu,\nu} f\right)(x) = \left(I_{a+}^{\nu(1-\mu)} \frac{d}{dx}\left(I_{a+}^{(1-\nu)(1-\mu)} f\right)\right)(x) \tag{2.4}$$

whenever the right hand side exists.

This generalization gives the classical Riemann-Liouville fractional differentiation operator if $\nu = 0$. For $\nu = 1$ it gives the fractional differential operator introduced by Caputo. We denote it by $D_{a+}^{\mu,1} f = {}^C D_{a+}^{\mu} f$.

Several authors (see [7, 16, 17]) called (2.4) the Hilfer fractional derivative. Applications of $D_{a+}^{\mu,\nu}$ are given in [7, 16, 17, 18].

The purpose of this paper is to give Opial-type integral inequalities involving different kinds of fractional differential operators. For $0 < \mu < 1$ and $0 < \nu \leqslant 1$, the Hilfer fractional differentiation operator $D_{a+}^{\mu,\nu}$ can be rewritten in the form

$$\begin{aligned} \left(D_{a+}^{\mu,\nu} f\right)(x) &= \left(I_{a+}^{\nu(1-\mu)}\left(D_{a+}^{\mu+\nu-\mu\nu} f\right)\right)(x) \qquad (2.5) \\ &= \frac{1}{\Gamma(\nu(1-\mu))} \int\limits_{a+}^{x} (x-\tau)^{\nu(1-\mu)-1}\left(D_{a+}^{\mu+\nu-\mu\nu} f\right)(\tau)\, d\tau. \end{aligned}$$

Definition of the generalized fractional integral operator containing Mittag–Leffler function is as follows.

DEFINITION 2.2. (Prabhakar [15]) Let μ,ν,γ be positive real numbers and $\omega\in\mathbb{R}$. Then the generalized fractional integral operator $\varepsilon^{\gamma}_{\mu,\nu,\omega,a+}$ for a real-valued continuous function f is defined by:

$$(\varepsilon^{\gamma}_{\mu,\nu,\omega,a+}f)(x)=\int_{a+}^{x}(x-t)^{\nu-1}E^{\gamma}_{\mu,\nu}(\omega(x-t)^{\mu})f(t)dt, \tag{2.6}$$

where the function $E^{\gamma}_{\mu,\nu}$ is generalized Mittag–Leffler function defined as

$$E^{\gamma}_{\mu,\nu}(t)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!\Gamma(\mu n+\nu)}t^n, \tag{2.7}$$

and $(\gamma)_n$ is the Pochhammer symbol: $(\gamma)_n=\gamma(\gamma+1)...(\gamma+n-1)$, $(\gamma)_0=1$.

Let $e^{\gamma}_{\mu,\nu}(t,\omega)=t^{\nu-1}E^{\gamma}_{\mu,\nu}(\omega t^{\mu})$. The integral operator $\varepsilon^{\gamma}_{\mu,\nu,\omega,a^+}$ is bounded in the space $C(I)$ where I is an interval in $\mathbb{R}$, with a finite norm $\|f\|_C=\max_{x\in I}|f(x)|$, and there exists a positive constant $M>0$, such that (see [10])

$$\left\|\varepsilon^{\gamma}_{\mu,\nu,\omega,a^+}f\right\|_C\leqslant M\|f\|_C.$$

We define a variant of Sobolev space:

$$W^{m,1}[a,b]=\left\{f\in L_1[a,b]:\frac{d^m}{dt^m}f\in L_1[a,b]\right\}. \tag{2.8}$$

DEFINITION 2.3. (Prabhakar derivative [15]) Let $f\in L_1[0,b]$, $0<t<b\leqslant\infty$, $\mu,\nu,\gamma>0$, and $f*e^{-\gamma}_{\mu,m-\nu,\omega}\in W^{m,1}[0,b]$, $m=[\nu]$ then the Prabhakar derivative is defined by following relation

$$\left(\mathbf{D}^{\gamma}_{\mu,\nu,\omega,0+}f\right)(t)=\frac{d^m}{dt^m}\varepsilon^{-\gamma}_{\mu,m-\nu,\omega,0+}f(t). \tag{2.9}$$

DEFINITION 2.4. (Caputo-Prabhakar derivative [15]) Let $f\in L_1[0,b]$, $0<t<b\leqslant\infty$, $\mu,\nu,\gamma>0$, $m=[\nu]$ then the Caputo-Prabhakar derivative for $f\in AC^m[0,b]$ is defined by following relation

$$\begin{aligned}\left({}^{C}D^{\gamma}_{\mu,\nu,\omega,0+}f\right)(t)&=\varepsilon^{-\gamma}_{\mu,m-\nu,\omega,0+}\frac{d^m}{dt^m}f(t) \\ &=\left(\mathbf{D}^{\gamma}_{\mu,\nu,\omega,0+}f\right)(t)-\sum_{k=0}^{m-1}t^{k-\mu}E^{-\gamma}_{\mu,k-\nu+1}(\omega t^{\mu})f^{(k)}(0+).\end{aligned} \tag{2.10}$$

REMARK 2.5. Let $\mu,\nu,\gamma>0$ and $f\in AC^m[0,b]$, $0<t<b\leqslant\infty$, then

$$\left({}^{C}D^{\gamma}_{\mu,\nu,\omega,0+}f\right)(t)=\mathbf{D}^{\gamma}_{\mu,\nu,\omega,0+}\left(f(t)-\sum_{k=0}^{m-1}\frac{t^k}{k!}f^{(k)}(0+)\right). \tag{2.11}$$

Moreover, if $f^{(k)}(0+) = 0,\ k = 0,1,2,...m-1$, then

$$\left({}^{C}D^{\gamma}_{\mu,\nu,\omega,0+}f\right)(t) = \left(\mathbf{D}^{\gamma}_{\mu,\nu,\omega,0+}f\right)(t).$$

Definition of the generalized fractional integral operator containing Mittag–Leffler function with six parameters is as follows.

DEFINITION 2.6. [16] Let $\mu,\nu,\delta,k,l,\gamma$ be positive real numbers, $\omega \in \mathbb{R}$. Then the generalized fractional integral operator containing Mittag–Leffler function $\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ for a real-valued continuous function f is defined by:

$$(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}f)(x) = \int_a^x (x-t)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(x-t)^{\mu})f(t)dt, \tag{2.12}$$

where the function $E^{\gamma,\delta,k}_{\mu,\nu,l}$ is generalized Mittag–Leffler function defined as

$$E^{\gamma,\delta,k}_{\mu,\nu,l}(t) = \sum_{n=0}^{\infty}\frac{(\gamma)_{kn}}{\Gamma(\mu n+\nu)}\frac{t^n}{(\delta)_{ln}}. \tag{2.13}$$

If $\delta = l = 1$ in (2.12), then the integral operator $\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ reduces to an integral operator containing generalized Mittag–Leffler function $E^{\gamma,1,k}_{\mu,\nu,1} = E^{\gamma,k}_{\mu,\nu}$ introduced by Srivastava and Tomovski in [17]. Along $\delta = l = 1$ in addition if $k = 1$ (2.12) reduces to an integral operator $\varepsilon^{\gamma}_{\mu,\nu,\omega,a^+}$ defined by Prabhakar in [15] containing Mittag-Leffler function $E^{\gamma,1,1}_{\mu,\nu,1} = E^{\gamma}_{\mu,\nu}$. For $\omega = 0$ in (2.12), integral operator $\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ would correspond essentially to the right-handed Riemann-Liouville fractional integral operator $\left(I^{\nu}_{a+}f\right)(x)$.

3. Opial-type inequalities for two functions

Here in this section we are interested to give Opial-type inequalities for different kinds of integral and differential operators. For example Hilfer differential operator, Prabhakar integral operator, Caputo Prabhakar derivative, generalized fractional integral operator. Also we have constructed some examples. First we give results for Hilfer differential operator and its special cases.

THEOREM 3.1. *Let $0 < \mu < 1$ and $0 < \nu \leqslant 1$, $r > 1$, $r > q > 0$ and $p \geqslant 0$. Then let $\varphi > 0$, $w \geqslant 0$ be measurable functions on $[a,x]$, and $D^{\mu+\nu-\mu\nu}_{a+}f_1$, $D^{\mu+\nu-\mu\nu}_{a+}f_2 \in L_r[a,b]$. Then the following inequality holds:*

$$\int_a^x w(t)\Big(\left|(D^{\mu,\nu}_{a+}f_1)(t)\right|^p\left|\left(D^{\mu+\nu-\mu\nu}_{a+}f_2\right)(t)\right|^q \tag{3.1}$$

$$+\left|(D^{\mu,\nu}_{a+}f_2)(t)\right|^p\left|\left(D^{\mu+\nu-\mu\nu}_{a+}f_1\right)(t)\right|^q\Big)dt$$

$$\leqslant \frac{2^{1-\frac{q}{r}}}{(\Gamma(\nu(1-\mu)))^p}\left(\frac{q}{p+q}\right)^{\frac{q}{r}}\left(d_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}\left(\int_a^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}}$$

$$\times\left(\int_a^x \varphi(\tau)\left[\left|\left(D_{a+}^{\mu+\nu-\mu\nu}f_1\right)(\tau)\right|^r+\left|\left(D_{a+}^{\mu+\nu-\mu\nu}f_2\right)(\tau)\right|^r\right]d\tau\right)^{\frac{p+q}{r}},$$

where

$$h(t)=w(t)\left[\int_a^t (t-\tau)^{[\nu(1-\mu)-1]\frac{r}{r-1}}\,\varphi(\tau)^{\frac{1}{1-r}}\,d\tau\right]^{\frac{p(r-1)}{r}}[\varphi(t)]^{\frac{-q}{r}}, \tag{3.2}$$

and $d_{\frac{p}{q}}$ *is defined by (1.4).*

Proof. If we take $g_1(t)=\left(D_{a+}^{\mu,\nu}f_1\right)(t)$ and $g_2(t)=\left(D_{a+}^{\mu,\nu}f_2\right)(t)$ in Theorem 1.1, by applying the integral representation (2.5) and using that the kernel

$$K(t,\tau)=\begin{cases}\frac{(t-\tau)^{\nu(1-\mu)-1}}{\Gamma(\nu(1-\mu))}: & a\leqslant\tau\leqslant t\\ 0: & t<\tau\leqslant b\end{cases}$$

is a non-negative measurable kernel for $\tau\in[a,t]$, we obtain (3.1). □

Specially, for $\nu=1$, we obtain the following corollary.

COROLLARY 3.2. *Let* $0<\mu<1$, $r>1$, $r>q>0$ *and* $p\geqslant 0$. *Then let* $\varphi>0$, $w\geqslant 0$ *be measurable functions on* $[a,x]$, *and* f_1', $f_2'\in L_r[a,b]$. *Then the following inequality holds:*

$$\int_a^x w(t)\left(\left|\left({}^C D_{a+}^{\mu}f_1\right)(t)\right|^p\left|f_2'(t)\right|^q+\left|\left({}^C D_{a+}^{\mu}f_2\right)(t)\right|^p\left|f_1'(t)\right|^q\right)dt \tag{3.3}$$

$$\leqslant \frac{2^{1-\frac{q}{r}}}{(\Gamma(1-\mu))^p}\left(\frac{q}{p+q}\right)^{\frac{q}{r}}\left(d_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}\left(\int_a^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}}$$

$$\times\left(\int_a^x \varphi(\tau)\left[\left|f_1'(\tau)\right|^r+\left|f_2'(\tau)\right|^r\right]d\tau\right)^{\frac{p+q}{r}},$$

where h *and* $d_{\frac{p}{q}}$ *are defined by (3.2) and (1.4), respectively.*

EXAMPLE 3.3. If we take $w(t)=1$, $\varphi(t)=1$, $\mu=\frac{1}{4}$, $r=2$, $a=0$, in (3.4) we obtain

$$h(t)=\left(\int_0^t \frac{d\tau}{\sqrt{t-\tau}}d\tau\right)^{\frac{p}{2}}=\left(2\sqrt{t}\right)^{\frac{p}{2}}$$

and

$$\left(\int_0^x [h(t)]^{\frac{2}{2-q}}\,dt\right)^{\frac{2-q}{2}} = \left(\int_0^x \left(2\sqrt{t}\right)^{\frac{p}{2-q}}\,dt\right)^{\frac{2-q}{2}} = \frac{2^{\frac{p}{2}}}{\left(\frac{p}{2(2-q)}+1\right)^{\frac{2-q}{2}}} x^{\frac{p+4-2q}{4}}.$$

where $0<q<2$ and $p \geqslant 0$. If $f_1',\ f_2' \in L_2[a,b]$, then for all $x \in [a,b]$ we obtain:

$$\int_0^x \left(\left|\left({}^C D_{a+}^{\mu} f_1\right)(t)\right|^p \left|f_2'(t)\right|^q + \left|\left({}^C D_{a+}^{\mu} f_2\right)(t)\right|^p \left|f_1'(t)\right|^q\right) dt \tag{3.4}$$

$$\leqslant \frac{2^{1-\frac{q}{2}+\frac{p}{2}}}{\left(\Gamma\left(\frac{3}{4}\right)\right)^p} \frac{\left(\frac{q}{p+q}\right)^{\frac{q}{2}} \left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{2}}}{\left(\frac{p}{2(2-q)}+1\right)^{\frac{2-q}{2}}} x^{\frac{p+4-2q}{4}} \left(\int_a^x \left[\left|f_1'(\tau)\right|^2 + \left|f_2'(\tau)\right|^2\right] d\tau\right)^{\frac{p+q}{2}}.$$

Analogously, by applying the Theorem 1.3 and using the technique of a proof of Theorem 3.1, we obtain:

THEOREM 3.4. *Let $0<\mu<1,\ 0<\nu\leqslant 1,\ r<0,\ q>0$ and $p\geqslant 0$. Then let $\varphi>0,\ w\geqslant 0$ be measurable functions on $[a,x]$, and $D_{a+}^{\mu+\nu-\mu\nu} f_1,\ D_{a+}^{\mu+\nu-\mu\nu} f_2 \in L_r[a,b]$ each of which is of fixed sign a.e. on $[a,b]$, with $\frac{1}{D_{a+}^{\mu+\nu-\mu\nu} f_1}, \frac{1}{D_{a+}^{\mu+\nu-\mu\nu} f_2} \in L_r[a,b]$. Then the following inequality holds:*

$$\int_a^x w(t)\Big(\left|\left(D_{a+}^{\mu,\nu} f_1\right)(t)\right|^p \left|\left(D_{a+}^{\mu+\nu-\mu\nu} f_2\right)(t)\right|^q \tag{3.5}$$

$$+\left|\left(D_{a+}^{\mu,\nu} f_2\right)(t)\right|^p \left|\left(D_{a+}^{\mu+\nu-\mu\nu} f_1\right)(t)\right|^q\Big) dt$$

$$\geqslant \frac{2^{1-\frac{q}{r}}}{\left(\Gamma\left(\nu\left(1-\mu\right)\right)\right)^p} \left(\frac{q}{p+q}\right)^{\frac{q}{r}} \left(c_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{r}} \left(\int_a^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}}$$

$$\times \left(\int_a^x \varphi(\tau)\left[\left|\left(D_{a+}^{\mu+\nu-\mu\nu} f_1\right)(\tau)\right|^r + \left|\left(D_{a+}^{\mu+\nu-\mu\nu} f_2\right)(\tau)\right|^r\right] d\tau\right)^{\frac{p+q}{r}},$$

h and $c_{\frac{p}{q}}$ are defined by (3.2) and (1.7), respectively.

COROLLARY 3.5. *Let $0<\mu<1,\ r<0,\ q>0$ and $p\geqslant 0$. Then let $\varphi>0,\ w\geqslant 0$ be measurable functions on $[a,x]$, and $f_1',\ f_2' \in L_r[a,b]$ each of which is of fixed sign* *a.e. on* $[a,b]$*, with* $\frac{1}{f_1'},\frac{1}{f_2'}\in L_r[a,b]$*. Then the following inequality holds:*

$$\int_a^x w(t)\left(\left|\left({}^C D^{\mu}_{a+}f_1\right)(t)\right|^p \left|f_2'(t)\right|^q + \left|\left({}^C D^{\mu}_{a+}f_2\right)(t)\right|^p \left|f_1'(t)\right|^q\right) dt \tag{3.6}$$

$$\geqslant \frac{2^{1-\frac{q}{r}}}{(\Gamma(1-\mu))^p}\left(\frac{q}{p+q}\right)^{\frac{q}{r}}\left(c_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}\left(\int_a^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}}$$

$$\times\left(\int_a^x \varphi(\tau)\left[\left|f_1'(\tau)\right|^r+\left|f_2'(\tau)\right|^r\right]d\tau\right)^{\frac{p+q}{r}},$$

where h *and* $c_{\frac{p}{q}}$ *are defined by (3.2) and (1.7), respectively.*

Let $f_1,\ f_2\in L_\infty[0,b]$, and $0<\mu<1$. Taking $g_1(t)=\left({}^C D^{\mu}_{0+}f_1\right)(t)$, $g_2(t)=\left({}^C D^{\mu}_{0+}f_1\right)(t)$, $\widetilde{g}_1(t)=\left({}^C D^{\mu}_{0+}f_2\right)(t)$, $\widetilde{g}_2(t)=\left({}^C D^{\mu}_{0+}f_2\right)(t)$ and

$$K_1(t,\tau)=K_2(t,\tau)=\begin{cases}\frac{(t-\tau)^{-\mu}}{\Gamma(1-\mu)}:0\leqslant\tau\leqslant t\\ 0:t<\tau\leqslant b\end{cases}$$

in Theorem 1.2, by computation

$$\int_0^x\left(\int_0^t K_1(t,\tau)\,d\tau\right)^{q_1}\left(\int_0^t K_1(t,\tau)\,d\tau\right)^{q_2}dt$$

$$=\frac{1}{(\Gamma(1-\mu))^{q_1+q_2}}\int_0^x\left(\int_0^t\frac{d\tau}{(t-\tau)^{\mu}}\right)^{q_1+q_2}dt$$

$$=\frac{x^{(q_1+q_2)(1-\mu)+1}}{(\Gamma(2-\mu))^{q_1+q_2}\left[(q_1+q_2)(1-\mu)+1\right]}$$

we obtain the following inequality:

THEOREM 3.6. *Let* $0<\mu<1$ *and* $p,q_1,q_2\geqslant 0$*. If* $w\geqslant 0$ *is measurable function on* $[0,x]$*,* $f_1',\ f_2'\in L_\infty[0,b]$*, then the following inequality holds:*

$$\int_0^x w(t)\left[\left|\left({}^C D^{\mu}_{0+}f_1\right)(t)\right|^{q_1}\left|\left({}^C D^{\mu}_{0+}f_2\right)(t)\right|^{q_2}\left|f_1(t)\right|^p\right. \tag{3.7}$$

$$\left.+\left|\left({}^C D^{\mu}_{0+}f_1\right)(t)\right|^{q_2}\left|\left({}^C D^{\mu}_{0+}f_2\right)(t)\right|^{q_1}\left|f_2(t)\right|^p\right]dt$$

$$\leqslant\|w\|_\infty\frac{x^{(q_1+q_2)(1-\mu)+1}}{(\Gamma(2-\mu))^{q_1+q_2}\left[(q_1+q_2)(1-\mu)+1\right]}$$

$$\times\frac{1}{2}\left[\left\|f_1'\right\|_\infty^{2(q_1+p)}+\left\|f_1'\right\|_\infty^{2q_2}+\left\|f_2'\right\|_\infty^{2q_2}+\left\|f_2'\right\|^{2(q_1+p)}\right].$$

Next we give results for generalized fractional integral operator involving generalized Mittag-Leffler function. Also we deduce results for Prabhakar integral operator, Caputo Prabhakar derivative and their special cases are discussed.

THEOREM 3.7. *Let $\mu,\nu,\delta,k,l,\gamma$ be positive real numbers, $\omega\in\mathbb{R}$, $r>1$, $r>q>0$ and $p\geqslant 0$. Let $\varphi>0$, $w\geqslant 0$ be measurable functions on $[a,x]$, and let f_1, $f_2\in L_r[a,b]$. Then the following inequality holds:*

$$\int_a^x w(t)\left(\left|\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}f_1(t)\right|^p|f_2(t)|^q+\left|\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}f_2(t)\right|^p|f_1(t)|^q\right)dt \tag{3.8}$$

$$\leqslant 2^{1-\frac{q}{r}}\left(\frac{q}{p+q}\right)^{\frac{q}{r}}\left(d_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}\left(\int_a^x[h(t)]^{\frac{r}{r-q}}dt\right)^{\frac{r-q}{r}}$$

$$\times\left(\int_a^x\varphi(\tau)\left[|f_1(\tau)|^r+|f_2(\tau)|^r\right]d\tau\right)^{\frac{p+q}{r}},$$

where

$$h(t)=w(t)\left[\int_a^t\left[(t-\tau)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(t-\tau)^{\mu})\right]^{\frac{r}{r-1}}\varphi(\tau)^{\frac{1}{1-r}}d\tau\right]^{\frac{p(r-1)}{r}}[\varphi(t)]^{\frac{-q}{r}}, \tag{3.9}$$

and $d_{\frac{p}{q}}$ is defined by (1.4).

EXAMPLE 3.8. For $\delta=k=l=1$, $w(t)=1,\varphi(t)=1,r=2$

$$h(t)=\left[\int_a^t\left[(t-\tau)^{\nu-1}E^{\gamma}_{\mu,\nu}(\omega(t-\tau)^{\mu})\right]^2d\tau\right]^{\frac{p}{2}},$$

where $0<q<2$ and $p\geqslant 0$. If f_1, $f_2\in L_2[0,b]$, then for all $x\in[a,b]$ we obtain:

$$\int_a^x\left(\left|\varepsilon^{\gamma}_{\mu,\nu,\omega,a^+}f_1(t)\right|^p|f_2(t)|^q+\left|\varepsilon^{\gamma}_{\mu,\nu,\omega,a^+}f_2(t)\right|^p|f_1(t)|^q\right)dt$$

$$\leqslant 2^{1-\frac{q}{2}}\left(\frac{q}{p+q}\right)^{\frac{q}{2}}\left(d_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{2}}\left(\int_a^x\left[\int_0^t\left[(t-\tau)^{\nu-1}E^{\gamma}_{\mu,\nu}(\omega(t-\tau)^{\mu})\right]^2d\tau\right]^{\frac{p}{2-q}}dt\right)^{\frac{2-q}{2}}$$

$$\times\left(\int_a^x\left[|f_1(\tau)|^2+|f_2(\tau)|^2\right]d\tau\right)^{\frac{p+q}{2}},$$

in particular for $\mu=2,\ w=-1,\ \nu=\gamma=1,\ a=0$ we have

$$h(t)=\left[\int_0^t\left[E^1_{2,1}(-(t-\tau)^2)\right]^2d\tau\right]^{\frac{p}{2}}=\left[\int_0^t Cos^2(t-\tau)d\tau\right]^{\frac{p}{2}}=\left[\frac{t}{2}+\frac{Sin2t}{4}\right]^{\frac{p}{2}},$$

and

$$\begin{aligned}\left(\int_0^x[h(t)]^{\frac{2}{2-q}}dt\right)^{\frac{2-q}{2}}&=\left(\int_0^x\left(\frac{t}{2}+\frac{Sin2t}{4}\right)^{\frac{p}{2-q}}dt\right)^{\frac{2-q}{2}}\\ &\int_0^x\left(\left|\varepsilon^1_{2,1,-1,0^+}f_1(t)\right|^p|f_2(t)|^q+\left|\varepsilon^1_{2,1,-1,0^+}f_2(t)\right|^p|f_1(t)|^q\right)dt\\ &\leqslant 2^{1-\frac{q}{2}}\left(\frac{q}{p+q}\right)^{\frac{q}{2}}\left(d_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{2}}\left(\int_0^x\left(\frac{t}{2}+\frac{Sin2t}{4}\right)^{\frac{p}{2-q}}dt\right)^{\frac{2-q}{2}}\\ &\times\left(\int_a^x\left[|f_1(\tau)|^2+|f_2(\tau)|^2\right]d\tau\right)^{\frac{p+q}{2}}.\end{aligned}$$

THEOREM 3.9. *Let $\mu,\nu,\delta,k,l,\gamma$ be positive real numbers, $\omega\in\mathbb{R}$, $r<0,\ q>0$ and $p\geqslant 0$. Let $\varphi>0,\ w\geqslant 0$ be measurable functions on $[a,x]$, and let $f_1,\ f_2\in L_r[a,b]$ each of which is of fixed sign a.e. on $[a,b]$, with $\frac{1}{f_1},\frac{1}{f_2}\in L_r[a,b]$. Then the following inequality holds:*

$$\begin{aligned}&\int_a^x w(t)\left(\left|\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}f_1(t)\right|^p|f_2(t)|^q+\left|\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}f_2(t)\right|^p|f_1(t)|^q\right)dt \qquad (3.10)\\ &\geqslant 2^{1-\frac{q}{r}}\left(\frac{q}{p+q}\right)^{\frac{q}{r}}\left(c_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}\left(\int_a^x[h(t)]^{\frac{r}{r-q}}dt\right)^{\frac{r-q}{r}}\\ &\times\left(\int_a^x\varphi(\tau)\left[|f_1(\tau)|^r+|f_2(\tau)|^r\right]d\tau\right)^{\frac{p+q}{r}},\end{aligned}$$

where h and $c_{\frac{p}{q}}$ are defined by (3.9) and (1.7), respectively.

COROLLARY 3.10. *Let $\nu>0,\ r>1,\ r>q>0$ and $p\geqslant 0$. If $\varphi>0,\ w\geqslant 0$ are measurable functions on $[a,x]$, and $f_1,\ f_2\in L_r[a,b]$, then the following inequality holds:*

$$\int_a^x w(t)\left(\left|\left(I_{a+}^{\nu}f_1\right)(t)\right|^p|f_2(t)|^q+\left|\left(I_{a+}^{\nu}f_2\right)(t)\right|^p|f_1(t)|^q\right)dt \tag{3.11}$$
$$\leqslant 2^{1-\frac{q}{r}}\left(\frac{q}{p+q}\right)^{\frac{q}{r}}\left(d_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}\left(\int_a^x [h(t)]^{\frac{r}{r-q}}dt\right)^{\frac{r-q}{r}}$$
$$\times\left(\int_a^x \varphi(\tau)\left[|f_1(\tau)|^r+|f_2(\tau)|^r\right]d\tau\right)^{\frac{p+q}{r}},$$

where h and $d_{\frac{p}{q}}$ are defined by (3.9) and (1.4), respectively.

THEOREM 3.11. *Let $\nu>0$, $r<0$, $q>0$ and $p\geqslant 0$. Let $\varphi>0$, $w\geqslant 0$ be measurable functions on $[a,x]$, and let f_1, $f_2\in L_r[a,b]$ each of which is of fixed sign a.e. on $[a,b]$, with $\frac{1}{f_1},\frac{1}{f_2}\in L_r[a,b]$. Then the following inequality holds:*

$$\int_a^x w(t)\left(\left|\left(I_{a+}^{\nu}f_1\right)(t)\right|^p|f_2(t)|^q+\left|\left(I_{a+}^{\nu}f_2\right)(t)\right|^p|f_1(t)|^q\right)dt \tag{3.12}$$
$$\geqslant 2^{1-\frac{q}{r}}\left(\frac{q}{p+q}\right)^{\frac{q}{r}}\left(c_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}\left(\int_a^x [h(t)]^{\frac{r}{r-q}}dt\right)^{\frac{r-q}{r}}$$
$$\times\left(\int_a^x \varphi(\tau)\left[|f_1(\tau)|^r+|f_2(\tau)|^r\right]d\tau\right)^{\frac{p+q}{r}},$$

where h and $c_{\frac{p}{q}}$ are defined by (3.9) and (1.7), respectively.

For f_1, $f_2\in L_\infty[0,b]$, $\mu,\nu,\delta,k,l,\gamma\in\mathbb{R}^+$, $\omega\geqslant 0$ if we take $g_1(t)=(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,0^+}f_1)(t)$, $g_2(t)=(\varepsilon^{\gamma}_{\mu,\nu,,\omega,0^+}f_1)(t)$, $\widetilde{g}_1(t)=\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,0^+}f_2\right)(t)$, $\widetilde{g}_2(t)=\left(\varepsilon^{\gamma}_{\mu,\nu,,\omega,0^+}f_2\right)(t)$ and

$$K_1(t,\tau)=\begin{cases}(t-\tau)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(t-\tau)^{\mu}): & 0\leqslant\tau\leqslant t\\ 0: & t<\tau\leqslant b\end{cases},$$

$$K_2(t,\tau)=\begin{cases}(t-\tau)^{\nu-1}E^{\gamma}_{\mu,\nu}(\omega(t-\tau)^{\mu}): & 0\leqslant\tau\leqslant t\\ 0: & t<\tau\leqslant b\end{cases}$$

in Theorem 1.2, by computation

$$\begin{aligned}
&\int_0^x \left(\int_0^t K_1(t,\tau)\,d\tau\right)^{q_1} \left(\int_0^t K_2(t,\tau)\,d\tau\right)^{q_2} dt \\
&= \int_0^x \left(\int_0^t (t-\tau)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(t-\tau)^{\mu})\,d\tau\right)^{q_1} \left(\int_0^t (t-\tau)^{\nu-1} E^{\gamma}_{\mu,\nu}(\omega(t-\tau)^{\mu})\,d\tau\right)^{q_2} dt \\
&= \int_0^x \left(t^{\nu} E^{\gamma,\delta,k}_{\mu,\nu+1,l}(\omega t^{\alpha})\right)^{q_1} \left(e^{\gamma}_{\mu,\nu+1}(\omega t^{\alpha})\right)^{q_2} dt
\end{aligned}$$

we obtain the following theorem.

THEOREM 3.12. *Let* $\mu,\nu,\delta,k,l,\gamma,\mathbb{R}^+,\ \omega,p,q_1,q_2 \geqslant 0$. *If* $w \geqslant 0$ *is measurable function on* $[0,x]$ *and* $f_1,\ f_2 \in L_\infty[0,b]$, *then the following inequality holds:*

$$\begin{aligned}
&\int_0^x w(t)\left[\left|(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,0^+} f_1)(t)\right|^{q_1} \left|\left(\varepsilon^{\gamma}_{\mu,\nu,,\omega,0^+} f_2\right)(t)\right|^{q_2} |f_1(t)|^p \right. \\
&\quad \left. + \left|(\varepsilon^{\gamma}_{\mu,\nu,\omega,0^+} f_1)(t)\right|^{q_2} \left|\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,0^+} f_2\right)(t)\right|^{q_1} |f_2(t)|^p\right] dt \\
&\leqslant \|w\|_\infty \int_0^x \left(t^{\nu} E^{\gamma,\delta,k}_{\mu,\nu+1,l}(\omega t^{\alpha})\right)^{q_1} \left(e^{\gamma}_{\mu,\nu+1}(\omega t^{\alpha})\right)^{q_2} dt \\
&\quad \times \frac{1}{2}\left[\|f_1\|_\infty^{2(q_1+p)} + \|f_1\|_\infty^{2q_2} + \|f_2\|_\infty^{2q_2} + \|f_2\|^{2(q_1+p)}\right].
\end{aligned} \tag{3.13}$$

COROLLARY 3.13. *Let* $\mu,\nu,\gamma \in \mathbb{R}^+,\ \omega,p,q_1,q_2 \geqslant 0$. *If* $f_1,\ f_2 \in L_\infty[0,b]$, *then the following inequality holds:*

$$\begin{aligned}
&\int_0^x \left[\left|(\varepsilon^{\gamma}_{\mu,\nu,\omega,0^+} f_1)(t)\right|^{q_1} \left|\left(\varepsilon^{\gamma}_{\mu,\nu,\omega,0^+} f_2\right)(t)\right|^{q_2} |f_1(t)|^p \right. \\
&\left. + \left|(\varepsilon^{\gamma}_{\mu,\nu,\omega,0^+} f_1)(t)\right|^{q_2} \left|\left(\varepsilon^{\gamma}_{\mu,\nu,\omega,0^+} f_2\right)(t)\right|^{q_1} |f_2(t)|^p\right] dt \\
&\leqslant \left(\int_0^x \left(e^{\gamma}_{\mu,\nu+1}(\omega t^{\alpha})\right)^{q_1+q_2} dt\right) \frac{1}{2}\left[\|f_1\|_\infty^{2(q_1+p)} + \|f_1\|_\infty^{2q_2} + \|f_2\|_\infty^{2q_2} + \|f_2\|^{2(q_1+p)}\right].
\end{aligned} \tag{3.14}$$

Taking $\omega = 0$ in (3.14), we get:

COROLLARY 3.14. *Let $\nu>0,\ p,q_1,q_2\geqslant 0$. If $w\geqslant 0$ is measurable function on $[0,x]$ and $f_1,\ f_2\in L_\infty[0,b]$, then the following inequality holds:*

$$\begin{aligned}&\int_0^x w(t)\Big[\left|(I^{\nu}_{0+}f_1)(t)\right|^{q_1}\left|\left(I^{\nu}_{0+}f_2\right)(t)\right|^{q_2}|f_1(t)|^p \\ &+\left|(I^{\nu}_{0+}f_1)(t)\right|^{q_2}\left|\left(I^{\nu}_{0+}f_2\right)(t)\right|^{q_1}|f_2(t)|^p\Big]dt \\ \leqslant\ &\frac{\|w\|_\infty}{(\Gamma(\nu+1))^{q_1+q_2}}\frac{x^{\nu(q_1+q_2)+1}}{\nu(q_1+q_2)+1} \\ &\times\frac{1}{2}\left[\|f_1\|_\infty^{2(q_1+p)}+\|f_1\|_\infty^{2q_2}+\|f_2\|_\infty^{2q_2}+\|f_2\|^{2(q_1+p)}\right].\end{aligned}\tag{3.15}$$

THEOREM 3.15. *Let $\mu,\nu,\gamma>0,\ r>1,\ r>q>0,\ p\geqslant 0$. If $\varphi>0,\ w\geqslant 0$ are measurable functions on $[0,x]$, $f_1,\ f_2\in L_r[0,b]$ and $f_1,f_2\in AC^m[0,b]$, $m=[\nu]$, then the following inequality holds:*

$$\begin{aligned}&\int_0^x w(t)\Big(\left|\left({}^C D^{\gamma}_{\mu,\nu,\omega,0+}f_1\right)(t)\right|^p\left|\frac{d^m}{dt^m}f_2(t)\right|^q \\ &+\left|\left({}^C D^{\gamma}_{\mu,\nu,\omega,0+}f_2\right)(t)\right|^p\left|\frac{d^m}{dt^m}f_1(t)\right|^q\Big)dt \\ \leqslant\ &2^{1-\frac{q}{r}}\left(\frac{q}{p+q}\right)^{\frac{q}{r}}\left(d_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}\left(\int_0^x [h(t)]^{\frac{r}{r-q}}dt\right)^{\frac{r-q}{r}} \\ &\times\left(\int_0^x\varphi(\tau)\left[\left|\frac{d^m}{d\tau^m}f_1(\tau)\right|^r+\left|\frac{d^m}{d\tau^m}f_2(\tau)\right|^r\right]d\tau\right)^{\frac{p+q}{r}},\end{aligned}\tag{3.16}$$

where

$$h(t)=w(t)\left[\int_0^t\left[e^{-\gamma}_{\mu,m-\nu}(t-\tau,\omega)\right]^{\frac{r}{r-1}}\varphi(\tau)^{\frac{1}{1-r}}d\tau\right]^{\frac{p(r-1)}{r}}[\varphi(t)]^{\frac{-q}{r}},\tag{3.17}$$

and $d_{\frac{p}{q}}$ is defined by (1.4).

THEOREM 3.16. *Let $\mu,\nu,\gamma>0,\ r>1,\ r>q>0,\ p\geqslant 0$. If $\varphi>0,\ w\geqslant 0$ be measurable functions on $[0,x]$, $f_1^{(m)},\ f_2^{(m)}\in L_r[0,b]$ and $f_1,f_2\in AC^m[0,b]$, $f_1*e^{-\gamma}_{\mu,m-\nu,\omega},\ f_2*e^{-\gamma}_{\mu,m-\nu,\omega}\in W^{m,1}[0,b]$, $m=[\nu]$, $f_1^{(k)}(0+)=f_2^{(k)}(0+)=0,\ k=$*

$0,1,2,\ldots m-1$, *then the following inequality holds:*

$$\int_0^x w(t)\Big(\Big|\Big(\mathbf{D}^{\gamma}_{\mu,\nu,\omega,0+}f_1\Big)(t)\Big|^p\Big|\frac{d^m}{dt^m}f_2(t)\Big|^q + \Big|\Big(\mathbf{D}^{\gamma}_{\mu,\nu,\omega,0+}f_2\Big)(t)\Big|^p\Big|\frac{d^m}{dt^m}f_1(t)\Big|^q\Big)dt \tag{3.18}$$
$$\leqslant 2^{1-\frac{q}{r}}\left(\frac{q}{p+q}\right)^{\frac{q}{r}}\left(d_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}\left(\int_0^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}}$$
$$\times\left(\int_0^x \varphi(\tau)\left[\left|\frac{d^m}{d\tau^m}f_1(\tau)\right|^r+\left|\frac{d^m}{d\tau^m}f_2(\tau)\right|^r\right]d\tau\right)^{\frac{p+q}{r}},$$

where h and $d_{\frac{p}{q}}$ are defined by (3.17) and (1.4), respectively.

THEOREM 3.17. *Let $\mu,\nu,\gamma,\omega>0$ $p,q_1,q_2\geqslant 0$, $w\geqslant 0$ be measurable function on $[0,x]$ and $f_1^{(m)}$, $f_2^{(m)}\in L_\infty[0,b]$, $m=[\nu]+1$. Then the following inequality holds:*

$$\int_0^x w(t)\Big[\big|\big({}^CD^{\nu}_{0+}f_1\big)(t)\big|^{q_1}\Big|\Big({}^CD^{\gamma}_{\mu,\nu,\omega,0+}f_2\Big)(t)\Big|^{q_2}\Big|\frac{d^m}{dt^m}f_1(t)\Big|^p + \Big|\Big({}^CD^{\gamma}_{\mu,\nu,\omega,0+}f_1\Big)(t)\Big|^{q_2}\big|\big({}^CD^{\nu}_{0+}f_2\big)(t)\big|^{q_1}\Big|\frac{d^m}{dt^m}f_2(t)\Big|^p\Big]dt \tag{3.19}$$
$$\leqslant \frac{\|w\|_\infty}{(\Gamma(m-\nu+1))^{q_1}}\int_0^x t^{(m-\nu)q_1}\left(e^{-\gamma}_{\mu,m-\nu+1}(\omega t^\mu)\right)^{q_2}dt$$
$$\times\frac{1}{2}\left[\left\|f_1^{(m)}\right\|_\infty^{2(q_1+p)}+\left\|f_1^{(m)}\right\|_\infty^{2q_2}+\left\|f_2^{(m)}\right\|_\infty^{2q_2}+\left\|f_2^{(m)}\right\|_\infty^{2(q_1+p)}\right].$$

Proof. If we take $g_1(t)=\big({}^CD^{\nu}_{0+}f_1\big)(t)$, $g_2(t)=\Big({}^CD^{\gamma}_{\mu,\nu,\omega,0+}f_2\Big)(t)$, $\widetilde{g}_1(t)=\big({}^CD^{\nu}_{0+}f_1\big)(t)$, $\widetilde{g}_2(t)=\Big({}^CD^{\gamma}_{\mu,\nu,\omega,0+}f_2\Big)(t)$,

$$K_1(t,\tau)=\begin{cases}\frac{(t-\tau)^{m-\nu-1}}{\Gamma(m-\nu)}: & 0\leqslant\tau\leqslant t\\ 0: & t<\tau\leqslant\tau\end{cases},$$
$$K_2(t,\tau)=\begin{cases}e^{-\gamma}_{\mu,m-\nu}\left(\omega(t-\tau)^\mu\right): & 0\leqslant\tau\leqslant t\\ 0: & t<\tau\leqslant\tau\end{cases},$$

since

$$\begin{aligned}&\int_0^x\left(\int_0^t K_1(t,\tau)\,d\tau\right)^{q_1}\left(\int_0^t K_2(t,\tau)\,d\tau\right)^{q_2}dt\\&=\frac{1}{(\Gamma(m-\nu))^{q_1}}\int_0^x\left(\int_0^t(t-\tau)^{m-\nu-1}\,d\tau\right)^{q_1}\left(\int_0^t e^{-\gamma}_{\mu,m-\nu}\left(\omega(t-\tau)^{\mu}\right)d\tau\right)^{q_2}dt\\&=\frac{1}{(\Gamma(m-\nu+1))^{q_1}}\int_0^x t^{(m-\nu)q_1}\left(e^{-\gamma}_{\mu,m-\nu+1}\left(\omega t^{\mu}\right)\right)^{q_2}dt,\end{aligned}$$

by applying the Theorem 1.2 we obtain (3.19). □

4. Multiple Opial-type inequalities

In this section we give multiple Opial-type inequalities for Hilfer differential operator, Prabhakar integral operator, Caputo Prabhakar derivative, generalized fractional integral operator. Iqbal et al. in [10] proved the following new multiple Opial-type inequalities for general kernels, which proofs are based on a technique from [5].

THEOREM 4.1. *Let* $y_i \in U(K_i,f)$, $i=1,2,...,N$, $N\in\mathbb{N}$. *Let* w_1 *and* w_2 *be continuous weight functions on* $[a,x]$ *with* $w_1 \geqslant 0$ *and* $w_2>0$. *Let* $r_i \geqslant 0$, $r=\sum_{i=1}^{N} r_i>0$, $p>0$, $q\geqslant 0$, $\sigma=\frac{1}{p+q}<1$ *and* $f\in L_{p+q}[a,b]$. *Then*

$$\begin{aligned}&\int_a^x w_1(t)\prod_{i=1}^{N}|y_i(t)|^{r_ip}\,|f(t)|^q\,dt \qquad (4.1)\\&\leqslant\left(\frac{q}{rp+q}\right)^{\sigma q}\left(\int_a^x[w_1(t)]^{\frac{1}{\sigma p}}\,[w_2(t)]^{\frac{-q}{p}}\right.\\&\left.\times\prod_{i=1}^{N}\left(\int_a^t[w_2(\tau)]^{-\frac{\sigma}{1-\sigma}}\,|K_i(t,\tau)|^{\frac{1}{1-\sigma}}\,d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}}dt\right)^{\sigma p}\left(\int_a^x w_2(t)\,|f(t)|^{\frac{1}{\sigma}}\,dt\right)^{\sigma(rp+q)}.\end{aligned}$$

THEOREM 4.2. *Suppose that assumptions of the Theorem 4.1 hold. Suppose also that* w_1 *is an increasing and* w_2 *is decreasing functions. Then*

$$\int_a^x w_1(t)\prod_{i=1}^N |y_i(t)|^{r_ip}|f(t)|^q\,dt \tag{4.2}$$
$$\leqslant \left(\frac{q}{rp+q}\right)^{\sigma q} w_1(x)\left[w_2(x)\right]^{-\sigma(rp+q)}$$
$$\times\left(\int_a^x \prod_{i=1}^N \left(\int_a^t |K_i(t,\tau)|^{\frac{1}{1-\sigma}}\,d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}} dt\right)^{\sigma p}\left(\int_a^x w_2(t)\,|f(t)|^{\frac{1}{\sigma}}\,dt\right)^{\sigma(rp+q)}.$$

Let $D_{a+}^{\mu_i+\nu_i-\mu_i\nu_i} f \in L_{p+q}[a,b]$, $0<\mu_i<1, 0<\nu_i\leqslant 1$, $i=1,2,..N$, $p>0$, $q\geqslant 0$. Replacing y_i by $D_{a+}^{\mu_i,\nu_i} f$, f by $D_{a+}^{\mu_i+\nu_i-\mu_i\nu_i} f$, $i=1,2,..,N$ and taking particular kernel

$$K_i(t,\tau)=\begin{cases}\frac{(t-\tau)^{\nu_i(1-\mu_i)-1}}{\Gamma(\nu_i(1-\mu_i))}, & a\leqslant\tau\leqslant t\\ 0, & t<\tau\leqslant b\end{cases} \tag{4.3}$$

in Theorem 4.1 and Theorem 4.2. Then we get following multiple Opial-type inequalities.

THEOREM 4.3. *Let* $0<\mu_i<1, 0<\nu_i\leqslant 1$, $i=1,2,..N$, $\omega\geqslant 0$, w_1 *and* w_2 *be continuous weight functions on* $[a,x]$ *with* $w_1\geqslant 0$ *and* $w_2>0$. *Let* $r_i\geqslant 0$, $r=\sum_{i=1}^N r_i>0$, $p>0$, $q\geqslant 0$, $\sigma=\frac{1}{p+q}<1$ *and* $D_{a+}^{\mu_i+\nu_i-\mu_i\nu_i} f\in L_{p+q}[a,b]$. *Then*

$$\int_a^x w_1(t)\prod_{i=1}^N \left|(D_{a+}^{\mu_i,\nu_i} f)(t)\right|^{r_ip}\left|D_{a+}^{\mu_i+\nu_i-\mu_i\nu_i} f(t)\right|^q dt \tag{4.4}$$
$$\leqslant\left(\frac{q}{rp+q}\right)^{\sigma q}\left(\int_a^x [w_1(t)]^{\frac{1}{\sigma p}}\,[w_2(t)]^{\frac{-q}{p}}\right.$$
$$\left.\times\prod_{i=1}^N\left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}}\left|\frac{(t-\tau)^{\nu_i(1-\mu_i)-1}}{\Gamma(\nu_i(1-\mu_i))}\right|^{\frac{1}{1-\sigma}}d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}}dt\right)^{\sigma p}$$
$$\times\left(\int_a^x w_2(t)\left|D_{a+}^{\mu_i+\nu_i-\mu_i\nu_i} f\right|^{\frac{1}{\sigma}}dt\right)^{\sigma(rp+q)}.$$

Moreover, if w_1 *is an increasing and* w_2 *is decreasing functions, then*

$$\int_a^x w_1(t)\prod_{i=1}^N \left|(D_{a+}^{\mu_i,\nu_i} f)(t)\right|^{r_ip}\left|D_{a+}^{\mu_i+\nu_i-\mu_i\nu_i} f(t)\right|^q dt \tag{4.5}$$
$$\leqslant\left(\frac{q}{rp+q}\right)^{\sigma q} w_1(x)\left[w_2(x)\right]^{-\sigma(rp+q)}$$

$$\times\left(\int_a^x \prod_{i=1}^N \left(\int_a^t \left|\frac{(t-\tau)^{\nu_i(1-\mu_i)-1}}{\Gamma(\nu_i(1-\mu_i))}\right|^{\frac{1}{1-\sigma}} d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}} dt\right)^{\sigma p}$$
$$\times\left(\int_a^x w_2(t)\left|D_{a+}^{\mu_i+\nu_i-\mu_i\nu_i} f\right|^{\frac{1}{\sigma}} dt\right)^{\sigma(rp+q)}.$$

COROLLARY 4.4. [10] *Let* $0<\mu_i<1,\ i=1,2,..N,\ \omega\geqslant 0,\ w_1$ *and* w_2 *are continuous weight functions on* $[a,x]$ *with* $w_1\geqslant 0$ *and* $w_2>0$. *Let* $r_i\geqslant 0,\ r=\sum_{i=1}^N r_i>0,$ $p>0,\ q\geqslant 0,\ \sigma=\frac{1}{p+q}<1$ *and* $f'\in L_{p+q}[a,b]$. *Then*

$$\int_a^x w_1(t)\prod_{i=1}^N \left|\left({}^C D_{a+}^{\mu_i} f\right)(t)\right|^{r_i p}\left|\frac{d}{dt}f(t)\right|^q dt \tag{4.6}$$
$$\leqslant\left(\frac{q}{rp+q}\right)^{\sigma q}\left(\int_a^x [w_1(t)]^{\frac{1}{\sigma p}}[w_2(t)]^{\frac{-q}{p}}\right.$$
$$\left.\times\prod_{i=1}^N\left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}}\left|\frac{(t-\tau)^{-\mu_i}}{\Gamma(1-\mu_i)}\right|^{\frac{1}{1-\sigma}} d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}} dt\right)^{\sigma p}$$
$$\times\left(\int_a^x w_2(t)\left|\frac{d}{dt}f(t)\right|^{\frac{1}{\sigma}} dt\right)^{\sigma(rp+q)}.$$

Moreover, if w_1 *is an increasing and* w_2 *is decreasing functions. Then*

$$\int_a^x w_1(t)\prod_{i=1}^N \left|\left({}^C D_{a+}^{\mu_i} f\right)(t)\right|^{r_i p}\left|\frac{d}{dt}f(t)\right|^q dt \tag{4.7}$$
$$\leqslant\left(\frac{q}{rp+q}\right)^{\sigma q} w_1(x)[w_2(x)]^{-\sigma(rp+q)}\left(\int_a^x\prod_{i=1}^N\left(\int_a^t\left|\frac{(t-\tau)^{-\mu_i}}{\Gamma(1-\mu_i)}\right|^{\frac{1}{1-\sigma}} d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}} dt\right)^{\sigma p}$$
$$\times\left(\int_a^x w_2(t)\left|\frac{d}{dt}f(t)\right|^{\frac{1}{\sigma}} dt\right)^{\sigma(rp+q)}.$$

Let $f\in L_{p+q}[a,b],\ p>0,\ q\geqslant 0$. Then by replacing y_i by $(\varepsilon_{\mu_i,\nu_i,l,\omega,a^+}^{\gamma_i,\delta,k} f)(t)$, where $\mu_i,\nu_i,\gamma_i,\ i=1,2,..,N$, k,l,δ are positive real numbers, $\omega\geqslant 0$ and taking particular kernel

$$K_i(t,\tau) = \begin{cases} (t-\tau)^{\nu-1} E^{\gamma_i,\delta,k}_{\mu_i,\nu_i,l}(\omega(t-\tau)^{\mu}), & a \leqslant \tau \leqslant t \\ 0, & t < \tau \leqslant b \end{cases} \tag{4.8}$$

in Theorem 4.1 and Theorem 4.2, we get:

THEOREM 4.5. *Let* $\mu_i, \nu_i, \gamma_i,\ i=1,2,..,N,\ k,l,\delta$ *be positive real numbers,* $\omega \geqslant 0$, w_1 *and* w_2 *be continuous weight functions on* $[a,x]$ *with* $w_1 \geqslant 0$ *and* $w_2 > 0$. *Let* $r_i \geqslant 0,\ r = \sum_{i=1}^{N} r_i > 0,\ p > 0,\ q \geqslant 0,\ \sigma = \frac{1}{p+q} < 1$ *and* $f \in L_{p+q}[a,b]$. *Then*

$$\begin{aligned} &\int_a^x w_1(t) \prod_{i=1}^{N} \left| (\varepsilon^{\gamma_i,\delta,k}_{\mu_i,\nu_i,l,\omega,a^+} f)(t) \right|^{r_i p} |f(t)|^q\, dt \\ &\leqslant \left(\frac{q}{rp+q} \right)^{\sigma q} \left(\int_a^x [w_1(t)]^{\frac{1}{\sigma p}} [w_2(t)]^{\frac{-q}{p}} \right. \\ &\quad \left. \times \prod_{i=1}^{N} \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} \left| (t-\tau)^{\nu_i-1} E^{\gamma_i,\delta,k}_{\mu_i,\nu_i,l}(\omega(t-\tau)^{\mu_i}) \right|^{\frac{1}{1-\sigma}} d\tau \right)^{\frac{(1-\sigma)r_i}{\sigma}} dt \right)^{\sigma p} \\ &\quad \times \left(\int_a^x w_2(t) |f(t)|^{\frac{1}{\sigma}}\, dt \right)^{\sigma(rp+q)}. \end{aligned} \tag{4.9}$$

Moreover, if w_1 *is an increasing and* w_2 *is decreasing functions. Then*

$$\begin{aligned} &\int_a^x w_1(t) \prod_{i=1}^{N} \left| (\varepsilon^{\gamma_i,\delta,k}_{\mu_i,\nu_i,l,\omega,a^+} f)(t) \right|^{r_i p} |f(t)|^q\, dt \\ &\leqslant \left(\frac{q}{rp+q} \right)^{\sigma q} w_1(x) [w_2(x)]^{-\sigma(rp+q)} \\ &\quad \times \left(\int_a^x \prod_{i=1}^{N} \left(\int_a^t \left| (t-\tau)^{\nu_i-1} E^{\gamma_i,\delta,k}_{\mu_i,\nu_i,l}(\omega(t-\tau)^{\mu_i}) \right|^{\frac{1}{1-\sigma}} d\tau \right)^{\frac{(1-\sigma)r_i}{\sigma}} dt \right)^{\sigma p} \\ &\quad \times \left(\int_a^x w_2(t) |f(t)|^{\frac{1}{\sigma}}\, dt \right)^{\sigma(rp+q)}. \end{aligned} \tag{4.10}$$

If we take $\omega = 0$, we obtain the following corollary.

COROLLARY 4.6. *Let* $\nu_i,\ i=1,2,..N$, *be positive real numbers,* $\omega \geqslant 0,\ w_1$ *and* w_2 *are continuous weight functions on* $[a,x]$ *with* $w_1 \geqslant 0$ *and* $w_2 > 0$. *Let* $r_i \geqslant 0$, $r=\sum_{i=1}^{N} r_i > 0,\ p>0,\ q\geqslant 0,\ \sigma=\frac{1}{p+q}<1$ *and* $f \in L_{p+q}[a,b]$. *Then*

$$\int_a^x w_1(t)\prod_{i=1}^{N}\left|(I_{a+}^{\nu_i}f)(t)\right|^{r_ip}|f(t)|^q\,dt \tag{4.11}$$
$$\leqslant \left(\frac{q}{rp+q}\right)^{\sigma q}\left(\int_a^x [w_1(t)]^{\frac{1}{\sigma p}}[w_2(t)]^{\frac{-q}{p}}\right.$$
$$\left.\times\prod_{i=1}^{N}\left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}}\left|\frac{(t-\tau)^{\nu_i-1}}{\Gamma(\nu_i)}\right|^{\frac{1}{1-\sigma}}d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}}dt\right)^{\sigma p}$$
$$\times\left(\int_a^x w_2(t)|f(t)|^{\frac{1}{\sigma}}\,dt\right)^{\sigma(rp+q)}.$$

Moreover, if w_1 *is an increasing and* w_2 *is decreasing functions. Then*

$$\int_a^x w_1(t)\prod_{i=1}^{N}\left|(I_{a+}^{\nu_i}f)(t)\right|^{r_ip}|f(t)|^q\,dt \tag{4.12}$$
$$\leqslant \left(\frac{q}{rp+q}\right)^{\sigma q} w_1(x)[w_2(x)]^{-\sigma(rp+q)}$$
$$\times\left(\int_a^x\prod_{i=1}^{N}\left(\int_a^t\left|\frac{(t-\tau)^{\nu_i-1}}{\Gamma(\nu_i)}\right|^{\frac{1}{1-\sigma}}d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}}dt\right)^{\sigma p}\left(\int_a^x w_2(t)|f(t)|^{\frac{1}{\sigma}}\,dt\right)^{\sigma(rp+q)}.$$

In the following we emphasize that results in [10] *can be seen for particular kernels as more fractional multiple Opial-type inequalities*

REMARK 4.7. Let $N\in\mathbb{N}$, w be continuous positive weight function on $[a,x]$. Let $r_i\geqslant 0,\ r=\sum_{i=1}^{N} r_i>0,\ p>0,\ q\geqslant 0,\ \sigma=\frac{1}{p+q}<1,\ D_{a+}^{\mu_i+\nu_i-\mu_i\nu_i}f\in L_{p+q}[a,b]$, $y_i=D_{a+}^{\mu_i,\nu_i}f,\ 0<\mu_i<1, 0<\nu_i\leqslant 1,\ i=1,2,...,N$ and particular kernel K_i is given by (4.3). Then by using Theorem 3.1 of [10] we can have multiple Opial-type fractional inequalities for Hilfer, Prabhakar, Caputo Prabhakar operators. Moreover if $r\geqslant 1$, w is decreasing function or $A\leqslant w(t)\leqslant B,\ t\in[a,x]$, then for the same functions y_i and same particular kernel K_i of (4.3) inequalities of Theorem 3.2 and Theorem 3.4 of [10] appear as fractional multiple Opial-type inequalities. The same results of Theorem 3.1, Theorem 3.2 and Theorem 3.4 of [10] provide multiple Opial-type fractional inequalities containing Mittag–Leffler function if y_i is replaced with generalized fractional integral operator $(\varepsilon^{\gamma_i,\delta,k}_{\mu_i,\nu_i,l,\omega,a^+}f)(t)$, where μ_i,ν_i,γ_i, $i=1,2,..,N$, k,l,δ are positive real numbers, $\omega\geqslant 0$, $f\in L_{p+q}[a,b]$ and particular kernel K_i is given by (4.8). Theorem 3.5 of [10] for non-weighted case of multiple Opial inequality also holds true in both cases of y_i and K_i, considered above.

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