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HERMITE INTERPOLATION OF COMPOSITION FUNCTION AND STEFFENSEN–TYPE INEQUALITIES

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Abstract. Hermite interpolation of composition function and some related inequalities are given. The obtained inequalities are closely related to a generalization of Steffensen's inequality given by the current authors in [2].

1. Introduction

Steffensen [7] proved the following inequality: if $f,h:[\alpha,\beta]\to\mathbb{R}$, $0\leqslant h\leqslant 1$ and f is decreasing, then

$$\int_\alpha^\beta f(t)h(t)\,dt \leqslant \int_\alpha^{\alpha+\gamma} f(t)\,dt, \qquad \text{where } \gamma=\int_\alpha^\beta h(t)\,dt. \tag{1}$$

A few hundred papers are devoted to studying generalizations of Steffensen's inequality (1). One recent is given by Rabier [6].

THEOREM 1. *Let $\phi:[0,\infty)\to\mathbb{R}$ be convex and continuous with $\phi(0)=0$. If $b>0$ and $h\in L^\infty(0,b), h\geqslant 0$ and $\|h\|_\infty\leqslant 1$, then $h\phi^{(1)}\in L^1(0,b)$ and*

$$\phi\Big(\int_0^b h(t)\,dt\Big)\leqslant \int_0^b h(t)\phi^{(1)}(t)\,dt \tag{2}$$

In fact, Rabier's result is closely related to another generalization of Steffensen's inequality given by Pečarić [4].

THEOREM 2. *Let $g:[a,b]\to\mathbb{R}$ be a nondecreasing and differentiable function and $f:I\to\mathbb{R}$ be a nondecreasing function (I is an interval in $\mathbb{R}$ such that $a,b,g(a)$, $g(b)\in I$).*

(a) If $g(x)\leqslant x$, then

$$\int_a^b f(t)g^{(1)}(t)\,dt \geqslant \int_{g(a)}^{g(b)} f(t)\,dt. \tag{3}$$

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(b) If $g(x) \geqslant x$, then the reverse of the above inequality holds.

The assumptions of Theorem 2 can be weakened and differentiability of g can be replaced with absolute continuity. Indeed, for a nondecreasing function f, the function $F(x) = \int_a^x f(t)\,dt$ is well defined and satisfies $F^{(1)} = f$ at all except at most countably many points. For absolutely continuous nondecrasing function g the substitution $z = g(t)$ in the integral is justified (see [3, Corollary 20.5]), so

$$F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(z)\,dz = \int_a^b f(g(t))g^{(1)}(t)\,dt \leqslant \int_a^b f(t)g^{(1)}(t)\,dt, \quad (4)$$

where the last inequality holds when $g(t) \leqslant t$.

Steffensen's inequality (1) follows from Theorem 2 by making substitutions $g(x) \mapsto \int_a^x h(t+\alpha-a)\,dt + a$ and $f(x) \mapsto -f(x+\alpha-a)$ and taking $b = \beta - \alpha + a$.

Moreover, a convex function ϕ from Theorem 1 has a nondecreasing right-sided derivative $\phi_+^{(1)}$ such that $\phi(x) = \int_0^x \phi_+^{(1)}(t)\,dt$. Furthermore, for a function $h : [0,b] \to [0,1]$, the function $g(x) = \int_0^x h(t)dt$ is absolutely continuous and satisfies $g(x) \leqslant x$ and $g^{(1)} = h$. Therefore, by taking $a = 0$, $f = \phi_+^{(1)}$ and $g(x) = \int_0^x h(t)\,dt$ in Theorem 2 (under the weaker assumptions) we get Theorem 1.

By replacing the equality

$$F(g(x)) = F(g(a)) + \int_{g(a)}^{g(x)} f(t)\,dt$$

with the n-th order Taylor expansion of the composition $F \circ g$, Fahad, Pečarić and Praljak [2] obtained the following generalization of Theorem 2.

THEOREM 3. *Let $n \in \mathbb{N}$. Let $g : [a,b] \to \mathbb{R}$ and $F : I \to \mathbb{R}$ (where I is an interval in $\mathbb{R}$ such that $a, b, g(a), g(b) \in I$) be two n times differentiable functions such that $g^{(1)}, g^{(2)}, \dots g^{(n)}$, $F^{(1)}, F^{(2)}, \dots, F^{(n)}$ are nondecreasing functions. If $g(x) \leqslant x$, then*

$$\begin{aligned} F(g(b)) \leqslant F(g(a)) &+ \sum_{k=1}^{n-1} F^{(k)}(g(a)) \sum_{i=k}^{n-1} B_{i,k}(g^{(1)}(a), \dots, g^{(i-k+1)}(a)) \frac{(b-a)^i}{i!} \\ &+ \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} \sum_{k=1}^{n} F^{(k)}(t) B_{n,k}(g^{(1)}(t), \dots, g^{(n-k+1)}(t))\,dt \end{aligned}$$

where $B_{m,n}(g^{(1)}(t), \dots, g^{(m-n+1)}(t))$ are the Bell polynomials.

The Bell polynomial $B_{m,k}(x_1, x_2, \dots, x_{m-k+1})$ with variables $x_1, x_2, \dots, x_{m-k+1}$ is

$$B_{m,k}(x_1, x_2, \dots, x_{m-k+1}) = \sum \frac{m!}{j_1! j_2! \cdots j_{m-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{m-k+1}}{(m-k+1)!}\right)^{j_{m-k+1}}$$

where the sum is taken over all sequences $j_1, j_2, \dots, j_{m-k+1}$ of non-negative integers such that

$$j_1 + j_2 + \ldots = k \quad \text{and} \quad j_1 + 2j_2 + 3j_3 + \ldots = m.$$

For the ease of notation later on, we also set $B_{0,0} = 1$ and $B_{m,0} = 0$ for $m \geqslant 1$.

The Bell polynomials appear in Faà di Bruno's formula which gives higher order derivatives of composition function $F \circ g$

$$\frac{d^m}{dx^m} F(g(x)) = \sum_{k=0}^{m} F^{(k)}(g(x)) B_{m,k}(g^{(1)}(x), \ldots, g^{(m-k+1)}(x)). \tag{5}$$

In this paper we will derive integral identities and related inequalities by replacing Taylor's expansion with Hermite interpolation.

We will first mention some results regarding Hermite interpolation used in this paper (for details see, e. g., [1]). Let $-\infty < a \leqslant a_1 < a_2 < \cdots < a_r \leqslant b < \infty$. For $H \in C^n[a,b]$ there exists a unique polynomial P_H of degree $n-1$, called the Hermite interpolating polynomial of the function H, satisfying

$$P_H^{(i)}(a_j) = H^{(i)}(a_j), \quad 0 \leqslant i \leqslant k_j,\ 1 \leqslant j \leqslant r,\ \sum_{j=1}^{r} k_j + r = n.$$

Explicit expression for P_H is

$$P_H(x) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(x) H^{(i)}(a_j),$$

where H_{ij} are the polynomials

$$H_{ij}(x) = \frac{1}{i!} \frac{w(x)}{(x-a_j)^{k_j+1-i}} \sum_{k=0}^{k_j-i} \frac{1}{k!} \frac{d^k}{dx^k} \left(\frac{(x-a_j)^{k_j+1}}{w(x)} \right) \Big|_{x=a_j} (x-a_j)^k,$$

where

$$w(x) = \prod_{j=1}^{r} (x-a_j)^{k_j+1}.$$

The error of the interpolation can be expressed as

$$e_H(x) = H(x) - P_H(x) = \int_a^b G_H(x,s) H^{(n)}(s),$$

where G_H is Green's function for Hermite interpolation given by

$$G_H(x,s) = \begin{cases} \sum_{j=1}^{l} \sum_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(x), & s \leqslant x, \\ -\sum_{j=l+1}^{r} \sum_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(x), & s \geqslant x \end{cases} \tag{6}$$

for all $a_l \leqslant s \leqslant a_{l+1}$, $l = 0, 1, \ldots, r$ $(a_0 = a, a_{r+1} = b)$.

Following are some special cases of Hermite interpolation of functions:

(i) Taylor's two-point condition: $m \in \mathbb{N}$, $n = 2m$, $r = 2$, $a_1 = a$, $a_2 = b$ and $k_1 = k_2 = m-1$. In this case

$$H(x) = \sum_{i=0}^{m-1} \sum_{k=0}^{m-i-1} \binom{m+k-1}{k} \Big[\frac{(x-a)^i}{i!} \left(\frac{x-b}{a-b}\right)^m \left(\frac{x-a}{b-a}\right)^k H^{(i)}(a) + \frac{(x-b)^i}{i!} \left(\frac{x-a}{b-a}\right)^m \left(\frac{x-b}{a-b}\right)^k H^{(i)}(b) \Big] + \int_a^b G_m(x,s) H^{(2m)}(s) ds,$$

where Green's function G_m is of the form

$$G_m(x,s) = \frac{(-1)^m}{(2m-1)!} \begin{cases} p^m(x,s) \sum_{k=0}^{m-1} \binom{m+k-1}{k} (x-s)^{m-1-k} q^k(x,s), & s \leqslant x, \\ q^m(x,s) \sum_{k=0}^{m-1} \binom{m+k-1}{k} (s-x)^{m-1-k} p^k(x,s), & x \leqslant s, \end{cases} \tag{7}$$

where $p(x,s) = \frac{(s-a)(b-x)}{(b-a)}$ and $q(x,s) = p(s,x)$.

(ii) $(m, n-m)$ conditions: $r = 2$, $a_1 = a$, $a_2 = b$, $1 \leqslant m \leqslant n-1$, $k_1 = m-1$ and $k_2 = n-m-1$. In this case

$$H(x) = \sum_{i=0}^{m-1} \tau_i(x) H^{(i)}(a) + \sum_{i=0}^{n-m-1} \eta_i(x) H^{(i)}(b) + \int_a^b G_{m,n}(x,s) H^{(n)}(s) ds,$$

where

$$\tau_i(x) = \frac{1}{i!}(x-a)^i \left(\frac{x-b}{a-b}\right)^{n-m} \sum_{k=0}^{m-1-i} \binom{n-m+k-1}{k} \left(\frac{x-a}{b-a}\right)^k,$$

$$\eta_i(x) = \frac{1}{i!}(x-b)^i \left(\frac{x-a}{b-a}\right)^{m} \sum_{k=0}^{n-m-1-i} \binom{m+k-1}{k} \left(\frac{x-b}{a-b}\right)^k$$

and Green's function $G_{m,n}$ is of the form

$$G_{m,n}(x,s) = \begin{cases} \sum_{j=0}^{m-1} \left[\sum_{p=0}^{m-1-j} \binom{n-m+p-1}{p} \left(\frac{x-a}{b-a}\right)^p \right] \frac{(x-a)^j (a-s)^{n-j-1}}{j!(n-j-1)!} \left(\frac{b-x}{b-a}\right)^{n-m}, & s \leqslant x, \\ -\sum_{i=0}^{n-m-1} \left[\sum_{q=0}^{n-m-1-i} \binom{m+q-1}{q} \left(\frac{b-x}{b-a}\right)^q \right] \frac{(x-b)^i (b-s)^{n-i-1}}{i!(n-i-1)!} \left(\frac{x-a}{b-a}\right)^{m}, & s \geqslant x \end{cases} \tag{8}$$

The following lemma describes positivity of Green's function (6) (see Lemma 2.3.3, page 75, in [1]).

LEMMA 1. *Green's function G_H given by (6) satisfies*

$$\frac{G_H(x,s)}{w(x)} > 0, \quad \text{for } a_1 \leqslant x \leqslant a_r,\ a_1 < s < a_r.$$

In this paper we will give Hermite interpolation of composition functions. Moreover, by using positivity of Green's function of Hermite interpolation and its special cases, we will give generalized Steffensen-type inequalities.

2. Main results

The following lemma gives Hermite interpolation for composition functions.

LEMMA 2. *Let* $-\infty < a \leqslant a_1 < a_2 < \cdots < a_r \leqslant b < \infty$, $\sum_{j=1}^{r} k_j + r = n$, *and let* $g : [a,b] \to I$ *and* $F : I \to \mathbb{R}$ *be two* n *times differentiable functions. Then*

$$F \circ g(x) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} \sum_{k=0}^{k_j-i} \frac{1}{i!k!} \frac{d^k}{dx^k} \Big(\frac{(x-a_j)^{k_j+1}}{w(x)} \Big)\Big|_{x=a_j} \frac{w(x)}{(x-a_j)^{k_j+1-i-k}} \sum_{l=0}^{i} F^{(l)}(g(a_j))$$

$$B_{i,l}(g^{(1)}(a_j),\ldots,g^{(i-l+1)}(a_j)) + \int_a^b G_H(x,s) \sum_{l=1}^{n} F^{(l)}(g(s)) B_{n,l}(g^{(1)}(s),\ldots,g^{(n-l+1)}(s))ds.$$

Proof. The proof of lemma can be obtained by applying Hermite interpolation for the function $H = F \circ g$ and then using Faà di Bruno's formula. □

We also have the following two identities corresponding to special cases of Hermite interpolation.

(i) Two-point Taylor form:

$$\begin{aligned} F \circ g(x) = \sum_{i=0}^{m-1} \sum_{k=0}^{m-i-1} \binom{m+k-1}{k} \Big[\frac{(x-a)^i}{i!} \Big(\frac{x-b}{a-b} \Big)^m \Big(\frac{x-a}{b-a} \Big)^k \\ \times \sum_{l=0}^{i} F^{(l)}(g(a)) B_{i,l}(g^{(1)}(a),\ldots,g^{(i-l+1)}(a)) \\ + \frac{(x-b)^i}{i!} \Big(\frac{x-a}{b-a} \Big)^m \Big(\frac{x-b}{a-b} \Big)^k \sum_{l=0}^{i} F^{(l)}(g(b)) B_{i,l}(g^{(1)}(b),\ldots,g^{(i-l+1)}(b)) \Big] \\ + \int_a^b G_m(x,s) \sum_{l=1}^{2m} F^{(l)}(g(s)) B_{2m,l}(g^{(1)}(s),\ldots,g^{(2m-l+1)}(s)))ds. \end{aligned}$$

(ii) $(m,n-m)$ form:

$$\begin{aligned} F \circ g(x) = \sum_{i=0}^{m-1} \tau_i(x) \sum_{l=0}^{i} F^{(l)}(g(a)) B_{i,l}(g^{(1)}(a),\ldots,g^{(i-l+1)}(a)) \\ + \sum_{i=0}^{n-m-1} \eta_i(x) \sum_{l=0}^{i} F^{(l)}(g(b)) B_{i,l}(g^{(1)}(b),\ldots,g^{(i-l+1)}(b)) \\ + \int_a^b G_{m,n}(x,s) \sum_{l=1}^{n} F^{(l)}(g(s)) B_{n,l}(g^{(1)}(s),\ldots,g^{(n-l+1)}(s))ds \end{aligned}$$

The following theorem gives our main result.

THEOREM 4. *Let* $-\infty < a_1 < a_2 < \cdots < a_r < \infty$, $\sum_{j=1}^{r} k_j + r = n$, $g : [a_1, a_r] \to \mathbb{R}$ *and* $F : I \to \mathbb{R}$ *(where* I *is an interval in* $\mathbb{R}$ *such that* $a_1, a_r, g(a_1), g(a_r) \in I$*) be two* n *times differentiable functions such that* $g, g^{(1)}, \ldots g^{(n-1)}$, $F^{(1)}, F^{(2)}, \ldots, F^{(n)}$ *are nondecreasing functions. Then:*

(a) *If* $g(y) \leqslant y$ *and* $k_2, k_3, \ldots, k_r$ *are odd, then for every* $x \in [a_1, a_r]$

$$F \circ g(x) \leqslant \sum_{j=1}^{r} \sum_{i=0}^{k_j} \sum_{k=0}^{k_j - i} \frac{1}{i!k!} \frac{d^k}{dx^k} \Big(\frac{(x-a_j)^{k_j+1}}{w(x)} \Big) \Big|_{x=a_j} \frac{w(x)}{(x-a_j)^{k_j+1-i-k}}$$
$$\times \sum_{l=0}^{i} F^{(l)}(g(a_j)) B_{i,l}(g^{(1)}(a_j), \ldots, g^{(i-l+1)}(a_j))$$
$$+ \int_{a_1}^{a_r} G_H(x,s) \sum_{l=1}^{n} F^{(l)}(s) B_{n,l}(g^{(1)}(s), \ldots, g^{(n-l+1)}(s)) ds.$$

(b) *If* $g(y) \leqslant y$ *and* $k_2, k_3, \ldots, k_{r-1}$ *are odd and* k_r *is even, then reverse of the inequality in* (a) *holds.*

(c) *If* $g(y) \geqslant y$ *and* $k_2, k_3, \ldots, k_r$ *are odd, then the reverse of the inequality in* (a) *holds.*

(d) *If* $g(y) \geqslant y$ *and* $k_2, k_3, \ldots, k_{r-1}$ *are odd and* k_r *is even, then the inequality in* (a) *holds.*

Proof. Applying Lemma 2 with $a = a_1$ and $b = a_r$ we get

$$F \circ g(x) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} \sum_{k=0}^{k_j - i} \frac{1}{i!k!} \frac{d^k}{dx^k} \Big(\frac{(x-a_j)^{k_j+1}}{w(x)} \Big) \Big|_{x=a_j} \frac{w(x)}{(x-a_j)^{k_j+1-i-k}}$$
$$\times \sum_{l=0}^{i} F^{(l)}(g(a_j)) B_{i,l}(g^{(1)}(a_j), \ldots, g^{(i-l+1)}(a_j))$$
$$+ \int_{a_1}^{a_r} G_H(x,s) \sum_{l=1}^{n} F^{(l)}(g(s)) B_{n,l}(g^{(1)}(s), \ldots, g^{(n-l+1)}(s)) ds.$$

By the assumptions of the theorem, $g^{(i)} \geqslant 0$ for $i = 1, \ldots, n$, so the Bell polynomials evaluated at the derivatives of g in the above expression are nonnegative. Furthermore, the assumptions in part (a) or (c) yield $w(x) \geqslant 0$, so $G_H(x,s) \geqslant 0$ by Lemma 1. Therefore, if $g(y) \leqslant y$ we have

$$\int_{a_1}^{a_r} G_H(x,s) \sum_{l=1}^{n} F^{(l)}(g(s)) B_{n,l}(g^{(1)}(s), \ldots, g^{(n-l+1)}(s)) ds$$
$$\leqslant \int_{a_1}^{a_r} G_H(x,s) \sum_{l=1}^{n} F^{(l)}(s) B_{n,l}(g^{(1)}(s), \ldots, g^{(n-l+1)}(s)) ds,$$

while the inequality is reversed if $g(y) \geqslant y$, which proves parts (a) and (c). Similarly, the assumptions in part (b) or (c) yield $w(x) \leqslant 0$, so we obtain the opposite inequalities. $\square$

By using the two-point Taylor form we obtain the following result.

COROLLARY 1. *Let $m \in \mathbb{N}$. Let $g : [a,b] \to \mathbb{R}$ and $F : I \to \mathbb{R}$ (where I is an interval in $\mathbb{R}$ such that $a,b,g(a),g(b) \in I$) be two $2m$ times differentiable functions such that $g, g^{(1)}, \dots, g^{(2m-1)}$, $F^{(1)}, F^{(2)}, \dots, F^{(2m)}$ are nondecreasing functions. Then:*

(a) *If $g(y) \leqslant y$ and m is even, then for every $x \in [a,b]$*

$$\begin{aligned}
F \circ g(x) \leqslant & \sum_{i=0}^{m-1} \sum_{k=0}^{m-i-1} \binom{m+k-1}{k} \Big[\frac{(x-a)^i}{i!} \left(\frac{x-b}{a-b}\right)^m \left(\frac{x-a}{b-a}\right)^k \\
& \times \sum_{l=0}^{i} F^{(l)}(g(a)) B_{i,l}(g^{(1)}(a), \dots, g^{(i-l+1)}(a)) \\
& + \frac{(x-b)^i}{i!} \left(\frac{x-a}{b-a}\right)^m \left(\frac{x-b}{a-b}\right)^k \sum_{l=0}^{i} F^{(l)}(g(b)) B_{i,l}(g^{(1)}(b), \dots, g^{(i-l+1)}(b)) \Big] \\
& + \int_a^b G_m(x,s) \sum_{l=1}^{2m} F^{(l)}(s) B_{2m,l}(g^{(1)}(s), \dots, g^{(2m-l+1)}(s)) ds.
\end{aligned}$$

(b) *If $g(y) \leqslant y$ and m is odd, then the above inequality is reversed.*

(c) *If $g(y) \geqslant y$ and m is even, then the reverse of inequality in (a) holds.*

(d) *If $g(y) \geqslant y$ and m is odd, then the inequality given in part (a) holds.*

By using the $(m, n-m)$ conditions we obtain the following result.

COROLLARY 2. *Let $m,n \in \mathbb{N}$. Let $g : [a,b] \to \mathbb{R}$ and $F : I \to \mathbb{R}$ (where I is an interval in $\mathbb{R}$ such that $a,b,g(a),g(b) \in I$) be two n times differentiable functions such that $g, g^{(1)}, \dots, g^{(n)}$, $F^{(1)}, F^{(2)}, \dots, F^{(n)}$ are nondecreasing functions. Then:*

(a) *If $g(y) \leqslant y$ and $n-m$ is even, then*

$$\begin{aligned}
F \circ g(x) \leqslant & \sum_{i=0}^{m-1} \tau_i(x) \sum_{l=0}^{i} F^{(l)}(g(a)) B_{i,l}(g^{(1)}(a), \dots, g^{(i-l+1)}(a)) \\
& + \sum_{i=0}^{n-m-1} \eta_i(x) \sum_{l=0}^{i} F^{(l)}(g(b)) B_{i,l}(g^{(1)}(b), \dots, g^{(i-l+1)}(b)) \\
& + \int_a^b G_{m,n}(x,s) \sum_{l=1}^{n} F^{(l)}(s) B_{n,l}(g^{(1)}(s), \dots, g^{(n-l+1)}(s)) ds.
\end{aligned}$$

(b) *If* $g(y) \leqslant y$ *and* $n-m$ *is odd, then reverse of the above inequality holds.*

(c) *If* $g(y) \geqslant y$ *and* $n-m$ *is even, then reverse of the inequality in part* (a) *holds.*

(d) *If* $g(y) \geqslant y$ *and* $n-m$ *is odd, then the inequality in part* (a) *holds.*

Further, by using the two-point Taylor and $(m,n-m)$ conditions we obtain the following results.

COROLLARY 3. *Let* $m \in \mathbb{N}$ *and let* $h:[0,b] \to [0,+\infty)$ *and* $F: I \to \mathbb{R}$ *(where* I *is an interval in* $\mathbb{R}$ *such that* $0, b, \int_0^b h(t)\,dt \in I$*) be two functions such that* $h, h^{(1)}, \dots, h^{(2m-2)}$, $F^{(1)}, F^{(2)}, \dots, F^{(2m)}$ *are nondecreasing functions. Then:*

(a) *If* $\int_0^y h(t)\,dt \leqslant y$ *and* m *is even, then for every* $x \in [0,b]$

$$
\begin{aligned}
F\left(\int_0^x h(t)dt\right) \leqslant \sum_{i=0}^{m-1}\sum_{k=0}^{m-i-1}\binom{m+k-1}{k}\Big[&\frac{x^{i+k}(b-x)^m}{i!b^{m+k}}\sum_{l=0}^{i}F^{(l)}(0)B_{i,l}(h(0),\dots,h^{(i-l)}(0)) \\
&+\frac{(-1)^i x^m (b-x)^{i+k}}{i!\,b^{m+k}}\sum_{l=0}^{i}F^{(l)}\left(\int_0^b h(t)\,dt\right)B_{i,l}(h(b),\dots,h^{(i-l)}(b))\Big] \\
&+\int_a^b G_m(x,s)\sum_{l=1}^{2m}F^{(l)}(s)B_{2m,l}(h(s),\dots,h^{(2m-l)}(s))ds.
\end{aligned}
$$

(b) *If* $\int_0^y h(t)\,dt \leqslant y$ *and* m *is odd, then reverse of the above inequality holds.*

(c) *If* $\int_0^y h(t)\,dt \geqslant y$ *and* m *is even, then reverse of the inequality in* (a) *holds.*

(d) *If* $\int_0^y h(t)\,dt \geqslant y$ *and* m *is odd, then the inequality from* (a) *holds.*

Proof. Follows from Corollary 1 by taking $a=0$ and $g(y)=\int_0^y h(t)\,dt$. $\square$

COROLLARY 4. *Let* $m,n \in \mathbb{N}$ *and let* $h:[0,b] \to [0,+\infty)$ *and* $F: I \to \mathbb{R}$ *(where* I *is an interval in* $\mathbb{R}$ *such that* $0, b, \int_0^b h(t)\,dt \in I$*) be two functions such that* $h, h^{(1)}, \dots, h^{(n-2)}$, $F^{(1)}, F^{(2)}, \dots, F^{(n)}$ *are nondecreasing functions. Then:*

(a) *If* $\int_0^y h(t)\,dt \leqslant y$ *and* $n-m$ *is even, then for every* $x \in [0,b]$

$$
\begin{aligned}
F\left(\int_0^x h(y)\,dy\right) \leqslant &\sum_{i=0}^{m-1}\tau_i(x)\sum_{l=0}^{i}F^{(l)}(0)B_{i,l}(h(0),\dots,h^{(i-l)}(0)) \\
&+\sum_{i=0}^{n-m-1}\eta_i(x)\sum_{l=0}^{i}F^{(l)}\left(\int_0^b h(t)\,dt\right)B_{i,l}(h(b),\dots,h^{(i-l)}(b)) \\
&+\int_0^b G_{m,n}(x,s)\sum_{l=1}^{n}F^{(l)}(s)B_{n,l}(h(s),h^{(1)}(s),\dots,h^{(n-l)}(s))ds.
\end{aligned}
$$

(b) *If* $\int_0^y h(t)\,dt \leqslant y$ *and* $n-m$ *is odd, then reverse of the above inequality holds.*

(c) *If* $\int_0^y h(t)\,dt \geqslant y$ *and* $n-m$ *is even, then reverse of the inequality in* (a) *holds.*

(d) *If* $\int_0^y h(t)\,dt \geqslant y$ *and* $n-m$ *is odd, then the inequality from* (a) *holds.*

By inserting the Bell polynomials and Green's function we get explicit forms of the inequalities from Corollary 1.

The relevant Bell polynomials are equal to:

$$
\begin{aligned}
n=2 &: B_{2,1}(x_1,x_2)=x_2, \quad B_{2,2}(x_1)=x_1^2,\\
n=4 &: B_{4,1}(x_1,x_2,x_3,x_4)=x_4, \quad B_{4,2}(x_1,x_2,x_3)=3x_1^3x_3+3x_2^2\\
&\quad B_{4,3}(x_1,x_2)=6x_1^2x_2, \quad B_{4,4}(x_1)=x_1^4
\end{aligned}
$$

Consequently, for $m=1$ we have

$$
\begin{aligned}
Fog(x) \leqslant{}& \frac{b-x}{b-a}F(g(a))+\frac{x-a}{b-a}F(g(b))\\
&-\frac{b-x}{b-a}\int_a^x (s-a)\Big(F^{(1)}(s)g^{(2)}(s)+F^{(2)}(s)g^{(1)}(s)^2\Big)ds\\
&-\frac{x-a}{b-a}\int_x^b (b-s)\Big(F^{(1)}(s)g^{(2)}(s)+F^{(2)}(s)g^{(1)}(s)^2\Big)ds
\end{aligned}
$$

and for $m=2$ we have

$$
\begin{aligned}
Fog(x) \leqslant{}& \frac{(b-x)^2}{(b-a)^3}\Big((b+2x-3a)F(g(a))+(b-a)(x-a)F^{(1)}(g(a))\Big)\\
&+\frac{(x-a)^2}{(a-b)^3}\Big((a+2x-3b)F(g(b))+(a-b)(x-b)F^{(1)}(g(b))\Big)\\
&+\frac{(b-x)^2}{6(b-a)^3}\int_a^x (s-a)^2\Big((x-s)(b-a)+2(x-a)(b-s)\Big)\\
&\times\Big(F^{(1)}(s)g^{(4)}(s)+3F^{(2)}(s)\big(g^{(1)}(s)^3g^{(3)}(s)+g^{(2)}(s)^2\big)\\
&+6F^{(3)}(s)g^{(1)}(s)^2g^{(2)}(s)+F^{(4)}(s)g^{(1)}(s)^4\Big)ds\\
&+\frac{(x-a)^2}{6(b-a)^3}\int_x^b (b-s)^2\Big((s-x)(b-a)+2(s-a)(b-x)\Big)\\
&\times\Big(F^{(1)}(s)g^{(4)}(s)+3F^{(2)}(s)\big(g^{(1)}(s)^3g^{(3)}(s)+g^{(2)}(s)^2\big)\\
&+6F^{(3)}(s)g^{(1)}(s)^2g^{(2)}(s)+F^{(4)}(s)g^{(1)}(s)^4\Big)ds.
\end{aligned}
$$

COROLLARY 5. *Let* m, h *and* F *be as in Corollary* 3 *and* $k:[0,b]\to[0,+\infty)$. *Then:*

(a) *If* $\int_0^y h(t)\,dt \leqslant y$ *and* m *is even, then for every* $x \in [0,b]$

$$\int_0^b k(x)F\left(\int_0^x h(y)\,dy\right)dx$$
$$\leqslant \int_0^b k(x)\sum_{i=0}^{m-1}\sum_{k=0}^{m-i-1}\binom{m+k-1}{k}\Big[\frac{x^{i+k}(b-x)^m}{i!\,b^{m+k}}\sum_{l=0}^{i}F^{(l)}(0)B_{i,l}(h(0),\dots,h^{(i-l)}(0))$$
$$+\frac{(-1)^i x^m(b-x)^{i+k}}{i!\,b^{m+k}}\sum_{l=0}^{i}F^{(l)}\left(\int_0^b h(t)\,dt\right)B_{i,l}(h(b),\dots,h^{(i-l)}(b))\Big]dx$$
$$+\int_0^b\int_0^b k(x)G_m(x,s)\sum_{l=1}^{2m}F^{(l)}(s)B_{2m,l}(h(s),\dots,h^{(2m-l)}(s))dxds.$$

(b) *If* $\int_0^y h(t)\,dt \leqslant y$ *and* m *is odd, then reverse of the above inequality holds.*

(c) *If* $\int_0^y h(t)\,dt \geqslant y$ *and* m *is even, then reverse of the inequality in* (a) *holds.*

(d) *If* $\int_0^y h(t)\,dt \geqslant y$ *and* m *is odd, then the inequality from* (a) *holds.*

Proof. Multiplying the inequality from Corollary 3 by $k(x) \geqslant 0$, integrating with respect to x from 0 to b and applying Fubini's theorem on the right hand side we obtain the stated inequality. □

Similarly, using the $(m,n-m)$ conditions, we obtain the following result.

COROLLARY 6. *Let* m, n, h *and* F *be as in Corollary 4 and* $k:[0,b]\to[0,+\infty)$. *Then:*

(a) *If* $\int_0^y h(t)\,dt \leqslant y$ *and* $n-m$ *is even, then for every* $x \in [0,b]$

$$\int_0^b k(x)F\left(\int_0^x h(y)\,dy\right)dx$$
$$\leqslant \int_0^b k(x)\Big[\sum_{i=0}^{m-1}\tau_i(x)\sum_{l=0}^{i}F^{(l)}(0)B_{i,l}(h(0),\dots,h^{(i-l)}(0))$$
$$+\sum_{i=0}^{n-m-1}\eta_i(x)\sum_{l=0}^{i}F^{(l)}\left(\int_0^b h(t)\,dt\right)B_{i,l}(h(b),\dots,h^{(i-l)}(b))\Big]dx$$
$$+\int_0^b\int_0^b k(x)G_{m,n}(x,s)\sum_{l=1}^{n}F^{(l)}(s)B_{n,l}(h(s),h^{(1)}(s),\dots,h^{(n-l)}(s))dxds.$$

(b) *If* $\int_0^y h(t)\,dt \leqslant y$ *and* $n-m$ *is odd, then reverse of the above inequality holds.*

(c) *If* $\int_0^y h(t)\,dt \geqslant y$ *and* $n-m$ *is even, then reverse of the inequality in* (a) *holds.*

(d) *If* $\int_0^y h(t)\,dt \geqslant y$ *and* $n-m$ *is odd, then the inequality from* (a) *holds.*

EXAMPLE 5. *Applying Corollary 5(a) with $F(t)=t^p$, $p>2m$, we obtain the inequality*

$$\int_0^b k(x)\left(\int_0^x h(t)\,dt\right)^p dx$$
$$\leqslant \int_0^b k(x)\Big[\sum_{i=0}^{m-1}\sum_{k=0}^{m-i-1}\binom{m+k-1}{k}\frac{(-1)^i x^m(b-x)^{i+k}}{i!b^{m+k}}\times$$
$$\times\sum_{l=0}^{i}(p)_l\left(\int_0^b h(t)\,dt\right)^{p-l}B_{i,l}(h(b),\dots,h^{(i-l)}(b))\Big]dx$$
$$+\int_0^b\int_0^b k(x)G_m(x,s)\sum_{l=1}^{2m}(p)_l s^{p-l}B_{2m,l}(h(s),\dots,h^{(2m-l)}(s))dxds,$$

where $(p)_l=p(p-1)\cdots(p-l+1)$ is the Pochhammer symbol.

EXAMPLE 6. *Applying Corollary 6(a) with $F(t)=t^p$, $p>n$, and using the explicit expression for $\eta_i(x)$, we obtain the inequality*

$$\int_0^b k(x)\left(\int_0^x h(t)\,dt\right)^p dx$$
$$\leqslant \int_0^b k(x)\Big[\sum_{i=0}^{n-m-1}\sum_{k=0}^{n-m-i-1}\binom{m+k-1}{k}\frac{(-1)^i x^m(b-x)^{i+k}}{i!\,b^{m+k}}$$
$$\times\sum_{l=0}^{i}(p)_l\left(\int_0^b h(t)\,dt\right)^{p-l}B_{i,l}(h(b),\dots,h^{(i-l)}(b))\Big]dx$$
$$+\int_0^b\int_0^b k(x)G_{m,n}(x,s)\sum_{l=1}^{n}(p)_l s^{p-l}B_{n,l}(h(s),\dots,h^{(n-l)}(s))dxds,$$

where $(p)_l=p(p-1)\cdots(p-l+1)$ is the Pochhammer symbol.

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