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THE FEKETE–SZEGÖ FUNCTIONAL PROBLEMS FOR SOME SUBCLASSES OF m–FOLD SYMMETRIC BI–UNIVALENT FUNCTIONS

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Abstract. In this paper, we introduce several new subclasses of the class of m-fold symmetric *bi-univalent* functions and obtain estimates of the Taylor-Maclaurin coefficients $|a_{m+1}|$, $|a_{2m+1}|$ and Fekete-Szegö functional problems for functions in these new subclasses. The results presented in this paper improve the earlier results of Ali *et al.* [1], Frasin and Aouf [6], and Srivastava *et al.* [14] in terms of the bounds as well as the ranges of the parameter under consideration. Our results also further generalize the results of Peng *et al.* [19].

1. Introduction and definitions

Let $\mathscr{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}), \tag{1.1}$$

which are *analytic* in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Further, by $\mathscr{S}$ we denote the class of all functions in $\mathscr{A}$ which are univalent in $\mathbb{U}$. For more details on univalent functions, see (for example) [5]. It is well known that every function $f \in \mathscr{S}$ has an inverse f^{-1} defined by

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U}) \tag{1.2}$$

and

$$f\big(f^{-1}(w)\big) = w \quad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right). \tag{1.3}$$

Indeed, the inverse function may have an analytic continuation to $\mathbb{U}$, with

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.4}$$

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A function $f \in \mathscr{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$, which are normalized by the equation (1.1).

An analytic function f is subordinate to another analytic function g, written $f(z) \prec g(z)$, provided that there is an analytic function w defined on $\mathbb{U}$ with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1$$

satisfying the following condition:

$$f(z) = g\big(w(z)\big).$$

Lewin [7] investigated the class Σ of bi-univalent functions and obtained a coefficient bound given by

$$|a_2| \leqq 1.51.$$

Subsequently, motivated by the work of Lewin [7], Brannan and Clunie [3] conjectured that

$$|a_2| \leqq \sqrt{2}.$$

Some examples of bi-univalent functions are given as follows (see also Srivastava *et al.* [14]):

$$\frac{z}{1-z}, \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \qquad \text{and} \qquad -\log(1-z).$$

The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \qquad (n \in \mathbb{N};\ n \geqq 3)$$

is still open (see, for details, [14]). In fact, in recent years, the study of bi-univalent functions was revived by (and has gained momentum) due mainly to the pioneering work of Srivastava *et al.* [14]. Many researchers (see [6, 12, 13, 14, 16, 17, 18, 20]) recently investigated several interesting subclasses of the class Σ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. For each function f in $\mathscr{S}$, the function h given by

$$h(z) = \sqrt[m]{f(z^m)} \qquad (m \in \mathbb{N})$$

is univalent and maps the unit disk $\mathbb{U}$ into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [9]; see also [15]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (m \in \mathbb{N};\ z \in \mathbb{U}). \tag{1.5}$$

We denote the class of m-fold symmetric univalent functions by $\mathscr{S}_m$, which are normalized by the above series expansion (1.5). In fact, the functions in the class $\mathscr{S}$ are one-fold symmetric (that is, $m = 1$). Analogous to the concept of m-fold symmetric univalent functions, one can think of the concept of m-fold symmetric bi-univalent functions in a natural way. Each function f in the class Σ generates an m-fold symmetric bi-univalent function for each positive integer m. The normalized form of f is given as in (1.5) and f^{-1} is given as follows:

$$g(w) = w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots, \tag{1.6}$$

where $f^{-1} = g$. We denote the class of m-fold symmetric bi-univalent functions by Σ_m. For $m = 1$, the formula (1.6) coincides with the formula (1.4) of the class Σ. Here, in this paper, we also denote by $\mathscr{P}$ the class of analytic functions of the form:

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \qquad (z \in \mathbb{U})$$

such that

$$\Re\big(p(z)\big) > 0 \qquad (z \in \mathbb{U}).$$

In view of the work of Pommerenke [9], the m-fold symmetric function p in the class $\mathscr{P}$ is of the form:

$$p(z) = 1 + c_m z^m + c_{2m} z^{2m} + c_{3m} z^{3m} + \cdots. \tag{1.7}$$

Throughout our present investigation, it is assumed that φ is an analytic function with positive real part in the unit disk $\mathbb{U}$ such that

$$\varphi(0) = 1 \qquad \text{and} \qquad \varphi'(0) > 0$$

and $\varphi(\mathbb{U})$ is symmetric with respect to the real axis. Such a function has a series expansion of the form:

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \qquad (B_1 > 0). \tag{1.8}$$

Let $u(z)$ and $v(z)$ be two analytic functions in the unit disk $\mathbb{U}$ with

$$u(0) = v(0) = 0 \qquad \text{and} \qquad \max\{|u(z)|, |v(z)|\} < 1.$$

We suppose also that

$$u(z) = b_m z^m + b_{2m} z^{2m} + b_{3m} z^{3m} + \cdots \tag{1.9}$$

and

$$v(w) = c_m w^m + c_{2m} w^{2m} + c_{3m} w^{3m} + \cdots. \tag{1.10}$$

We observe that

$$|b_m| \leqq 1, \quad |b_{2m}| \leqq 1 - |b_m|^2, \quad |c_m| \leqq 1 \qquad \text{and} \qquad |c_{2m}| \leqq 1 - |c_m|^2. \tag{1.11}$$

By simple computations, we have

$$\varphi\big(u(z)\big) = 1 + B_1 b_m z^m + (B_1 b_{2m} + B_2 b_m^2) z^{2m} + \cdots \qquad (|z| < 1) \tag{1.12}$$

and

$$\varphi\big(v(w)\big) = 1 + B_1 c_m w^m + (B_1 c_{2m} + B_2 c_m^2) w^{2m} + \cdots \qquad (|w| < 1). \tag{1.13}$$

Motivated essentially by the work of Ma and Minda [8], we introduce here some new subclasses of m-fold symmetric bi-univalent functions and obtain bounds for the Taylor-Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ and Fekete-Szegö functional problems for functions in these new classes. The results presented in this paper improve the earlier results of Ali *et al.* [1], Frasin and Aouf [6], and Srivastava *et al.* [14] in terms of the bounds as well as the ranges of the parameter under consideration and also further generalize the results of Peng *et al.* [19].

2. Coefficient bounds for the function class $\mathcal{H}_{\Sigma,m}(\varphi)$

DEFINITION 1. A function $f(z)$, given by (1.5), is said to be in the class $\mathcal{H}_{\Sigma,m}(\varphi)$, if the following conditions are satisfied:

$$f \in \Sigma_m, \quad f'(z) \prec \varphi(z) \qquad \text{and} \qquad g'(w) \prec \varphi(w) \qquad \big(g(w) = f^{-1}(w)\big),$$

where the function $g(w)$ is defined by (1.6).

For various special choices of the function $\varphi(z)$ and for the case when $m = 1$, our function class $\mathcal{H}_{\Sigma,m}(\varphi)$ reduces to the following known function classes.

1. For $m = 1$, the function class given by

$$\mathcal{H}_{\Sigma,m}(\varphi) \equiv \mathcal{H}_{\Sigma,1}(\varphi) = \mathcal{H}_{\Sigma}(\varphi)$$

was studied by Ali *et al.* [1].

2. For $m = 1$ and

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} \qquad (0 \leqq \gamma < 1),$$

the function class given by

$$\mathcal{H}_{\Sigma,m}(\varphi) \equiv \mathcal{H}_{\Sigma,1}\left(\left(\frac{1+z}{1-z}\right)^{\gamma}\right)$$

was studied by Srivastava *et al.* [14].

3. For $m = 1$ and

$$\varphi(z) = \frac{1+(1-2\gamma)z}{1-z} \qquad (0 \leqq \gamma < 1),$$

the function class given by

$$\mathscr{H}_{\Sigma,m}(\varphi) \equiv \mathscr{H}_{\Sigma,1}\left(\frac{1+(1-2\gamma)z}{1-z}\right)$$

was studied by Srivastava *et al.* [14].

We first state and prove the following theorem.

THEOREM 1. *Let the function $f(z)$, given by (1.5), be in the class $\mathscr{H}_{\Sigma,m}(\varphi)$. Then*

$$|a_{m+1}| \leqq \frac{B_1\sqrt{2B_1}}{\sqrt{(m+1)\left[2(m+1)B_1+\left|(2m+1)B_1^2-2(1+m)B_2\right|\right]}} \tag{2.1}$$

and $|a_{2m+1}|$

$$\leqq \begin{cases} \left(1-\frac{2(m+1)}{(2m+1)B_1}\right)\frac{B_1^3}{\left[2(m+1)B_1+\left|(2m+1)B_1^2-2(m+1)B_2\right|\right]}+\frac{B_1}{2m+1} & \left(B_1 \geqq \frac{2(m+1)}{2m+1}\right) \\ \frac{B_1}{2m+1} & \left(B_1 < \frac{2(m+1)}{2m+1}\right). \end{cases} \tag{2.2}$$

Proof. Let $f \in \mathscr{H}_{\Sigma,m}(\varphi)$ and $g = f^{-1}$. Then there are analytic functions $u : \mathbb{U} \to \mathbb{U}$ and $v : \mathbb{U} \to \mathbb{U}$, with

$$u(0) = v(0) = 0,$$

satisfying the following conditions:

$$f'(z) = \varphi\big(u(z)\big) \qquad \text{and} \qquad g'(w) = \varphi\big(v(w)\big). \tag{2.3}$$

Since

$$f'(z) = 1+(m+1)a_{m+1}z^m+(2m+1)a_{2m+1}z^{2m}+\cdots$$

and

$$g'(w) = 1-(m+1)a_{m+1}w^m+(2m+1)\left[(m+1)a_{m+1}^2-a_{2m+1}\right]w^{2m}+\cdots,$$

it follows from (1.12), (1.13) and (2.3) that

$$(m+1)a_{m+1} = B_1b_m, \tag{2.4}$$

$$(2m+1)a_{2m+1} = B_1b_{2m}+B_2b_m^2, \tag{2.5}$$

$$-(m+1)a_{m+1} = B_1c_m \tag{2.6}$$

and

$$(2m+1)\left[(m+1)a_{m+1}^2-a_{2m+1}\right] = B_1c_{2m}+B_2c_m^2. \tag{2.7}$$

From (2.4) and (2.6), we get

$$c_m = -b_m. \tag{2.8}$$

By adding (2.5) and (2.7) and, upon some computations using (2.4) and (2.8), we get

$$(m+1)\left[(2m+1)B_1^2 - 2(1+m)B_2\right] a_{m+1}^2 = B_1^3 (b_{2m} + c_{2m}). \tag{2.9}$$

Further, the equations (2.8) and (2.9), together with the equation (1.11), yield

$$\left|(m+1)\left[(2m+1)B_1^2 - 2(1+m)B_2\right] a_{m+1}^2\right| \leqq 2B_1^3 \left(1 - |b_m|^2\right). \tag{2.10}$$

Now, from (2.4) and (2.10), we get

$$|a_{m+1}| \leqq \frac{B_1\sqrt{2B_1}}{\sqrt{(m+1)\left[2(m+1)B_1 + \left|(2m+1)B_1^2 - 2(1+m)B_2\right|\right]}},$$

as asserted in (2.1).

By subtracting (2.7) from (2.5), we get

$$2(2m+1)\,a_{2m+1} = (m+1)(2m+1)\,a_{m+1}^2 + B_1 (b_{2m} - c_{2m}). \tag{2.11}$$

From (1.11), (2.4), (2.8) and (2.11), it follows that

$$\begin{aligned} |a_{2m+1}| &\leqq \frac{(m+1)}{2}|a_{m+1}|^2 + \frac{B_1}{2(2m+1)}\left(|b_{2m}| + |c_{2m}|\right) \\ &\leqq \frac{(m+1)}{2}|a_{m+1}|^2 + \frac{B_1}{(2m+1)}\left(1 - |b_m|^2\right) \\ &= \left(\frac{m+1}{2} - \frac{(m+1)^2}{(2m+1)B_1}\right)|a_{m+1}|^2 + \frac{B_1}{2m+1}, \end{aligned}$$

which implies the assertion (2.2). This completes the proof of Theorem 1. □

REMARK 1. For the case of one-fold symmetric functions, Theorem 1 reduces to the corresponding results of Peng *et al.* [19], which we recall here as Corollary 1 below.

COROLLARY 1. (see [19]) *Let the function* $f(z)$*, given by* (1.5)*, be in the class* $\mathscr{H}_\Sigma(\varphi)$*. Then*

$$|a_2| \leqq \frac{B_1\sqrt{B_1}}{\sqrt{4B_1 + |3B_1{}^2 - 4B_2|}} \tag{2.12}$$

and

$$|a_3| \leqq \begin{cases} \left(1 - \frac{4}{3B_1}\right)\frac{B_1^3}{(4B_1 + |3B_1^2 - 4B_2|)} + \frac{B_1}{3} & \left(B_1 \geqq \frac{4}{3}\right) \\ \frac{B_1}{3} & \left(B_1 < \frac{4}{3}\right). \end{cases} \tag{2.13}$$

For the case of one-fold symmetric functions and for the class of strongly starlike functions, the function φ is given by

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} = 1 + 2\gamma z + 2\gamma^2 z^2 + \cdots \quad (0 < \gamma \leqq 1), \tag{2.14}$$

which gives

$$B_1 = 2\gamma \qquad \text{and} \qquad B_2 = 2\gamma^2.$$

Hence Theorem 1 gives the following corollary.

COROLLARY 2. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathscr{H}_{\Sigma,1}\left(\left(\frac{1+z}{1-z}\right)^{\gamma}\right)$. *Then*

$$|a_2| \leqq \frac{\gamma\sqrt{2}}{\sqrt{2+\gamma}} \tag{2.15}$$

and

$$|a_3| \leqq \begin{cases} \dfrac{8\gamma^2}{6+3\gamma} & \left(\frac{2}{3} \leqq \gamma \leqq 1\right) \\ \dfrac{2\gamma}{3} & \left(0 < \gamma < \frac{2}{3}\right). \end{cases} \tag{2.16}$$

REMARK 2. The estimate for $|a_3|$ asserted by Corollary 2 is more accurate than those given by Theorem 2 in Srivastava *et al.* [14].

Next, for the case of one-fold symmetric functions and for the class of starlike functions of order γ, the function φ is given by

$$\varphi(z) = 1 + 2(1-\gamma)z + 2(1-\gamma)z^2 + \cdots,$$

so that

$$B_1 = B_2 = 2(1-\gamma),$$

and Theorem 1 would lead us to the following corollary.

COROLLARY 3. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathscr{H}_{\Sigma,1}\left(\frac{1+(1-2\gamma)z}{1-z}\right)$. *Then*

$$|a_2| \leqq \frac{\sqrt{2}(1-\gamma)}{\sqrt{2+|1-3\gamma|}} \tag{2.17}$$

and

$$|a_3| \leqq \begin{cases} \frac{8-12\gamma}{9} & \left(0 \leqq \gamma \leqq \frac{1}{3}\right) \\ \frac{2(1-\gamma)}{3} & \left(\frac{1}{3} < \gamma < 1\right). \end{cases} \tag{2.18}$$

REMARK 3. The estimates for $|a_2|$ and $|a_3|$ asserted by Corollary 3 are more accurate than those given by Theorem 2 in Srivastava *et al.* [14].

REMARK 4. For the case of one-fold symmetric functions, the estimates for $|a_2|$ and $|a_3|$ given by the equations (2.1) and (2.2) are smaller than that given by Theorem 2.1 in Ali *et al.* [1].

THEOREM 2. *Let the function $f(z)$, given by (1.5), be in the class $\mathscr{H}_{\Sigma,m}(\varphi)$. Then*

$$\left|a_{2m+1}-\delta a_{m+1}^2\right| \leqq \begin{cases} \frac{B_1}{2m+1} & \text{for } 0 \leqq |h(\delta)| < \frac{1}{2(2m+1)} \\ 2B_1\,|h(\delta)| & \text{for } |h(\delta)| \geqq \frac{1}{2(2m+1)} \end{cases}, \tag{2.19}$$

where

$$h(\delta)=\frac{B_1^2\,(m+1-2\delta)}{2\,(m+1)\left[(2m+1)B_1^2-2(1+m)B_2\right]}.$$

Proof. From the equations (2.9) and (2.11), we get

$$a_{m+1}^2=\frac{B_1^3\,(b_{2m}+c_{2m})}{(m+1)\left[(2m+1)B_1^2-2(1+m)B_2\right]} \tag{2.20}$$

and

$$a_{2m+1}=\frac{(m+1)}{2}a_{m+1}^2+\frac{B_1\,(b_{2m}-c_{2m})}{2\,(2m+1)}. \tag{2.21}$$

From the equations (2.20) and (2.21), it follows that

$$a_{2m+1}-\delta a_{m+1}^2=B_1\left[\left(h(\delta)+\frac{1}{2\,(2m+1)}\right)b_{2m}+\left(h(\delta)-\frac{1}{2\,(2m+1)}\right)c_{2m}\right],$$

where

$$h(\delta)=\frac{B_1^2\,(m+1-2\delta)}{2\,(m+1)\left[(2m+1)B_1^2-2(1+m)B_2\right]}.$$

Since all B_i are real and $B_1>0$, which implies the assertion (2.19). This completes the proof of Theorem 2. □

For the case of one-fold symmetric functions, Theorem 2 reduces to the following Corollary 4.

COROLLARY 4. *Let the function $f(z)$, given by (1.5), be in the class $\mathscr{H}_{\Sigma}(\varphi)$. Then*

$$\left|a_3-\delta a_2^2\right| \leqq \begin{cases} \frac{B_1}{3} & \text{for } 0 \leqq |h(\delta)| < \frac{1}{6} \\ 2B_1\,|h(\delta)| & \text{for } |h(\delta)| \geqq \frac{1}{6} \end{cases}.$$

Taking $\delta=1$ and $\delta=0$ in Theorem 2, we have the following corollaries.

COROLLARY 5. *Let the function $f(z)$, given by (1.5), be in the class $\mathscr{H}_{\Sigma,m}(\varphi)$. Then*

$$\left|a_{2m+1}-a_{m+1}^2\right| \leqq \begin{cases} \frac{B_1}{2m+1} & \text{for } 0 \leqq |h(\delta)| < \frac{1}{2(2m+1)} \\ 2B_1\,|h(\delta)| & \text{for } \quad |h(\delta)| \geqq \frac{1}{2(2m+1)} \end{cases}.$$

For the case of one-fold symmetric functions, Corollary 5 reduces to the following corollary.

COROLLARY 6. *Let the function $f(z)$, given by (1.5), be in the class $\mathscr{H}_{\Sigma,1}(\varphi)$. Then*

$$\left|a_3-a_2^2\right| \leqq \frac{B_1}{3}.$$

COROLLARY 7. *Let the function $f(z)$, given by (1.5), be in the class $\mathscr{H}_{\Sigma,m}(\varphi)$. Then*

$$|a_{2m+1}| \leqq \begin{cases} \frac{B_1}{2m+1} & \text{for} \quad \frac{B_2}{B_1^2} \in (-\infty,0)\cup\left(\frac{2m+1}{m+1},\infty\right) \\ \frac{B_1^3}{(2m+1)B_1^2-2(1+m)B_2} & \text{for } \frac{B_2}{B_1^2} \in \left(\frac{2m+1}{2(m+1)},\frac{2m+1}{m+1}\right)\cup\left(0,\frac{2m+1}{2(m+1)}\right) \end{cases}.$$

For the case of one-fold symmetric functions, Corollary 7 reduces to the following corollary.

COROLLARY 8. *Let the function $f(z)$, given by (1.5), be in the class $\mathscr{H}_{\Sigma,1}(\varphi)$. Then*

$$|a_3| \leqq \begin{cases} \frac{B_1}{3} & \text{for } \frac{B_2}{B_1^2} \in (-\infty,0)\cup\left(\frac{3}{2},\infty\right) \\ \frac{B_1^3}{3B_1^2-4B_2} & \text{for } \frac{B_2}{B_1^2} \in \left(\frac{3}{4},\frac{3}{2}\right)\cup\left(0,\frac{3}{4}\right) \end{cases}.$$

3. Coefficient bounds for the function class $\mathscr{N}_{\Sigma,m}(\lambda,\varphi)$

DEFINITION 2. A function $f(z)$, given by (1.5), is said to be in the class $\mathscr{N}_{\Sigma,m}(\lambda,\varphi)$ if the following conditions are satisfied:

$$f \in \Sigma_m, \qquad \frac{zf'(z)}{f(z)}+\frac{\lambda z^2 f''(z)}{f(z)} \prec \varphi(z)$$

and

$$\frac{wg'(w)}{g(w)}+\frac{\lambda w^2 g''(w)}{g(w)} \prec \varphi(w) \qquad \left(g(w)=f^{-1}(w)\right),$$

where the function $g(w)$ is defined by (1.6).

For various special choices of the function $\varphi(z)$, and for the case when $m=1$ and $\lambda=0$, the above-defined class $\mathcal{N}_{\Sigma,m}(\lambda,\varphi)$ reduces to the following function classes.

1. For $m=1$, the class given by

$$\mathcal{N}_{\Sigma,m}(\lambda,\varphi) \equiv \mathcal{N}_{\Sigma,1}(\lambda,\varphi)$$

   was studied by Ali *et al.* [1].

2. For $m=1,\ \lambda=0$ and

$$\varphi(z)=\left(\frac{1+z}{1-z}\right)^{\gamma} \qquad (0 \leqq \gamma < 1),$$

   the class given by

$$\mathcal{N}_{\Sigma,m}\left(0,\left(\frac{1+z}{1-z}\right)^{\gamma}\right) \equiv \mathcal{S}\mathcal{T}_{\Sigma}^{*}(\gamma)$$

   coincides with the class of bi-strongly starlike functions of order γ, which was studied by Brannan and Taha [4].

3. For $m=1,\ \lambda=0$ and

$$\varphi(z)=\frac{1+(1-2\gamma)z}{1-z} \qquad (0 \leqq \gamma < 1),$$

   the class given by

$$\mathcal{N}_{\Sigma,m}\left(0,\frac{1+(1-2\gamma)z}{1-z}\right) \equiv \mathcal{S}_{\Sigma}^{*}(\gamma)$$

   coincides with the class of bi-starlike functions of order γ, which was also studied by Brannan and Taha [4].

For functions belonging to the class $\mathcal{N}_{\Sigma,m}(\lambda,\varphi)$, we prove the following theorem.

THEOREM 3. *Let the function $f(z)$, given by (1.5), be in the class $\mathcal{N}_{\Sigma,m}(\lambda,\varphi)$. Also let $\lambda \geqq 0$. Then*

$$|a_{m+1}| \leqq \frac{B_1\sqrt{B_1}}{\sqrt{\left|(1+2\lambda+2\lambda m)B_1^2-[1+\lambda(m+1)]^2B_2\right|+[1+\lambda(m+1)]^2B_1}} \tag{3.1}$$

and

$$|a_{2m+1}| \leqq \begin{cases} \dfrac{(m+1)B_1}{2m^2(1+2\lambda+2\lambda m)} & (|B_2| \leqq B_1) \\[2ex] \dfrac{(m+1)\left|(1+2\lambda+2\lambda m)B_1^2-[1+\lambda(m+1)]^2B_2\right|B_1+(m+1)[1+\lambda(m+1)]^2|B_2|B_1}{2m^2(1+2\lambda+2\lambda m)\left(\left|(1+2\lambda+2\lambda m)B_1^2-[1+\lambda(m+1)]^2B_2\right|+[1+\lambda(m+1)]^2B_1\right)} & (|B_2| > B_1). \end{cases} \tag{3.2}$$

Proof. Let $f \in \mathcal{N}_{\Sigma,m}(\lambda,\varphi)$. Then there are analytic functions $u:\mathbb{U}\to\mathbb{U}$ and $v:\mathbb{U}\to\mathbb{U}$, with

$$u(0)=v(0)=0,$$

satisfying the following conditions:

$$\frac{zf'(z)}{f(z)}+\frac{\lambda z^2 f''(z)}{f(z)}=\varphi\big(u(z)\big) \tag{3.3}$$

and

$$\frac{wg'(w)}{g(w)}+\frac{\lambda w^2 g''(w)}{g(w)}=\varphi(v(w)) \qquad \big(g(w)=f^{-1}(w)\big). \tag{3.4}$$

Since

$$\begin{aligned}\frac{zf'(z)}{f(z)}+\frac{\lambda z^2 f(z)}{f(z)}&=1+m[1+\lambda(m+1)]a_{m+1}z^m\\ &+\big(2m[1+\lambda(2m+1)]a_{2m+1}-m[1+\lambda(m+1)]a_{m+1}^2\big)z^{2m}+\cdots\end{aligned}$$

and

$$\begin{aligned}&\frac{wg'(w)}{g(w)}+\frac{\lambda w^2 g''(w)}{g(w)}=1-m[1+\lambda(m+1)]a_{m+1}w^m\\ &+\Big([m(2m+1)+\lambda m(m+1)(4m+1)]a_{m+1}^2-2m[1+\lambda(2m+1)]a_{2m+1}\Big)w^{2m}+\cdots,\end{aligned}$$

from (1.12), (1.13), (3.3) and (3.4), we find that

$$m[1+\lambda(m+1)]a_{m+1}=B_1b_m, \tag{3.5}$$

$$2m[1+\lambda(2m+1)]a_{2m+1}-m[1+\lambda(m+1)]a_{m+1}^2=B_1b_{2m}+B_2b_m^2, \tag{3.6}$$

$$-m[1+\lambda(m+1)]a_{m+1}=B_1c_m \tag{3.7}$$

and

$$\begin{aligned}&[m(2m+1)+\lambda m(m+1)(4m+1)]a_{m+1}^2-2m[1+\lambda(2m+1)]a_{2m+1}\\ &\quad=B_1c_{2m}+B_2c_m^2.\end{aligned} \tag{3.8}$$

From (3.5) and (3.7), we get

$$c_m=-b_m. \tag{3.9}$$

By adding the equations (3.6) and (3.8) and, upon some computations using (3.5) and (3.9), we obtain

$$2m^2\Big[(1+2\lambda+2\lambda m)B_1^2-(1+\lambda(m+1))^2B_2\Big]a_{m+1}^2=B_1^3(b_{2m}+c_{2m}). \tag{3.10}$$

Further, the equations (3.9), (3.10), together with the equation (1.11), yield

$$\left|m^2\Big((1+2\lambda+2\lambda m)B_1^2-[1+\lambda(m+1)]^2B_2\Big)a_{m+1}^2\right|=B_1^3\left(1-|b_m|^2\right). \tag{3.11}$$

Now, from the equations (3.5) and (3.11), we get

$$|a_{m+1}| \leqq \frac{B_1\sqrt{B_1}}{\sqrt{\left|(1+2\lambda+2\lambda m)B_1^2-[1+\lambda(m+1)]^2B_2\right|+[1+\lambda(m+1)]^2B_1}}.$$

as asserted in (3.1).

By simple calculations from (3.6) and (3.8), and using the equations (3.5) and (3.9), we find that

$$\begin{aligned}&4m^2[1+\lambda(2m+1)](1+2\lambda+2\lambda m)a_{2m+1}\\&\quad=[2m+1+\lambda(m+1)(4m+1)]B_1b_{2m}\\&\qquad+(1+\lambda(m+1))B_1c_{2m}+2\left(1+\lambda+m+3\lambda m+2\lambda m^2\right)B_2b_m^2.\end{aligned} \tag{3.12}$$

Thus, by using the equation (1.11) in (3.12), we get

$$\begin{aligned}&2m^2[1+\lambda(2m+1)](1+2\lambda+2\lambda m)|a_{2m+1}|\\&\quad\leqq\left[1+\lambda+m+3\lambda m+2\lambda m^2\right]B_1-\left(1+\lambda+m+3\lambda m+2\lambda m^2\right)B_1|b_m|^2\\&\qquad+\left[1+\lambda+m+3\lambda m+2\lambda m^2\right]|B_2||b_m|^2.\end{aligned} \tag{3.13}$$

Since

$$|b_m|^2 \leqq \frac{[1+\lambda(m+1)]^2B_1}{\left|(1+2\lambda+2\lambda m)B_1^2-[1+\lambda(m+1)]^2B_2\right|+[1+\lambda(m+1)]^2B_1}, \tag{3.14}$$

upon substituting from (3.14) into (3.13), we are led easily to the assertion (3.2) of Theorem 3. This evidently completes the demonstration of Theorem 3. □

For the case of one-fold symmetric functions, Theorem 3 reduces to the results of Peng *et al.* [19].

COROLLARY 9. (see [19]) *Let the function* $f(z)$*, given by* (1.5)*, be in the class* $\mathcal{N}_{\Sigma,1}(\lambda,\varphi)$*. Then*

$$|a_2| \leqq \frac{B_1\sqrt{B_1}}{\sqrt{\left|(1+4\lambda)B_1{}^2-(1+2\lambda)^2B_2\right|+(1+2\lambda)^2B_1}}$$

and

$$|a_3| \leqq \begin{cases}\frac{B_1}{1+4\lambda} & (|B_2|\leqq B_1)\\[2mm] \frac{\left|(1+4\lambda)B_1^2-(1+2\lambda)^2B_2\right|B_1+(1+2\lambda)^2|B_2|B_1}{(1+4\lambda)\left(\left|(1+4\lambda)B_1^2-(1+2\lambda)^2B_2\right|+(1+2\lambda)^2B_1\right)} & (|B_2|>B_1).\end{cases}$$

For the case of one-fold symmetric functions with $\lambda = 0$ and for the class of strongly starlike functions, the function φ is given by

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} = 1 + 2\gamma z + 2\gamma^2 z^2 + \ldots \ (0 < \gamma \leqslant 1) \tag{3.15}$$

which gives

$$B_1 = 2\gamma \qquad \text{and} \qquad B_2 = 2\gamma^2.$$

Hence Theorem 3 reduces to the following result.

COROLLARY 10. (see [4]) *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathscr{S}\mathscr{T}^*_{\Sigma}(\gamma)$. *Then*

$$|a_2| \leqq \frac{\gamma}{\sqrt{1+\gamma}} \tag{3.16}$$

and

$$|a_3| \leqq 2\gamma. \tag{3.17}$$

REMARK 5. The estimate for $|a_3|$ given by Corollary 10 is more accurate than the bound given by Theorem 2.1 in Brannan and Taha [4].

For the case of one-fold symmetric functions with $\lambda = 0$ and for the class of bi-starlike functions, the function φ is given by

$$\varphi(z) = 1 + 2(1-\gamma)z + 2(1-\gamma)z^2 + \cdots,$$

so that

$$B_1 = B_2 = 2(1-\gamma).$$

Thus, clearly, Theorem 3 yields the following result.

COROLLARY 11. (see [4]) *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathscr{S}^*_{\Sigma}(\gamma)$. *Then*

$$|a_2| \leqq \frac{2(1-\gamma)}{\sqrt{1+|1-2\gamma|}} \tag{3.18}$$

and

$$|a_3| \leqq 2(1-\gamma). \tag{3.19}$$

REMARK 6. The estimate for $|a_2|$ given by Corollary 11 is more accurate than the bound given by Theorem 3.1 in Brannan and Taha [4].

REMARK 7. For the case of one-fold symmetric functions, the estimates for $|a_2|$ and $|a_3|$ given by the equations (3.1) and (3.2) are smaller than those given by Theorem 3.1 in Ali *et al.* [1].

THEOREM 4. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathcal{N}_{\Sigma,m}(\lambda,\varphi)$. *Also let* $\lambda \geqq 0$. *Then*

$$\left|a_{2m+1}-\delta a_{m+1}^{2}\right| \leqq \begin{cases} \frac{B_1}{2m(2\lambda m+\lambda+1)} & \text{for } 0 \leqq |h(\delta)| < \frac{1}{4m(2\lambda m+\lambda+1)} \\ 2B_1\,|h(\delta)| & \text{for } \quad |h(\delta)| \geqq \frac{1}{4m(2\lambda m+\lambda+1)} \end{cases}, \tag{3.20}$$

where

$$h(\delta) = \frac{B_1^2(m+1-2\delta)}{4m^2\left[(2\lambda m+2\lambda+1)B_1^2-(\lambda m+\lambda+1)^2 B_2\right]}.$$

Proof. From the equation (3.10), we get

$$a_{m+1}^{2} = \frac{B_1^3(b_{2m}+c_{2m})}{2m^2\left[(2\lambda m+2\lambda+1)B_1^2-(\lambda m+\lambda+1)^2 B_2\right]}. \tag{3.21}$$

Subract (3.6) from the (3.8), we get

$$a_{2m+1} = \frac{(m+1)}{2}a_{m+1}^{2} + \frac{B_1(b_{2m}-c_{2m})}{4m(2\lambda m+\lambda+1)}. \tag{3.22}$$

From the equations (3.21) and (3.22), it follows that

$$a_{2m+1}-\delta a_{m+1}^{2}=B_1\left[\left(h(\delta)+\frac{1}{4m(2\lambda m+\lambda+1)}\right)b_{2m}+\left(h(\delta)-\frac{1}{4m(2\lambda m+\lambda+1)}\right)c_{2m}\right].$$

where

$$h(\delta) = \frac{B_1^2(m+1-2\delta)}{4m^2\left[(2\lambda m+2\lambda+1)B_1^2-(\lambda m+\lambda+1)^2 B_2\right]}.$$

Since all B_i are real and $B_1 > 0$, which implies the assertion (3.20). This completes the proof of Theorem 4. □

For the case of one-fold symmetric functions, Theorem 4 reduces to the following Corollary 12.

COROLLARY 12. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathcal{N}_{\Sigma,1}(\lambda,\varphi)$. *Then*

$$\left|a_3-\delta a_2^2\right| \leqq \begin{cases} \frac{B_1}{2(3\lambda+1)} & \text{for } 0 \leqq |h(\delta)| < \frac{1}{4(3\lambda+1)} \\ 2B_1\,|h(\delta)| & \text{for } \quad |h(\delta)| \geqq \frac{1}{4(3\lambda+1)} \end{cases}.$$

Taking $\delta = 1$ and $\delta = 0$ in Theorem 4, we have the following corollaries.

COROLLARY 13. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathcal{N}_{\Sigma,m}(\lambda,\varphi)$. *Also let* $\lambda \geqq 0$. *Then*

$$\left|a_{2m+1}-a_{m+1}^2\right| \leqq \begin{cases} \frac{B_1}{2m(2\lambda m+\lambda+1)} & \text{for } 0 \leqq |h(\delta)| < \frac{1}{4m(2\lambda m+\lambda+1)} \\ 2B_1|h(\delta)| & \text{for } \quad |h(\delta)| \geqq \frac{1}{4m(2\lambda m+\lambda+1)} \end{cases}.$$

For the case of one-fold symmetric functions, Corollary 13 reduces to the following corollary.

COROLLARY 14. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathcal{N}_{\Sigma,1}(\lambda,\varphi)$. *Then*

$$\left|a_3-a_2^2\right| \leqq \frac{B_1}{2(3\lambda+1)}.$$

COROLLARY 15. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathcal{N}_{\Sigma,m}(\lambda,\varphi)$. *Also let* $\lambda \geqq 0$. *Then*

$$|a_{2m+1}| \leqq \begin{cases} \frac{B_1}{2m(2\lambda m+\lambda+1)} & \text{for } \frac{B_2}{B_1^2} \in \left(-\infty, -\frac{1}{m(\lambda m+\lambda+1)}\right) \cup (\sigma_1,\infty) \\ \frac{(m+1)B_1^3}{2m^2\left[(2\lambda m+2\lambda+1)B_1^2-(\lambda m+\lambda+1)^2B_2\right]} & \text{for } \frac{B_2}{B_1^2} \in \left(-\frac{1}{m(\lambda m+\lambda+1)},\sigma_2\right) \cup (\sigma_2,\sigma_1) \end{cases}.$$

where

$$\sigma_1 = \frac{4\lambda m^2+5\lambda m+2m+\lambda+1}{m(\lambda m+\lambda+1)^2}$$

and

$$\sigma_2 = \frac{(2\lambda m+2\lambda+1)}{(\lambda m+\lambda+1)^2}.$$

For the case of one-fold symmetric functions, Corollary 15 reduces to the following corollary.

COROLLARY 16. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathcal{N}_{\Sigma,1}(\lambda,\varphi)$. *Then*

$$|a_3| \leqq \begin{cases} \frac{B_1}{2(3\lambda+1)} & \text{for } \quad \frac{B_2}{B_1^2} \in \left(-\infty,-\frac{1}{1+2\lambda}\right) \cup \left(\frac{10\lambda+3}{(1+2\lambda)^2},\infty\right) \\ \frac{B_1^3}{(4\lambda+1)B_1^2-(1+2\lambda)^2B_2} & \text{for } \frac{B_2}{B_1^2} \in \left(-\frac{1}{1+2\lambda},\frac{4\lambda+1}{(1+2\lambda)^2}\right) \cup \left(\frac{4\lambda+1}{(1+2\lambda)^2},\frac{10\lambda+3}{(1+2\lambda)^2}\right) \end{cases}.$$

For the case of one-fold symmetric functions and $\lambda = 0$, Corollary 16 reduces to the following corollary.

COROLLARY 17. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathcal{N}_{\Sigma,1}(0,\varphi)$. *Then*

$$|a_3| \leqq \begin{cases} \frac{B_1}{2} & \text{for } \frac{B_2}{B_1^2} \in (-\infty,-1) \cup (3,\infty) \\ \frac{B_1^3}{B_1^2 - B_2} & \text{for } \frac{B_2}{B_1^2} \in (-1,1) \cup (1,3) \end{cases}.$$

4. Coefficient bounds for the function class $\mathcal{M}_{\Sigma,m}(\lambda,\varphi)$

DEFINITION 3. A function $f(z)$, given by (1.5), is said to be in the class $\mathcal{M}_{\Sigma,m}(\lambda,\varphi)$ if the following conditions are satisfied:

$$f \in \Sigma_m, \qquad (1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z)$$

and

$$(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \varphi(w) \qquad \left(g(w) = f^{-1}(w)\right),$$

where the function $g(w)$ is defined by (1.6).

Following the case of one-fold symmetric functions, a function in the class $\mathcal{M}_{\Sigma,m}(\lambda,\varphi)$ is called bi-Mocanu-convex function of Ma-Minda type that essentially unifies the classes $\mathcal{S}^*_{\Sigma}(\varphi)$ and $\mathcal{C}_{\Sigma}(\varphi)$ studied by Ali *et al.* [1] for the case of bi-univalent functions.

For various special choices of the function $\varphi(z)$ and the parameter λ, and for the case when $m = 1$, the above-defined class $\mathcal{M}_{\Sigma,m}(\lambda,\varphi)$ reduces to the following function classes.

1. For $m = 1$, the function class given by

$$\mathcal{M}_{\Sigma,m}(\lambda,\varphi) \equiv \mathcal{M}_{\Sigma,1}(\lambda,\varphi) = \mathcal{M}_{\Sigma}(\varphi)$$

   was studied by Ali *et al.* [1].

2. For $m = 1$, $\lambda = 1$ and

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} \qquad (0 < \gamma \leqq 1),$$

   the function class given by

$$\mathcal{M}_{\Sigma,1}\left(1, \left(\frac{1+z}{1-z}\right)^{\gamma}\right) \equiv \mathcal{S}\mathcal{T}^*_{\Sigma}(\gamma)$$

   coincides with the class of strongly bi-starlike functions of order γ, which was studied by Brannan and Taha [4].

3. For $m=1,\ \lambda=0$ and
$$\varphi(z)=\left(\frac{1+z}{1-z}\right)^{\gamma} \qquad (0<\gamma\leqq 1),$$
the function class given by
$$\mathscr{M}_{\Sigma,1}\left(1,\left(\frac{1+z}{1-z}\right)^{\gamma}\right)\equiv \mathscr{CV}_{\Sigma}(\gamma) \qquad (0<\gamma\leqq 1)$$
coincides with the class of strongly bi-convex functions of order γ, which was studied by Brannan and Taha [4].

4. For $m=1,\ \lambda=1$ and
$$\varphi(z)=\frac{1+(1-2\gamma)z}{1-z} \qquad (0\leqq\gamma<1),$$
the function class given by
$$\mathscr{M}_{\Sigma,1}\left(1,\frac{1+(1-2\gamma)z}{1-z}\right)\equiv \mathscr{C}_{\Sigma}(\gamma) \qquad (0\leqq\gamma<1)$$
coincides with the class of bi-convex functions of order γ, which was studied by Brannan and Taha [4].

5. For $m=1,\ \lambda=0$ and
$$\varphi(z)=\frac{1+(1-2\alpha)z}{1-z} \qquad (0\leqq\gamma<1),$$
the function class given by
$$\mathscr{M}_{\Sigma,1}\left(1,\frac{1+(1-2\gamma)z}{1-z}\right)\equiv \mathscr{S}^{*}_{\Sigma}(\gamma) \qquad (0\leqq\gamma<1)$$
cincides with the class of bi-starlike functions of order α, which was studied by Brannan and Taha [4].

THEOREM 5. *Let the function $f(z)$, given by (1.5), be in the class $\mathscr{M}_{\Sigma,m}(\lambda,\varphi)$. Also let $\lambda\geqq 0$. Then*
$$|a_{m+1}|\leqq \frac{B_1\sqrt{B_1}}{m\sqrt{(1+\lambda m)\left|B_1{}^2-(1+\lambda m)B_2\right|+(1+\lambda m)^2B_1}} \tag{4.1}$$
and
$$|a_{2m+1}|\leqq \begin{cases} \frac{(1+m)B_1}{2m^2(1+\lambda m)} & (|B_2|\leqq B_1) \\ \frac{(1+m)\left|B_1^2-(1+\lambda m)B_2\right|B_1+(1+m)(1+\lambda m)|B_2|B_1}{2m^2(1+\lambda m)\left(\left|B_1^2-(1+\lambda m)B_2\right|+(1+\lambda m)^2B_1\right)} & (|B_2|>B_1). \end{cases} \tag{4.2}$$

Proof. Let $f \in \mathscr{M}_{\Sigma,m}(\lambda,\varphi)$. Then there are analytic functions $u : \mathbb{U} \to \mathbb{U}$ and $v : \mathbb{U} \to \mathbb{U}$, with

$$u(0) = v(0) = 0$$

satisfying the following conditions:

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) = \varphi\big(u(z)\big) \tag{4.3}$$

and

$$(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) = \varphi\big(v(w)\big). \tag{4.4}$$

Since

$$\begin{aligned}(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) &= 1 + m(1+\lambda m)\,a_{m+1}z^m \\ &+ \left[2m(1+2\lambda m)\,a_{2m+1} - m\left(1+2\lambda m+\lambda m^2\right)a_{m+1}^2\right]z^{2m} + \cdots\end{aligned}$$

and

$$\begin{aligned}(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) &= 1 - m(1+\lambda m)\,a_{m+1}w^m \\ &+ \left[m\left(1+2\lambda m+2m+3\lambda m^2\right)a_{m+1}^2 - 2m(1+2\lambda m)\,a_{2m+1}\right]w^{2m} + \cdots,\end{aligned}$$

from (1.12), (1.13), (4.3) and (4.4), we find that

$$m(1+\lambda m)\,a_{m+1} = B_1 b_m, \tag{4.5}$$

$$2m(1+2\lambda m)\,a_{2m+1} - m\left(1+2\lambda m+\lambda m^2\right)a_{m+1}^2 = B_1 b_{2m} + B_2 b_m^2, \tag{4.6}$$

$$-m(1+\lambda m)\,a_{m+1} = B_1 c_m \tag{4.7}$$

and

$$m\left(1+2\lambda m+2m+3\lambda m^2\right)a_{m+1}^2 - 2m(1+2\lambda m)\,a_{2m+1} = B_1 c_{2m} + B_2 c_m^2. \tag{4.8}$$

Equations (4.5) and (4.7) now yield

$$c_m = -b_m. \tag{4.9}$$

By adding the equations (4.6) and (4.8), and after some computations using (4.5) and (4.9), we get

$$2m^2(1+\lambda m)\left({B_1}^2 - (1+\lambda m)B_2\right)a_{m+1}^2 = B_1^3\,(b_{2m}+c_{2m}). \tag{4.10}$$

Further, from the equations (4.9) and (4.10), together with (1.11), we have

$$\left|m^2(1+\lambda m)\left({B_1}^2 - (1+\lambda m)B_2\right)a_{m+1}^2\right| \leqq B_1^3\left(1-|b_m|^2\right). \tag{4.11}$$

Now, from (4.5) and (4.11), we get

$$|a_{m+1}| \leqq \frac{B_1\sqrt{B_1}}{m\sqrt{(1+\lambda m)\left|B_1{}^2-(1+\lambda m)B_2\right|+(1+\lambda m)^2 B_1}}$$

as asserted in (4.1).

By simple calculations using (4.6) and (4.8), together with the equations (4.5) and (4.9), we obtain

$$\begin{aligned} 4m^2(1+2\lambda m)(1+\lambda m)a_{2m+1} &= \left(1+2m+2\lambda m+3\lambda m^2\right)B_1 b_{2m} \\ &+ \left(1+2\lambda m+\lambda m^2\right)B_1 c_{2m} + 2(m+1)(1+2\lambda m)B_2 b m^2. \end{aligned} \tag{4.12}$$

Thus, by using the equation (1.11) in (4.12), we get

$$|a_{2m+1}| \leqq \frac{(1+m)B_1}{2m^2(1+\lambda m)} + \frac{(1+m)(|B_2|-B_1)|b_m|^2}{2m^2(1+\lambda m)}. \tag{4.13}$$

Since

$$|b_m|^2 \leqq \frac{m^2(1+\lambda m)^2 B_1}{m^2(1+\lambda m)\left(\left|B_1^2-(1+\lambda m)B_2\right|+(1+\lambda m)^2 B_1\right)}, \tag{4.14}$$

upon substituting from (4.14) into (4.13), we are led fairly easily to the assertion (4.2) of Theorem 5. Our demonstration of Theorem 5 is thus completed. □

REMARK 8. For the case of one-fold symmetric functions, Theorem 5 reduces to the following known results due to Peng *et al.* [19].

COROLLARY 18. (see [19]) *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathscr{M}_{\Sigma,1}(\lambda,\varphi)$. *Then*

$$|a_2| \leqq \frac{B_1\sqrt{B_1}}{\sqrt{(1+\lambda)\left|B_1{}^2-(1+\lambda)B_2\right|+(1+\lambda)^2 B_1}}$$

and

$$|a_3| \leqq \begin{cases} \frac{B_1}{1+\lambda} & (|B_2| \leqq B_1) \\ \frac{\left|B_1^2-(1+\lambda)B_2\right|B_1+(1+\lambda)|B_2|B_1}{(1+\lambda)\left[\left|B_1^2-(1+\lambda)B_2\right|+(1+\lambda)B_1\right]} & (|B_2| > B_1). \end{cases}$$

REMARK 9. For the case of one-fold symmetric functions, the estimates for $|a_2|$ and $|a_3|$ given by the equations (4.1) and (4.2) are smaller than those given by Theorem 2.3 in Ali *et al.* [1].

THEOREM 6. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathscr{M}_{\Sigma,m}(\lambda,\varphi)$. *Also let* $\lambda \geqq 0$. *Then*

$$\left|a_{2m+1}-\delta a_{m+1}^{2}\right| \leqq \begin{cases} \frac{B_1}{2m(2\lambda m+1)} & \text{for } 0 \leqq |h(\delta)| < \frac{1}{4m(2\lambda m+1)} \\ 2B_1\,|h(\delta)| & \text{for } \quad |h(\delta)| \geqq \frac{1}{4m(2\lambda m+1)} \end{cases}, \tag{4.15}$$

where

$$h(\delta)=\frac{B_1^2\,(m+1-2\delta)}{4m^2\left[(\lambda m+1)\,B_1^2-(\lambda m+1)^2 B_2\right]}.$$

Proof. From the equation (4.10), we get

$$a_{m+1}^{2}=\frac{B_1^3\,(b_{2m}+c_{2m})}{2m^2\left[(\lambda m+1)\,B_1^2-(\lambda m+1)^2 B_2\right]}. \tag{4.16}$$

Subract (3.6) from the (3.8), we get

$$a_{2m+1}=\frac{(m+1)}{2}a_{m+1}^{2}+\frac{B_1\,(b_{2m}-c_{2m})}{4m\,(2\lambda m+1)}. \tag{4.17}$$

From the equations (4.16) and (4.17), it follows that

$$a_{2m+1}-\delta a_{m+1}^{2}=B_1\left[\left(h(\delta)+\frac{1}{4m\,(2\lambda m+1)}\right)b_{2m}+\left(h(\delta)-\frac{1}{4m\,(2\lambda m+1)}\right)c_{2m}\right].$$

where

$$h(\delta)=\frac{B_1^2\,(m+1-2\delta)}{4m^2\left[(\lambda m+1)\,B_1^2-(\lambda m+1)^2 B_2\right]}.$$

Since all B_i are real and $B_1>0$, which implies the assertion (4.15). This completes the proof of Theorem 6. □

For the case of one-fold symmetric functions, Theorem 6 reduces to the following corollary.

COROLLARY 19. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathscr{M}_{\Sigma,1}(\lambda,\varphi)$. *Then*

$$\left|a_{3}-\delta a_{2}^{2}\right| \leqq \begin{cases} \frac{B_1}{2(2\lambda+1)} & \text{for } 0 \leqq |h(\delta)| < \frac{1}{4(2\lambda+1)} \\ 2B_1\,|h(\delta)| & \text{for } \quad |h(\delta)| \geqq \frac{1}{4(2\lambda+1)} \end{cases}.$$

Taking $\delta=1$ and $\delta=0$ in Theorem 6, we have the following corollaries.

COROLLARY 20. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathscr{M}_{\Sigma,m}(\lambda,\varphi)$. *Also let* $\lambda \geqq 0$. *Then*

$$\left|a_{2m+1}-a_{m+1}^2\right| \leqq \begin{cases} \frac{B_1}{2m(2\lambda m+1)} & \text{for } 0 \leqq |h(\delta)| < \frac{1}{4m(2\lambda m+1)} \\ 2B_1\,|h(\delta)| & \text{for } \quad |h(\delta)| \geqq \frac{1}{4m(2\lambda m+1)} \end{cases}.$$

For the case of one-fold symmetric functions, Corollary 20 reduces to the following corollary.

COROLLARY 21. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathscr{M}_{\Sigma,1}(\lambda,\varphi)$. *Then*

$$\left|a_3-a_2^2\right| \leqq \frac{B_1}{2(2\lambda+1)}.$$

COROLLARY 22. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathscr{M}_{\Sigma,m}(\lambda,\varphi)$. *Also let* $\lambda \geqq 0$. *Then*

$$|a_{2m+1}| \leqq \begin{cases} \frac{B_1}{2m(2\lambda m+1)} \\ \qquad \text{for } \frac{B_2}{B_1^2} \in \left(-\infty, -\frac{m(2\lambda m+\lambda+1)}{(\lambda m+1)^2}\right) \cup \left(\frac{2\lambda m^2+3\lambda m+m+2}{(\lambda m+1)^2}, \infty\right) \\ \frac{(m+1)B_1^3}{2m^2\left[(\lambda m+1)B_1^2-(\lambda m+1)^2B_2\right]} \\ \qquad \text{for } \frac{B_2}{B_1^2} \in \left(-\frac{m(2\lambda m+\lambda+1)}{(\lambda m+1)^2}, \frac{1}{(\lambda m+1)}\right) \cup \left(\frac{1}{(\lambda m+1)}, \frac{2\lambda m^2+3\lambda m+m+2}{(\lambda m+1)^2}\right) \end{cases}.$$

For the case of one-fold symmetric functions, Corollary 22 reduces to the following corollary.

COROLLARY 23. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathscr{M}_{\Sigma,1}(\lambda,\varphi)$. *Then*

$$|a_3| \leqq \begin{cases} \frac{B_1}{2(2\lambda+1)} & \text{for } \frac{B_2}{B_1^2} \in \left(-\infty, -\frac{(1+3\lambda)}{(1+\lambda)^2}\right) \cup \left(\frac{5\lambda+3}{(1+\lambda)^2}, \infty\right) \\ \frac{B_1^3}{(\lambda+1)B_1^2-(1+\lambda)^2B_2} & \text{for } \frac{B_2}{B_1^2} \in \left(-\frac{(1+3\lambda)}{(1+\lambda)^2}, \frac{1}{1+\lambda}\right) \cup \left(\frac{1}{1+\lambda}, \frac{5\lambda+3}{(1+\lambda)^2}\right) \end{cases}.$$

For the case of one-fold symmetric functions and $\lambda = 0$, Corollary 23 reduces to the following corollary.

COROLLARY 24. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathscr{M}_{\Sigma,1}(0,\varphi)$. *Then*

$$|a_3| \leqq \begin{cases} \frac{B_1}{2} & \text{for } \frac{B_2}{B_1^2} \in (-\infty,-1) \cup (3,\infty) \\ \frac{B_1^3}{B_1^2-B_2} & \text{for } \frac{B_2}{B_1^2} \in (-1,1) \cup (1,3) \end{cases}.$$

For the case of one-fold symmetric functions and $\lambda = 1$, Corollary 23 reduces to the following corollary.

COROLLARY 25. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathscr{M}_{\Sigma,1}(1,\varphi)$. *Then*

$$|a_3| \leqq \begin{cases} \frac{B_1}{6} & \text{for } \frac{B_2}{B_1^2} \in (-\infty,-1) \cup (2,\infty) \\ \frac{B_1^3}{2B_1^2-4B_2} & \text{for } \frac{B_2}{B_1^2} \in \left(-1,\frac{1}{2}\right) \cup \left(\frac{1}{2},3\right) \end{cases}.$$

5. Coefficient bounds for the function class $\mathscr{B}_{\Sigma,m}(\lambda,\varphi)$

We begin this section by introducing the function class $\mathscr{B}_{\Sigma,m}(\lambda,\varphi)$ as follows.

DEFINITION 4. A function $f(z)$, given by (1.5), is said to be in the class $\mathscr{B}_{\Sigma,m}(\lambda,\varphi)$ if the following conditions are satisfied:

$$f \in \Sigma_m, \qquad (1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec \varphi(z)$$

and

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) \prec \varphi(w),$$

where the function $g(w)$ is defined by (1.6).

For various special choices of the function $\varphi(z)$, and in the case when $m = 1$, the above-defined function class $\mathscr{B}_{\Sigma,m}(\lambda,\varphi)$ reduces to the following function classes.

1. For $m = 1$, the function class given by

$$\mathscr{B}_{\Sigma,m}(\lambda,\varphi) \equiv \mathscr{B}_{\Sigma,1}(\lambda,\varphi)$$

   was studied by Ali *et al.* [1].

2. For $m = 1$ and

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} \qquad (0 \leqq \gamma < 1),$$

   the function class given by

$$\mathscr{B}_{\Sigma,m}(\lambda,\varphi) \equiv \mathscr{B}_{\Sigma,1}\left(\lambda, \left(\frac{1+z}{1-z}\right)^{\gamma}\right)$$

   was studied by Srivastava *et al.* [14].

3. For $m = 1$ and
$$\varphi(z) = \frac{1+(1-2\gamma)z}{1-z} \qquad (0 \leqq \gamma < 1),$$
the function class given by
$$\mathscr{B}_{\Sigma,m}(\lambda,\varphi) \equiv \mathscr{B}_{\Sigma,1}\left(\lambda, \frac{1+(1-2\gamma)z}{1-z}\right)$$
was studied by Srivastava *et al.* [14].

We first state and prove the following theorem.

THEOREM 7. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathscr{B}_{\Sigma,m}(\lambda,\varphi)$. *Then*

$$|a_{m+1}| \leqq \frac{\mathscr{B}_1\sqrt{2\mathscr{B}_1}}{\sqrt{\left|(m+1)(1+2\lambda m)B_1^2 - 2(1+\lambda m)^2 B_2\right| + 2(\lambda m+1)^2 B_1}} \tag{5.1}$$

and

$$|a_{2m+1}| \leqq \begin{cases} \frac{B_1}{1+2\lambda m} & \left(B_1 \leqq \frac{2(1+\lambda m)^2}{(m+1)(1+2\lambda m)}\right) \\ \frac{\left|(m+1)(1+2\lambda m)B_1^2-2(1+\lambda m)^2B_2\right|B_1+(m+1)(1+2\lambda m)B_1^3}{(1+2\lambda m)\left(\left|(m+1)(1+2\lambda m)B_1^2-2(1+\lambda m)^2B_2\right|+2(\lambda m+1)^2B_1\right)} & \left(B_1 > \frac{2(1+\lambda m)^2}{(m+1)(1+2\lambda m)}\right). \end{cases} \tag{5.2}$$

Proof. Let $f \in \mathscr{B}_{\Sigma,m}(\lambda,\varphi)$ and $g(w) = f^{-1}(w)$. Then there are analytic functions $u : \mathbb{U} \to \mathbb{U}$ and $v : \mathbb{U} \to \mathbb{U}$, with
$$u(0) = v(0) = 0,$$
satisfying the following conditions:
$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = \varphi\big(u(z)\big) \tag{5.3}$$
and
$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) = \varphi\big(v(w)\big). \tag{5.4}$$
Since
$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = 1 + (\lambda m+1)a_{m+1}z^m + (2\lambda m+1)a_{2m+1}z^{2m} + \cdots$$

and

$$(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)$$
$$=1-(\lambda m+1)a_{m+1}w^m+(2\lambda m+1)\left[(m+1)a_{m+1}^2-a_{2m+1}\right]w^{2m}+\cdots,$$

it follows from (1.12), (1.13), (5.3) and (5.4) that

$$(\lambda m+1)a_{m+1}=B_1b_m, \tag{5.5}$$

$$(2\lambda m+1)a_{2m+1}=B_1b_{2m}+B_2b_m^2, \tag{5.6}$$

$$-(\lambda m+1)a_{m+1}=B_1c_m \tag{5.7}$$

and

$$(2\lambda m+1)\left[(m+1)a_{m+1}^2-a_{2m+1}\right]=B_1c_{2m}+B_2c_m^2. \tag{5.8}$$

From (5.5) and (5.7), we obtain

$$c_m=-b_m. \tag{5.9}$$

By adding (5.6) and (5.8), we get

$$\left((m+1)(2\lambda m+1)B_1^2-2(1+\lambda m)^2B_2\right)a_{m+1}^2=B_1^3(b_{2m}+c_{2m}). \tag{5.10}$$

Also, the equations (5.9) and (5.10), together with (1.11), imply that

$$\left|\left((m+1)(2\lambda m+1)B_1^2-2(1+\lambda m)^2B_2\right)a_{m+1}^2\right|\leqq B_1^3\left(1-|b_m|^2\right). \tag{5.11}$$

Now, from the equations (5.5) and (5.11), we deduce that

$$|a_{m+1}|\leqq\frac{B_1\sqrt{2B_1}}{\sqrt{\left|(m+1)(1+2\lambda m)B_1^2-2(1+\lambda m)^2B_2\right|+2(\lambda m+1)^2B_1}},$$

as asserted in (5.1).

Next, upon subtracting (5.8) from (5.6), we get

$$2(2\lambda m+1)a_{2m+1}=(2\lambda m+1)(m+1)a_{m+1}^2+B_1(b_{2m}-c_{2m}). \tag{5.12}$$

Using the equations (1.11) and (5.9) in (5.12), it follows that

$$2(2\lambda m+1)|a_{2m+1}|\leqq(m+1)(2\lambda m+1)|a_{m+1}|^2+B_1(|b_{2m}|+|c_{2m}|),$$

which, in view of (5.5), implies that

$$2(2\lambda m+1)B_1|a_{2m+1}|$$
$$\leqq\left((m+1)(2\lambda m+1)B_1-2(\lambda m+1)^2\right)|a_{m+1}|^2+2B_1^2. \tag{5.13}$$

Finally, by applying the equation (5.1) in (5.13), we arrive at the assertion (5.2) of Theorem 7. This obviously completes the proof of Theorem 7. □

REMARK 10. For the case of one-fold symmetric functions, Theorem 7 reduces to the corresponding results of Peng *et al.* [19], which we recall here as Corollary 26 below.

COROLLARY 26. (see [19]) *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathscr{B}_{\Sigma,1}(\lambda,\varphi)$. *Then*

$$|a_2| \leqq \frac{B_1\sqrt{B_1}}{\sqrt{(\lambda+1)^2 B_1 + \left|(2\lambda+1)B_1{}^2 - (\lambda+1)^2 B_2\right|}} \tag{5.14}$$

and

$$|a_3| \leqq \begin{cases} \frac{B_1}{1+2\lambda} & \left(B_1 \leqq \frac{(1+\lambda)^2}{1+2\lambda}\right) \\ \frac{\left|(1+2\lambda)B_1^2-(1+\lambda)^2 B_2\right|B_1+(1+2\lambda)B_1^3}{(1+2\lambda)\left(\left|(1+2\lambda)B_1^2-(1+\lambda)^2 B_2\right|+(\lambda+1)^2 B_1\right)} & \left(B_1 > \frac{(1+\lambda)^2}{1+2\lambda}\right). \end{cases} \tag{5.15}$$

For the case of one-fold symmetric functions and for the class of strongly starlike functions, the function φ is given by

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} = 1 + 2\gamma z + 2\gamma^2 z^2 + \cdots \qquad (0 < \gamma \leqq 1), \tag{5.16}$$

which gives

$$B_1 = 2\gamma \qquad \text{and} \qquad B_2 = 2\gamma^2.$$

Hence Theorem 7 gives us the following corollary.

COROLLARY 27. *Let the function* $f(z)$, *given by* (1.5), *be in the class* $\mathscr{B}_{\Sigma,1}\left(\lambda,\left(\frac{1+z}{1-z}\right)^{\gamma}\right)$. *Then*

$$|a_2| \leqq \frac{2\gamma}{\sqrt{(\lambda+1)^2 + |1+2\lambda-\lambda^2|\gamma}} \tag{5.17}$$

and

$$|a_3| \leqq \begin{cases} \frac{2\gamma}{1+2\lambda} & \left(0 < \gamma \leqq \frac{(1+\lambda)^2}{2(1+2\lambda)}\right) \\ \frac{2\gamma^2\left|1+2\lambda-\lambda^2\right|+4(1+2\lambda)\gamma^2}{(1+2\lambda)\left(\left|1+2\lambda-\lambda^2\right|\gamma+(\lambda+1)^2\right)} & \left(\frac{(1+\lambda)^2}{2(1+2\lambda)} < \gamma \leqq 1\right). \end{cases} \tag{5.18}$$

REMARK 11. The estimates for $|a_2|$ and $|a_3|$ given by Corollary 27 are more accurate than those given by Theorem 2.2 in Frasin and Aouf [6].

For the case of one-fold symmetric functions and for the class of starlike functions of order γ, the function φ is given by

$$\varphi(z) = 1 + 2(1-\gamma)z + 2(1-\gamma)z^2 + \cdots,$$

so that

$$B_1 = B_2 = 2(1-\gamma).$$

Clearly, therefore, Theorem 7 yields the following corollary.

COROLLARY 28. *Let the function $f(z)$, given by (1.5), be in the class $\mathscr{B}_{\Sigma,1}\left(\lambda, \frac{1+(1-2\gamma)z}{1-z}\right)$. Then*

$$|a_2| \leqq \frac{2(1-\gamma)}{\sqrt{(\lambda+1)^2 + \left|2(1+2\lambda)(1-\gamma) - (\lambda+1)^2\right|}} \tag{5.19}$$

and

$$|a_3| \leqq \begin{cases} \frac{2(1-\gamma)}{1+2\lambda} & \left(\frac{1+2\lambda-\lambda^2}{2(1+2\lambda)} \leqq \gamma < 1\right) \\ \frac{4(1+2\lambda)(1-\gamma)-(\lambda+1)^2}{(1+2\lambda)^2} & \left(0 \leqq \gamma < \frac{1+2\lambda-\lambda^2}{2(1+2\lambda)}\right). \end{cases} \tag{5.20}$$

REMARK 12. The estimates for $|a_2|$ and $|a_3|$ given by Corollary 28 are more accurate than those given by Theorem 3.2 in Frasin and Aouf [6].

THEOREM 8. *Let the function $f(z)$, given by (1.5), be in the class $\mathscr{B}_{\Sigma,m}(\lambda,\varphi)$. Then*

$$\left|a_{2m+1} - \delta a_{m+1}^2\right| \leqq \begin{cases} \frac{B_1}{(2\lambda m+1)} & \text{for } 0 \leqq |h(\delta)| < \frac{1}{2(2\lambda m+1)} \\ 2B_1|h(\delta)| & \text{for } \quad |h(\delta)| \geqq \frac{1}{2(2\lambda m+1)} \end{cases}, \tag{5.21}$$

where

$$h(\delta) = \frac{B_1^2(m+1-2\delta)}{2\left[(m+1)(2\lambda m+1)B_1^2 - 2(\lambda m+1)^2 B_2\right]}.$$

Proof. Adding the equations (5.6) and (5.8) we get

$$a_{m+1}^2 = \frac{B_1^3(b_{2m}+c_{2m})}{(m+1)(2\lambda m+1)B_1^2 - 2(\lambda m+1)^2 B_2}. \tag{5.22}$$

Subract (5.8) from the (5.6), we get

$$a_{2m+1} = \frac{(m+1)}{2} a_{m+1}^2 + \frac{B_1(b_{2m}-c_{2m})}{2(2\lambda m+1)}. \tag{5.23}$$

From the equations (5.22) and (5.23), it follows that

$$a_{2m+1}-\delta a_{m+1}^2=B_1\left[\left(h(\delta)+\frac{1}{2(2\lambda m+1)}\right)b_{2m}+\left(h(\delta)-\frac{1}{2(2\lambda m+1)}\right)c_{2m}\right],$$

where

$$h(\delta)=\frac{B_1^2(m+1-2\delta)}{2\left[(m+1)(2\lambda m+1)B_1^2-2(\lambda m+1)^2B_2\right]}.$$

Since all B_i are real and $B_1>0$, which implies the assertion (5.21). This completes the proof of Theorem 8. □

For the case of one-fold symmetric functions, Theorem 8 reduces to the following corollary.

COROLLARY 29. *Let the function $f(z)$, given by (1.5), be in the class $\mathscr{B}_{\Sigma,1}(\lambda,\varphi)$. Then*

$$\left|a_3-\delta a_2^2\right|\leqq\begin{cases}\frac{B_1}{(2\lambda+1)} & \text{for } 0\leqq|h(\delta)|<\frac{1}{2(2\lambda+1)}\\ 2B_1|h(\delta)| & \text{for } \quad |h(\delta)|\geqq\frac{1}{2(2\lambda+1)}\end{cases}.$$

Taking $\delta=1$ and $\delta=0$ in Theorem 8, we have the following corollaries.

COROLLARY 30. *Let the function $f(z)$, given by (1.5), be in the class $\mathscr{B}_{\Sigma,m}(\lambda,\varphi)$. Then*

$$\left|a_{2m+1}-a_{m+1}^2\right|\leqq\begin{cases}\frac{B_1}{(2\lambda m+1)} & \text{for } 0\leqq|h(\delta)|<\frac{1}{2(2\lambda m+1)}\\ 2B_1|h(\delta)| & \text{for } \quad |h(\delta)|\geqq\frac{1}{2(2\lambda m+1)}\end{cases}.$$

For the case of one-fold symmetric functions, Corollary 30 reduces to the following corollary.

COROLLARY 31. *Let the function $f(z)$, given by (1.5), be in the class $\mathscr{B}_{\Sigma,1}(\lambda,\varphi)$. Then*

$$\left|a_3-a_2^2\right|\leqq\frac{B_1}{(2\lambda+1)}.$$

COROLLARY 32. *Let the function $f(z)$, given by (1.5), be in the class $\mathscr{B}_{\Sigma,m}(\lambda,\varphi)$.*

Then

$$|a_{2m+1}| \leqq \begin{cases} \frac{B_1}{(2\lambda m+1)} \\ \qquad \text{for} \quad \frac{B_2}{B_1^2} \in (-\infty,0) \cup \left(\frac{(2\lambda m+1)(m+1)}{(\lambda m+1)^2},\infty\right) \\ \frac{(m+1)B_1^3}{(m+1)(2\lambda m+1)B_1^2-2(\lambda m+1)^2 B_2} \\ \qquad \text{for} \quad \frac{B_2}{B_1^2} \in \left(\frac{(2\lambda m+1)(m+1)}{2(\lambda m+1)^2},\frac{(2\lambda m+1)(m+1)}{(\lambda m+1)^2}\right) \cup \left(\frac{(2\lambda m+1)(m+1)}{2(\lambda m+1)^2},0\right) \end{cases}.$$

For the case of one-fold symmetric functions, Corollary 32 reduces to the following corollary.

COROLLARY 33. *Let the function $f(z)$, given by (1.5), be in the class $\mathscr{B}_{\Sigma,1}(\lambda,\varphi)$. Then*

$$|a_3| \leqq \begin{cases} \frac{B_1}{(2\lambda+1)} & \text{for} \quad \frac{B_2}{B_1^2} \in (-\infty,0) \cup \left(\frac{2(2\lambda+1)}{(1+\lambda)^2},\infty\right) \\ \frac{B_1^3}{(2\lambda+1)B_1^2-(1+\lambda)^2 B_2} & \text{for} \quad \frac{B_2}{B_1^2} \in \left(\frac{(2\lambda+1)}{(1+\lambda)^2},\frac{2(2\lambda+1)}{(1+\lambda)^2}\right) \cup \left(0,\frac{(2\lambda+1)}{(1+\lambda)^2}\right) \end{cases}.$$

For the case of one-fold symmetric functions and $\lambda = 0$, Corollary 33 reduces to the following corollary.

COROLLARY 34. *Let the function $f(z)$, given by (1.5), be in the class $\mathscr{B}_{\Sigma,1}(0,\varphi)$. Then*

$$|a_3| \leqq \begin{cases} B_1 & \text{for} \quad \frac{B_2}{B_1^2} \in (-\infty,0) \cup (2,\infty) \\ \frac{B_1^3}{B_1^2-B_2} & \text{for} \quad \frac{B_2}{B_1^2} \in (0,1) \cup (1,2) \end{cases}.$$

For the case of one-fold symmetric functions and $\lambda = 1$, Corollary 33 reduces to the following corollary.

COROLLARY 35. *Let the function $f(z)$, given by (1.5), be in the class $\mathscr{B}_{\Sigma,1}(1,\varphi)$. Then*

$$|a_3| \leqq \begin{cases} \frac{B_1}{3} & \text{for} \quad \frac{B_2}{B_1^2} \in (-\infty,0) \cup \left(\frac{3}{2},\infty\right) \\ \frac{B_1^3}{3B_1^2-4B_2} & \text{for} \quad \frac{B_2}{B_1^2} \in \left(0,\frac{3}{4}\right) \cup \left(\frac{3}{4},\frac{3}{2}\right) \end{cases}.$$

6. Concluding remarks and observations

In our present investigation, we have introduced and studied the coefficient problems associated with each of the following four new subclasses:

$$\mathscr{H}_{\Sigma,m}(\varphi), \quad \mathscr{N}_{\Sigma,m}(\lambda,\varphi), \quad \mathscr{M}_{\Sigma,m}(\lambda,\varphi) \qquad \text{and} \qquad \mathscr{B}_{\Sigma,m}(\lambda,\varphi)$$

of the class of m-fold symmetric bi-univalent functions in the open unit disk $\mathbb{U}$. These m-fold symmetric bi-univalent function classes are given by Definitions 1 to 4, respectively. For functions in each of these four m-fold symmetric bi-univalent function classes, we have derived the estimates of the Taylor-Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ and Fekete-Szegö functional problems for functions belonging to these new subclasses. The results presented in this paper have been shown to considerably improve the earlier results of Ali *et al.* [1], Frasin and Aouf [6], and Srivastava *et al.* [14] in terms of the bounds as well as the ranges of the parameter under consideration. Our results also further generalize the results of Peng *et al.* [19].

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