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AN INEQUALITY ABOUT PAIRS OF CONJUGATE HÖLDER NUMBERS

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Abstract. Most of the inequalities that we encounter in mathematics are based on a monotonicity or convexity argument. The functions that are constructed during a proof are monotone or convex (concave) throughout their domains. However, there are functions that change their monotonicity and convexity on their domains. In this case, a chopping of the domain into intervals on which a function is monotone or convex (concave) is necessary. Two inequalities about pairs of Hölder conjugate numbers are presented. One follows very elegantly from Young inequality, and the other requires chopping the domain into three subintervals, and proving the inequality differently on each of them.

1. Introduction

Pairs of Hölder conjugate numbers are important in Analysis for many reasons: they are involved in the classical Hölder inequality and dualities that exist between the L^p spaces, for $p \geqslant 1$. Moreover, in the recent years, sharp results about fundamental inequalities in Harmonic Analysis, involve some powers of Hölder conjugate numbers. For example, the classical Hausdorff–Young inequality says that for any number p in the interval $[1, 2]$, if q denotes its Hölder conjugate, then the Fourier transform is a bounded linear operator from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$, of operatorial norm less than or equal to 1. The constant 1 equals the operatorial norm only for $p = 1$ and $p = 2$. For p in $(1, 2)$, the exact operatorial norm was computed first in [2], at it was shown to be $(p^{1/p}/q^{1/q})^{1/2}$. Since the operatorial norm was known to be no more than 1, we obtain the following interesting inequality, involving Hölder conjugate numbers raised to powers depending on themselves:

$$p^{1/p} \leqslant q^{1/q},$$

for $1 \leqslant p \leqslant 2$ and $(1/p) + (1/q) = 1$.

Also, the sharp constant, giving the operatorial norm of the convolution product as a bounded bilinear operator from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ to $L^r(\mathbb{R})$, where p, q, and r are related by the condition:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1, \tag{1}$$

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which according to Young inequality, had been known to be at most 1, was proven by many authors to be:

$$\sqrt{\frac{p^{1/p}q^{1/q}r'^{1/r'}}{p'^{1/p'}q'^{1/q'}r^{1/r}}},$$

where p', q', and r' denote the Hölder conjugates of p, q, and r, respectively, see [1], [2] and [3], for example. Again, as a corollary, we obtain the inequality:

$$p^{1/p}q^{1/q}r'^{1/r'} \leqslant p'^{1/p'}q'^{1/q'}r^{1/r},$$

for p, q, and r related by the condition (1). We obtain thus a new inequality involving powers in which both the base and exponent depend on the same number. Of course, these inequalities about these powers are much simpler facts than the deep and hard results about the norms of various linear or multi–linear operators from Harmonic Analysis, but they present an interest per se, and as, we hope, this paper will demonstrate they are not always easy to prove.

We have used excuses from results about sharp constants in classical inequalities from Harmonic Analysis, to study inequalities involving Hölder conjugate numbers, that appear in both the bases and exponents of some powers. In this paper, we will prove a double inequality. While one of them follows smoothly from some convexity argument, the other one is much harder to prove.

2. An inequality about Hölder conjugate numbers

In this section we prove the result of this paper.

THEOREM 1. (Main) *Let p and q be positive Hölder conjugate numbers, that means:*

$$\frac{1}{p}+\frac{1}{q}=1.$$

Then the following inequalities hold:

$$\left(\frac{1}{p}\right)^{p-1}+\left(\frac{1}{q}\right)^{q-1} \leqslant 1 \leqslant \left(\frac{1}{p}\right)^{q-1}+\left(\frac{1}{q}\right)^{p-1}. \tag{2}$$

Proof. We will prove first the second inequality, namely, for all p, q positive and Hölder conjugate, we have:

$$\left(\frac{1}{p}\right)^{q-1}+\left(\frac{1}{q}\right)^{p-1} \geqslant 1.$$

Indeed, using the Young inequality:

$$\frac{1}{q}A^q+\frac{1}{p}B^p \geqslant AB,$$

for $A := 1/p$ and $B := 1/q$, we have;

$$\left(\frac{1}{p}\right)^{q-1} + \left(\frac{1}{q}\right)^{p-1} = pq\left[\frac{1}{q}\left(\frac{1}{p}\right)^{q} + \frac{1}{p}\left(\frac{1}{q}\right)^{p}\right]$$
$$\geqslant pq\cdot\left(\frac{1}{p}\cdot\frac{1}{q}\right) = 1.$$

Now, we will prove the first inequality:

$$\left(\frac{1}{p}\right)^{p-1} + \left(\frac{1}{q}\right)^{q-1} \leqslant 1.$$

Without loss of generality, let us assume that $p \leqslant q$, and thus, since p and q are Hölder conjugate, we have: $1 < p \leqslant 2$ and $2 \leqslant q < \infty$. We are going to split the interval $[2,\infty)$ corresponding to q, into three intervals:

$$[2,\infty) = I_q^1 \cup I_q^2 \cup I_q^3,$$

where:

$$I_q^1 := \left[\frac{12}{5},\infty\right), \quad I_q^2 := \left[\frac{11}{5},\frac{12}{5}\right], \quad I_q^3 := \left[2,\frac{11}{5}\right].$$

Accordingly, since:

$$p := \frac{q}{q-1},$$

the interval $(1,\, 2]$ corresponding to p, is split as:

$$(1,2] = I_p^1 \cup I_p^2 \cup I_p^3,$$

where:

$$I_p^1 := \left(1,\frac{12}{7}\right], \quad I_p^2 := \left[\frac{12}{7},\frac{11}{6}\right], \quad I_p^3 := \left[\frac{11}{6},2\right].$$

We analyze three cases:

Case 1. If $q \in I_q^1$, which means $p \in I_p^1$, then using the Bernoulli inequality:

$$(1+x)^r \leqslant 1 + rx,$$

for all $0 < r < 1$ and $x \geqslant -1$, we have:

$$\left(\frac{1}{p}\right)^{p-1} + \left(\frac{1}{q}\right)^{q-1} = \left(1-\frac{1}{q}\right)^{p-1} + \left(\frac{1}{q}\right)^{q-1}$$
$$\leqslant 1-(p-1)\frac{1}{q} + \left(\frac{1}{q}\right)^{q-1}$$
$$= 1 - \frac{1}{q(q-1)} + \frac{1}{q^{q-1}},$$

since:

$$(p-1)(q-1)=1.$$

If we can show now that:

$$1-\frac{1}{q(q-1)}+\frac{1}{q^{q-1}}\leqslant 1,$$

the proof in this case will be done. This last inequality is equivalent to:

$$q^{q-2}\geqslant q-1,$$

for all $q\geqslant 12/5$. Considering the function, $f:[12/5,\infty)\to\mathbb{R}$,

$$f(q)=q^{q-2}-q+1,$$

we can see that its derivative is:

$$f'(q)=q^{q-2}\left(\ln q+1-\frac{2}{q}\right)-1,$$

which is an increasing function since all the functions $q\mapsto q^{q-2}$, $q\mapsto \ln q$, and $q\mapsto -2/q$ are increasing. Thus, for all $q\in[12/5,\infty)$, we have:

$$\begin{aligned}
f'(q) &\geqslant f'\left(\frac{12}{5}\right)\\
&\geqslant 1\cdot\left[\ln\left(\frac{12}{5}\right)+1-\frac{5}{6}\right]-1\\
&=\frac{5}{6}\left[\ln\left(\frac{12}{5}\right)^{6/5}-1\right]\\
&>\frac{5}{6}(\ln 2.8-1)\\
&>\frac{5}{6}(\ln e-1)=0.
\end{aligned}$$

In the above inequality we used the fact that:

$$\left(\frac{12}{5}\right)^{6/5}>2.8,$$

that means:

$$\left(\frac{12}{5}\right)^{6/5}>\frac{14}{5},$$

which is equivalent to:

$$2^7\cdot 3^6>5\cdot 7^5,$$

that means:

$$93312 > 84035,$$

which is true.

Therefore, f is increasing on the interval $[12/5,\ \infty)$. Thus, for all $q \in I_q^1$, we have:

$$\begin{aligned} q^{q-2} - q + 1 &= f(q) \\ &\geqslant f\left(\frac{12}{5}\right) \\ &= \left(\frac{12}{5}\right)^{12/5-2} - \frac{12}{5} + 1 \\ &= \left(\frac{12}{5}\right)^{2/5} - \frac{7}{5} > 0. \end{aligned}$$

Here we used the fact that:

$$\left(\frac{12}{5}\right)^{2/5} > \frac{7}{5},$$

which is equivalent to:

$$12^2 \cdot 5^3 > 7^5.$$

This means:

$$18000 > 16807,$$

which is true. Thus, the proof is done in this case.

Case 2. If $q \in [11/5,\ 12/5]$, or equivalently $p \in [12/7,\ 11/6]$, then we have:

$$\left(\frac{1}{p}\right)^{p-1} + \left(\frac{1}{q}\right)^{q-1} = \left(\frac{1}{p}\right)^{p-1} + \frac{1}{q}\left(1 - \frac{1}{p}\right)^{q-2}.$$

Using now Taylor formula, with Lagrange remainder, for the function:

$$g(x) = (1-x)^{q-2},$$

there exists $c \in (0,\ 1/p)$, such that:

$$\begin{aligned} g\left(\frac{1}{p}\right) &= g(0) + g'(0) \cdot \frac{1}{p} + \frac{1}{2} g''(c) \cdot \frac{1}{p^2} \\ &= 1 - (q-2)\frac{1}{p} + \frac{(q-2)(q-3)}{2}(1-c)^{q-4}\frac{1}{p^2} \\ &= 1 - \frac{q-2}{p} - \frac{(q-2)(3-q)}{2(1-c)^{4-q}p^2} \\ &< 1 - \frac{q-2}{p} - \frac{(q-2)(3-q)}{2(1)^{4-q}p^2}, \end{aligned}$$

since $q-2>0$, $3-q>0$, and $4-q>0$.

Therefore, we have:

$$\begin{aligned}\left(\frac{1}{p}\right)^{p-1}+\left(\frac{1}{q}\right)^{q-1} &= \left(\frac{1}{p}\right)^{p-1}+\frac{1}{q}g\left(\frac{1}{p}\right)\\ &\leqslant \left(\frac{1}{p}\right)^{p-1}+\frac{1}{q}\left[1-\frac{q-2}{p}-\frac{(q-2)(3-q)}{2p^2}\right].\end{aligned}$$

If we can show now that:

$$\left(\frac{1}{p}\right)^{p-1}+\frac{1}{q}\left[1-\frac{q-2}{p}-\frac{(q-2)(3-q)}{2p^2}\right]\leqslant 1,$$

then the proof, in this case, will be done. This last inequality is equivalent to:

$$\begin{aligned}\left(\frac{1}{p}\right)^{p-1} &\leqslant \frac{1}{p}+\frac{q-2}{pq}+\frac{(q-2)(3-q)}{2p^2q}\\ &= \frac{1}{p}\cdot 2\cdot\frac{q-1}{q}+\frac{(q-2)(3-q)}{2p^2q}\\ &= 2\frac{1}{p^2}+\frac{(q-2)(3-q)}{2p^2q}.\end{aligned}$$

Multiplying, both sides by p^2, we must prove that, for all $p\in I_p^2$, we have:

$$p^{3-p}\leqslant 2+\frac{(q-2)(3-q)}{2q}.$$

We are going to show a little bit more than that, namely:

Claim 1.

$$\max_{p\in I_p^2} p^{3-p} < \min_{q\in I_q^2}\left[2+\frac{(q-2)(3-q)}{2q}\right].$$

To prove this claim, we will prove two sub–claims:

Subclaim 1.1. The function $u : I_p^2 \to \mathbb{R}$,

$$u(p) = p^{3-p}$$

is increasing.

To show this, we need to prove that the function $U(p) := \ln u(p)$ is increasing on I_p^2. Indeed, we have:

$$\begin{aligned}U'(p) &= \frac{d}{dp}[(3-p)\ln p]\\ &= -\ln p+\frac{3}{p}-1.\end{aligned}$$

We can see from here that U' is decreasing, since $p \mapsto -\ln p$ and $p \mapsto 3/p$ are both decreasing functions. Thus, for all $p \in [12/7,\, 11/6]$, we have:

$$\begin{aligned} U'(p) &\geqslant U'\left(\frac{11}{6}\right) \\ &= -\ln\left(\frac{11}{6}\right) + \frac{7}{11} \\ &= \frac{7}{11}\left[1 - \ln\left(\frac{11}{6}\right)^{11/7}\right] > 0, \end{aligned}$$

since he following inequalities hold:

$$\left(\frac{11}{6}\right)^{11/7} < \left(1 + \frac{1}{11}\right)^{11} < e. \tag{3}$$

Indeed, the second inequality from above follows from the fact that the sequence $x_n = (1+1/n)^n$, $n \geqslant 1$, is increasing and convergent to e. The first inequality of (3) is equivalent to:

$$11^8 < 6 \cdot 12^7,$$

that means:

$$214358881 < 214990848,$$

which is true.

Therefore, u is an increasing function, and so:

$$\begin{aligned} \max_{p \in I_p^2} p^{3-p} &= u\left(\frac{11}{6}\right) \\ &= \left(\frac{11}{6}\right)^{7/6}. \end{aligned} \tag{4}$$

Subclaim 1.2. The function $v : I_q^2 \to \mathbb{R}$,

$$v(q) = 2 + \frac{(q-2)(3-q)}{2q}$$

is increasing on I_q^2. Indeed, its derivative is:

$$v'(q) = \frac{6-q^2}{2q^2} > 0,$$

since $\sqrt{6} > 12/5 \geqslant q$, for all $q \in I_q^2$. Because v is increasing on I_q^2, we have:

$$\begin{aligned}\min_{q \in I_q^2}\left[2+\frac{(q-2)(3-q)}{2q}\right] &= v\left(\frac{11}{5}\right)\\ &= 2+\frac{(11/5-2)(3-11/5)}{2\cdot 11/5} \\ &= \frac{112}{55}.\end{aligned} \tag{5}$$

We can see from formulas (4) and (5) that to prove Claim 1, we have to show that:

$$\left(\frac{11}{6}\right)^{7/6} < \frac{112}{55}. \tag{6}$$

Inequality (6) is equivalent to:

$$\left(\frac{11}{6}\right)^{7} < \left(\frac{112}{55}\right)^{6}. \tag{7}$$

To prove (7) we will show that:

$$\left(\frac{11}{6}\right)^{7} < 70 < \left(\frac{112}{55}\right)^{6}. \tag{8}$$

Indeed, the first inequality of (8) is equivalent to:

$$11^7 < 70 \cdot 6^7,$$

which means:

$$19487171 < 19595520,$$

which is clearly true. Since $112/55 > 2$, the second inequality of (8) can be proven in the following way:

$$\begin{aligned}\left(\frac{112}{55}\right)^{6} &> 2\cdot\left(\frac{112}{55}\right)^{5}\\ &= 2\cdot 7\cdot\frac{16\cdot 112^4}{55^5}\\ &> 2\cdot 7\cdot 5 = 70,\end{aligned}$$

since

$$16\cdot 112^4 > 5\cdot 55^5,$$

which means:

$$2517630976 > 2516421875.$$

Therefore, the proof is also complete in this case.

Case 3. If $q \in I_q^3$, and accordingly $p \in I_p^3$, we can say that both p and q are in $I_p^3 \cup I_q^3 = [11/6,\ 11/5]$. We keep p and q conjugate. If we can show that the function $h : [11/6,\ 11/5] \to \mathbb{R}$,

$$h(x) = x^{2-x}$$

is concave, then using Jensen inequality we have:

$$\begin{aligned}
\left(\frac{1}{p}\right)^{p-1} + \left(\frac{1}{q}\right)^{q-1} &= \frac{1}{p}p^{2-p} + \frac{1}{q}q^{2-q} \\
&= \frac{1}{p}h(p) + \frac{1}{q}h(q) \\
&\leqslant h\left(\frac{1}{p}p + \frac{1}{q}q\right) \\
&= h(2) \\
&= 1,
\end{aligned}$$

and so, the proof will be complete. Therefore, to finish the proof, we have to prove the following:

Claim 2. The function h is concave on $I := [11/6,\ 11/5]$.

Indeed, its first derivative is:

$$h'(x) = h(x)\left(-\ln x + \frac{2}{x} - 1\right).$$

Differentiating one more time, we get:

$$\begin{aligned}
h''(x) &= h(x)\left[\left(-\ln x + \frac{2}{x} - 1\right)^2 - \left(\frac{1}{x} + \frac{2}{x^2}\right)\right] \\
&= \frac{h(x)}{x^2}\left[(x\ln x - 2 + x)^2 - (x+2)\right].
\end{aligned}$$

If we can show the following subclaim, then $h''(x) \leqslant 0$, and we will be done:

Subclaim 2.1. For all $x \in I$, we have:

$$0 \leqslant x\ln x - 2 + x \leqslant \sqrt{x+2}.$$

Indeed, since the function $r : I \to \mathbb{R}$,

$$r(x) := x\ln x - 2 + x$$

has an increasing first derivative:

$$r'(x) = \ln x + 2,$$

r is convex on I. It is obvious that the function $s : I \to \mathbb{R}$,

$$s(x) := \sqrt{x+2}$$

is concave. We will prove first the following inequalities:

$$r\left(\frac{11}{6}\right) < \frac{3}{2} < s\left(\frac{11}{6}\right) \tag{9}$$

and

$$r\left(\frac{11}{5}\right) < 2 < s\left(\frac{11}{5}\right). \tag{10}$$

Indeed, to prove (9), using the well known inequality:

$$\ln x \leqslant x-1,$$

for all $x > 0$, we have:

$$\begin{aligned} r\left(\frac{11}{6}\right) &= \frac{11}{6}\ln\left(\frac{11}{6}\right) - 2 + \frac{11}{6} \\ &\leqslant \frac{11}{6}\left(\frac{11}{6}-1\right) - \frac{1}{6} \\ &= \frac{49}{36} < \frac{3}{2} \\ &< \sqrt{\frac{11}{6}+2} = s\left(\frac{11}{6}\right). \end{aligned}$$

To prove (10), we proceed as follows. Since

$$\begin{aligned} e^{4/5} &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{4}{5}\right)^n \\ &> \sum_{n=0}^{3} \frac{1}{n!}\left(\frac{4}{5}\right)^n \\ &= 1 + \frac{4}{5} + \frac{8}{25} + \frac{32}{375} \\ &> 1 + \frac{4}{5} + \frac{8}{25} + \frac{30}{375} \\ &= \frac{11}{5}, \end{aligned}$$

we have:

$$\ln\left(\frac{11}{5}\right) < \frac{4}{5} < \frac{9}{11}. \tag{11}$$

Using inequality (11), we obtain:

$$\begin{aligned} r\left(\frac{11}{5}\right) &= \frac{11}{5}\ln\left(\frac{11}{5}\right) - 2 + \frac{11}{5} \\ &< \frac{11}{5}\cdot\frac{9}{11} + \frac{1}{5} \\ &= 2 \\ &< \sqrt{\frac{11}{5}+2} = s\left(\frac{11}{5}\right). \end{aligned}$$

Since r is convex and s is concave, the graph of r is below the line segment joining the points $(11/6,\ r(11/6))$ and $(11/5,\ r(11/5))$, which in turn is below the line segment joining the points $(11/6,\ s(11/6))$ and $(11/5,\ s(11/5))$, which in turn is below the graph of s. Thus for all $x \in I$, we have:

$$r(x) \leqslant s(x).$$

It remains to prove only that for all $x \in I$, we have:

$$r(x) > 0.$$

Since obviously r is an increasing function, for all $x \in I$, we have:

$$\begin{aligned} r(x) &\geqslant r(11/6) \\ &= \ln\left(\left(1+\frac{5}{6}\right)^{11/6}\right) - \frac{1}{6} \\ &> \ln\left(1+\frac{11}{6}\cdot\frac{5}{6}\right) - \frac{1}{6} \\ &> \ln(2) - \frac{1}{2} > 0, \end{aligned}$$

since $e < 4$. The proof is now complete. □

The double inequality (2) can be extended through continuity to the case $p = 1$ and $q = \infty$. In that case, since:

$$\lim_{p\to 1^+}\left(\frac{1}{p}\right)^{p-1} = 1,$$

$$\lim_{q\to\infty}\left(\frac{1}{q}\right)^{q-1} = 0,$$

$$\begin{aligned} \lim_{p\to 1^+}\left(\frac{1}{p}\right)^{q-1} &= \lim_{p\to 1^+}\left(\frac{1}{p}\right)^{1/(p-1)} \\ &= \lim_{p\to 1^+}\left[\left(1+\frac{1-p}{p}\right)^{p/(1-p)}\right]^{-1/p} \\ &= e^{-1}, \end{aligned}$$

and

$$\lim_{q\to\infty}\left(\frac{1}{q}\right)^{p-1} = \lim_{q\to\infty}\left(\frac{1}{q}\right)^{1/(q-1)}$$
$$= 1,$$

the double inequality (2) becomes:

$$1 \leqslant 1 \leqslant 1+\frac{1}{e}.$$

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