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SOME VOLTERRA–FREDHOLM TYPE NONLINEAR INEQUALITIES INVOLVING FOUR ITERATED INFINITE INTEGRAL AND APPLICATION

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Abstract. In this paper, we establish some four iterated infinite integral inequalities, which includes a nonconstant term outside the integrals. The upper bound of the embedded unknown function is estimated explicitly by adopting novel analysis techniques, such as: change of variable, amplification method, differential and integration. The derived result can be applied in the study of qualitative properties of solutions of infinite integral equations.

1. Introduction

In the study of qualitative properties of solutions of differential equations, integral equations and difference equations, one often deals with certain integral inequalities and their discrete versions. The well-known Gronwall-Bellman inequality [1, 2] can be equivalently regarded as the following

$$u(x) \leqslant c + \int_a^x f(s)u(s)ds, \quad x \in [a, a+X], \tag{1}$$

where $c \geqslant 0, a$ are constants, the function f is given, nonnegative and continuous, and u is the unknown. In 1956 Bihari [3] discussed the nonlinear integral inequality

$$u(x) \leqslant c + \int_0^x f(s)w(u(s))ds. \tag{2}$$

In 1990 Pinto [4] investigated the integral inequality with summing

$$u(x) \leqslant c(x) + \sum_{i=1}^{n} \int_a^x g_i(s)w_i(u(s))ds. \tag{3}$$

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Replacing the upper limit x of the integral with a function $b(x)$ in (2), in 2000 Lipovan [5] improved Bihari's results by studying the following retarded integral inequalities

$$u(x) \leqslant c + \int_{b(x_0)}^{b(x)} f(s)w(u(s))ds, \tag{4}$$

and

$$u(x) \leqslant c + \int_{t_0}^{x} f(s)w(u(s))ds + \int_{b(t_0)}^{b(x)} g(s)w(u(s))ds. \tag{5}$$

In 2005 Agarwal, Deng and Zhang [6] generally discussed the retarded Gronwall-like integral inequality

$$u(x) \leqslant c(x) + \sum_{i=1}^{n} \int_{b_i(x_0)}^{b_i(x)} g_i(t,s)w_i(u(s))ds, \quad x_0 \leqslant x < X. \tag{6}$$

In 2011, Abdeldaim et al. [7] studied the following iterated integral inequality

$$u(x) \leqslant u_0 + \int_0^x g(s)u(s)\Big[u(s) + \int_0^s h(\tau)\Big[u(\tau) + \int_0^\tau r(\xi)u(\xi)d\xi\Big]d\tau\Big]ds. \tag{7}$$

In order to investigate the boundary value problem of differential equation, Bainov and Simeonov [8] studied a Volterra-Fredholm tpye integral inequality,

$$u(t) \leqslant c + \int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{X} g(s)u(s)ds, \ \ t \in [t_0, X]. \tag{8}$$

In 2002, Pachpatte [9] generalized the above inequality to the inequality

$$u(t) \leqslant c + \int_{t_0}^{t} f(t,s)u(s)ds + \int_{t_0}^{X} g(t,s)u(s)ds, \ \ t \in [t_0, X]. \tag{9}$$

In 2004, Pachpatte [10] further studied the following Volterra-Fredholm type inequality

$$\begin{aligned} u(t) \leqslant c &+ \int_{\alpha(t_0)}^{\alpha(t)} a(t,s)\Big[f(s)u(s)ds + \int_{\alpha(t_0)}^{s} c(s,\tau)u(\tau)d\tau\Big]ds \\ &+ \int_{\alpha(t_0)}^{\alpha(X)} b(t,s)u(s)ds, \ \ t \in [t_0, X]. \end{aligned} \tag{10}$$

In 2008 Ma and Pečarić [11] discussed more generally the Volterra-Fredholm type inequality

$$\begin{aligned} u(t) \leqslant c &+ \int_{\alpha(t_0)}^{\alpha(t)} \sigma_1(s)\Big[f(s)w(u(s)) + \int_{\alpha(t_0)}^{s} \sigma_2(\tau)w(u(\tau))d\tau\Big]ds \\ &+ \int_{\alpha(t_0)}^{\alpha(X)} \sigma_1(s)\Big[f(s)w(u(s)) + \int_{\alpha(t_0)}^{s} \sigma_2(\tau)w(u(\tau))d\tau\Big]ds, \ \ t \in [t_0, X]. \end{aligned} \tag{11}$$

In 2012, Zheng and Fu [12] studied the discrete version of Volterra-Fredholm type inequality involving infinite sums

$$\begin{aligned}\phi(u(m,n)) \leqslant a(m,n) + \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\Big[b_i(s,t,m,n)\varphi(u(s,t))\\ + \sum_{\xi=s+1}^{\infty}\sum_{\eta=t+1}^{\infty} c_i(\xi,\eta,s,t)\varphi(u(\xi,\eta))\Big]\\ + \sum_{i=1}^{l_2}\sum_{M+1}^{\infty}\sum_{N+1}^{\infty}\Big[d_i(s,t,x,y)\phi(u(s,t))\\ + \sum_{\xi=s+1}^{\infty}\sum_{\eta=t+1}^{\infty} e_i(\xi,\eta,s,t)\phi(u(\xi,\eta))\Big]. \qquad (12)\end{aligned}$$

As required in estimation for solutions, invariant sets, and stability, many generalized versions of the Gronwall-Bellman inequality were given with an invariant decomposition ([13, 14, 15]), a singular kernel ([16, 17]), and maxima ([18, 19]). More results about integral inequalities of single variable and multivariables can be found from for example ([20, 21, 22, 23, 24]). The results about the discrete version of Volterra-Fredholm type inequality can be found from for example ([25, 26, 27, 28, 29]).

In this paper, based on the works of [6, 11, 12], we discuss some four iterated infinite integral inequalities.

2. Main result

Throughout this paper, let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+ := [0,\infty)$, $X,Y \in \mathbb{R}_+, \Omega = [X,\infty)\times[Y,\infty)$.

LEMMA 1. *Suppose that $u,b \in C(\Omega,\mathbf{R}_+)$, $a \in C(\Omega,\mathbf{R}_+)$ is a nonincreasing function in the first variable, and $\alpha \in C^1([X,\infty),[X,\infty))$ is nondecreasing such that $\alpha(x) \geqslant x$, $\alpha(X) = X$, $\alpha(\infty) = \infty$. If $u(x,y)$ satisfies*

$$u(x,y) \leqslant a(x,y) + \int_{\alpha(x)}^{\infty} b(s,y)u(s,y)ds, \quad x \in [X,\infty), \qquad (13)$$

then

$$u(x,y) \leqslant a(x,y)\exp\Big(\int_{\alpha(x)}^{\infty} b(s,y)ds\Big), \quad x \in [X,\infty). \qquad (14)$$

Proof. Fixed $X_1 \in [X,\infty)$. From (13), we have

$$\begin{aligned}u(x,y) &\leqslant a(x,y) + \int_{\alpha(x)}^{\infty} b(s,y)u(s,y)ds\\ &\leqslant a(X_1,y) + \int_{\alpha(x)}^{\infty} b(s,y)u(s,y)ds, \quad x \in [X_1,\infty). \qquad (15)\end{aligned}$$

Let

$$z_1(x,y) = a(X_1,y) + \int_{\alpha(x)}^{\infty} b(s,y)u(s,y)ds, \ \ x \in [X_1,\infty). \tag{16}$$

Then

$$z_1(\infty,y) = a(X_1,y), u(x,y) \leqslant z_1(x,y). \tag{17}$$

Differentiating $z_1(x,y)$ with respect to x, using (17) we have

$$\begin{aligned} z'_{1x}(x,y) &= -\alpha'(x)b(\alpha(x),y)u(\alpha(x),y) \\ &\geqslant -\alpha'(x)b(\alpha(x),y)z_1(\alpha(x),y) \\ &\geqslant -\alpha'(x)b(\alpha(x),y)z_1(x,y), \ \ x \in [X_1,\infty). \end{aligned} \tag{18}$$

From (18), we have

$$\frac{z'_{1x}(x,y)}{z_1(x,y)} \geqslant -\alpha'(x)b(\alpha(x),y), \ \ x \in [X_1,\infty). \tag{19}$$

Integrating both sides of the above inequality in the first variable from x to ∞, we have

$$\ln z_1(\infty,y) - \ln z_1(x,y) \geqslant -\int_{\alpha(x)}^{\infty} b(s,y)ds, \ \ x \in [X_1,\infty), \tag{20}$$

that is

$$\ln z_1(x,y) \leqslant \ln z_1(\infty,y) + \int_{\alpha(x)}^{\infty} b(s,y)ds, \ \ x \in [X_1,\infty). \tag{21}$$

Using (17) and (21), we obtain

$$u(x,y) \leqslant z_1(x,y) \leqslant a(X_1,y)\exp\Big(\int_{\alpha(x)}^{\infty} b(s,y)ds\Big), \ \ x \in [X_1,\infty). \tag{22}$$

since X_1 is chosen arbitrarily, from (22) we get the desired estimation (14). □

LEMMA 2. *Suppose that* $u \in C(\Omega,\mathbf{R}_+)$, $H \in C(\Omega,\mathbf{R}_+)$ *is a nonincreasing function in the first variable, and* $b \in C(\Omega^2,\mathbf{R}_+)$ *is nonincreasing in the third variable with* $H(x,y) > 0$. $\phi,\varphi \in C(\mathbf{R}_+,\mathbf{R}_+)$ *are strictly increasing with* $\phi(r) > 0$, $\varphi(r) > 0$ *for* $r > 0$. *Suppose that* $\alpha \in C^1([X,\infty),[X,\infty))$ $(\beta \in C^1([Y,\infty),[Y,\infty)))$ *is nondecreasing such that* $\alpha(x) \geqslant x$, $\alpha(X) = X$, $\alpha(\infty) = \infty$ $(\beta(y) \geqslant y$, $\beta(Y) = Y$, $\beta(\infty) = \infty)$. *If* $u(x,y)$ *satisfies*

$$u(x,y) \leqslant H(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t,x,y)\varphi(\phi^{-1}(u(s,t)))dtds, \ \ (x,y) \in \Omega. \tag{23}$$

Then

$$u(x,y) \leqslant \Phi_1^{-1}\Big(\Phi_1(H(x,y)) + \textstyle\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t,x,y)dtds\Big), \ \ (x,y) \in \Omega, \tag{24}$$

where

$$\Phi_1(z) = \int_c^z \frac{ds}{\varphi(\phi^{-1}(s))}, \quad z \geqslant c > 0. \tag{25}$$

Proof. Fixed $X_1 \in [X,\infty)$. From (23), we have

$$\begin{aligned} u(x,y) &\leqslant H(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t,x,y)\varphi(\phi^{-1}(u(s,t)))dtds \\ &\leqslant H(X_1,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t,X_1,y)\varphi(\phi^{-1}(u(s,t)))dtds, \end{aligned} \tag{26}$$

for all $(x,y) \in [X_1,\infty)\times[Y,\infty)$. Let

$$z_2(x,y) = H(X_1,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t,X_1,y)\varphi(\phi^{-1}(u(s,t)))dtds, \tag{27}$$

for all $(x,y) \in [X_1,\infty)\times[Y,\infty)$. Then

$$u(x,y) \leqslant z_2(x,y), z(\infty,y) = H(X_1,y), \forall (x,y) \in [X_1,\infty)\times[Y,\infty). \tag{28}$$

Obviously, $z_2(x,y)$ is a nonincreasing function in every variable. Since $\alpha(x) \geqslant x$ and $\beta(y) \geqslant y$, we have $z_2(\alpha(x),\beta(y)) \leqslant z(x,y)$.

Differentiating $z_2(x,y)$ with respect to x, using (28) we have

$$\begin{aligned} z'_{2x}(x,y) &= -\alpha'(x)\int_{\beta(y)}^{\infty} b(\alpha(x),t,X_1,y)\varphi(\phi^{-1}(u(\alpha(x),t)))dt \\ &\geqslant -\alpha'(x)\int_{\beta(y)}^{\infty} b(\alpha(x),t,X_1,y)\varphi(\phi^{-1}(z_2(\alpha(x),t)))dt \\ &\geqslant -\alpha'(x)\varphi(\phi^{-1}(z_2(\alpha(x),\beta(y))))\int_{\beta(y)}^{\infty} b(\alpha(x),t,X_1,y)dt \\ &\geqslant -\alpha'(x)\varphi(\phi^{-1}(z_2(x,y)))\int_{\beta(y)}^{\infty} b(\alpha(x),t,X_1,y)dt, \end{aligned} \tag{29}$$

for all $(x,y) \in [X_1,\infty)\times[Y,\infty)$. From (29) we get

$$\frac{z'_{2x}(x,y)}{\varphi(\phi^{-1}(z_2(x,y)))} \geqslant -\alpha'(x)\textstyle\int_{\beta(y)}^{\infty} b(\alpha(x),t,X_1,y)dt. \tag{30}$$

Integrating both sides of the above inequality in the first variable from x to ∞, we obtain

$$\Phi_1(z_2(\infty,y)) - \Phi_1(z_2(x,y)) \geqslant -\textstyle\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t,X_1,y)dtds, \tag{31}$$

i.e.

$$\begin{aligned} \Phi_1(z_2(x,y)) &\leqslant \Phi_1(z_2(\infty,y)) + \textstyle\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t,X_1,y)dtds \\ &\leqslant \Phi_1(H(X_1,y)) + \textstyle\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t,X_1,y). \end{aligned} \tag{32}$$

It implies that

$$z_2(x,y) \leqslant \Phi_1^{-1}\Big(\Phi_1(H(X_1,y)) + \textstyle\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} b(s,t,X_1,y)dtds\Big), \tag{33}$$

for all $(x,y) \in [X_1,\infty)\times[Y,\infty)$. since X_1 is chosen arbitrarily, from (33) we get the desired estimation (24). □

THEOREM 1. *Suppose that* $u \in C(\Omega, \mathbf{R}_+)$, $a \in C(\Omega, \mathbf{R}_+)$ *is a nonincreasing function in every variable, and* $b_i,\ c_i \in C(\Omega^2, \mathbf{R}_+)$, $i = 1,2,\cdots,\ l_1$, $d_i,\ e_i \in C(\Omega^2, \mathbf{R}_+)$, $i = 1,2,\cdots,l_2$ *are nonincreasing in the third and fourth variables, and there is at least one function among* $d_i,\ e_i,\ i = 1,2,\cdots,l_2$ *not equivalent to zero,* $\phi,\ \varphi,\ \alpha,\ \beta$ *are defined as in Lemma 2. If* $u(x,y)$ *satisfies*

$$\begin{aligned}\phi(u(x,y)) \leqslant a(x,y) &+ \sum_{i=1}^{l_1} \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} \Big[b_i(s,t,x,y)\varphi(u(s,t)) \\ &+ \int_s^{\infty} \int_t^{\infty} c_i(\xi,\eta,s,t)\varphi(u(\xi,\eta))d\eta d\xi \Big] dtds \\ &+ \sum_{i=1}^{l_2} \int_{\alpha(X)}^{\infty} \int_{\beta(y)}^{\infty} \Big[d_i(s,t,x,y)\phi(u(s,t)) \\ &+ \int_s^{\infty} \int_t^{\infty} e_i(\xi,\eta,s,t)\phi(u(\xi,\eta))d\eta d\xi \Big] dtds, \quad (x,y) \in \Omega. \end{aligned} \tag{34}$$

Then

$$\begin{aligned} u(x,y) \leqslant \phi^{-1}\Big\{ \Phi_1^{-1}\Big\{ \Phi_1 \Big[\Psi_1^{-1} \Big[\int_{\alpha(X)}^{\infty} \int_{\beta(Y)}^{\infty} \sum_{i=1}^{l_1} \Big(b_i(s,t,X,Y) \\ + \int_s^{\infty} \int_t^{\infty} c_i(\xi,\eta,s,t)d\eta d\xi \Big) dtds \Big]\Big] + \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} \sum_{i=1}^{l_1} \Big[b_i(s,t,x,y) \\ + \int_s^{\infty} \int_t^{\infty} c_i(\xi,\eta,s,t)d\eta d\xi \Big] dtds \Big\}\Big\}, \quad (x,y) \in \Omega, \end{aligned} \tag{35}$$

provided that Ψ_1 *is increasing, where* Φ_1 *is defined as in Lemma 2, and*

$$\Psi_1(u) = \Phi_1\Big(\frac{u-\mu_1}{\mu_2}\Big) - \Phi_1(u), \tag{36}$$

$$\mu_1 = a(X,Y), \tag{37}$$

$$\mu_2 = \sum_{i=1}^{l_2} \int_{\alpha(X)}^{\infty} \int_{\beta(Y)}^{\infty} \Big[d_i(s,t,X,Y) + \int_s^{\infty} \int_t^{\infty} e_i(\xi,\eta,s,t)d\eta d\xi \Big] dtds. \tag{38}$$

Proof. Define a function $z_3(x,y)$ by the function on the right-hand side of (34). Then $z_3(x,y)$ is nonincreasing in every variable,

$$u(x,y) \leqslant \phi^{-1}(z_3(x,y)), \quad \forall (x,y) \in [X,\infty) \times [Y,\infty), \tag{39}$$

and

$$\begin{aligned} z_3(x,y) \leqslant a(x,y) &+ \sum_{i=1}^{l_1} \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} \Big[b_i(s,t,x,y)\varphi(\phi^{-1}(z_3(s,t))) \\ &+ \int_s^{\infty} \int_t^{\infty} c_i(\xi,\eta,s,t)\varphi(\phi^{-1}(z_3(\xi,\eta)))d\eta d\xi \Big] dtds \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{l_2}\int_{\alpha(X)}^{\infty}\int_{\beta(y)}^{\infty}\Big[d_i(s,t,x,y)z_3(s,t)\\
&+\int_s^{\infty}\int_t^{\infty}e_i(\xi,\eta,s,t)z_3(\xi,\eta)d\eta d\xi\Big]dtds\\
\leqslant\ & a(X,Y)+\sum_{i=1}^{l_1}\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\Big[b_i(s,t,x,y)\\
&+\int_s^{\infty}\int_t^{\infty}c_i(\xi,\eta,s,t)d\eta d\xi\Big]\varphi(\phi^{-1}(z_3(s,t)))dtds\\
&+\sum_{i=1}^{l_2}\int_{\alpha(X)}^{\infty}\int_{\beta(Y)}^{\infty}\Big[d_i(s,t,x,y)\\
&+\int_s^{\infty}\int_t^{\infty}e_i(\xi,\eta,s,t)d\eta d\xi\Big]z_3(s,t)dtds\\
=\ & H(X,Y)+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\sum_{i=1}^{l_1}\Big[b_i(s,t,x,y)\\
&+\int_s^{\infty}\int_t^{\infty}c_i(\xi,\eta,s,t)d\eta d\xi\Big]\varphi(\phi^{-1}(z_3(s,t)))dtds, \qquad (40)
\end{aligned}
$$

for all $(x,y)\in[X,\infty)\times[Y,\infty)$, where

$$
\begin{aligned}
H(X,Y) = a(X,Y)&+\sum_{i=1}^{l_2}\int_{\alpha(X)}^{\infty}\int_{\beta(Y)}^{\infty}\Big[d_i(s,t,X,Y)\\
&+\int_s^{\infty}\int_t^{\infty}e_i(\xi,\eta,s,t)d\eta d\xi\Big]z_3(s,t)dtds>0. \qquad (41)
\end{aligned}
$$

By a suitable application of Lemma 2, from (40) we obtain

$$
\begin{aligned}
z_3(x,y)\leqslant \Phi_1^{-1}\Big(&\Phi_1(H(X,Y))+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\sum_{i=1}^{l_1}\Big[b_i(s,t,x,y)\\
&+\int_s^{\infty}\int_t^{\infty}c_i(\xi,\eta,s,t)d\eta d\xi\Big]dtds\Big), \qquad (42)
\end{aligned}
$$

and

$$
\begin{aligned}
H(X,Y) =\ & a(X,Y)+\sum_{i=1}^{l_2}\int_{\alpha(X)}^{\infty}\int_{\beta(Y)}^{\infty}\Big[d_i(s,t,X,Y)\\
&+\int_s^{\infty}\int_t^{\infty}e_i(\xi,\eta,s,t)d\eta d\xi\Big]z_3(s,t)dtds\\
\leqslant\ & a(X,Y)+z_3(X,Y)\sum_{i=1}^{l_2}\int_{\alpha(X)}^{\infty}\int_{\beta(Y)}^{\infty}\Big[d_i(s,t,X,Y)\\
&+\int_s^{\infty}\int_t^{\infty}e_i(\xi,\eta,s,t)d\eta d\xi\Big]dtds\\
\leqslant\ & a(X,Y)+\Phi_1^{-1}\Big(\Phi_1(H(X,Y))+\int_{\alpha(X)}^{\infty}\int_{\beta(Y)}^{\infty}\sum_{i=1}^{l_1}\Big[b_i(s,t,X,Y)
\end{aligned}
$$

$$+\int_s^\infty\int_t^\infty c_i(\xi,\eta,s,t)d\eta d\xi\Big]dtds\Big)\sum_{i=1}^{l_2}\int_{\alpha(X)}^\infty\int_{\beta(Y)}^\infty\Big[d_i(s,t,X,Y)$$

$$+\int_s^\infty\int_t^\infty e_i(\xi,\eta,s,t)d\eta d\xi\Big]dtds$$

$$=\mu_1+\Phi_1^{-1}\Big(\Phi_1(H(X,Y))+\int_{\alpha(X)}^\infty\int_{\beta(Y)}^\infty\sum_{i=1}^{l_1}\Big[b_i(s,t,X,Y)$$

$$+\int_s^\infty\int_t^\infty c_i(\xi,\eta,s,t)d\eta d\xi\Big]dtds\Big)\mu_2. \tag{43}$$

It implies that

$$\Phi_1\Big(\frac{H(X,Y)-\mu_1}{\mu_2}\Big)\leqslant\Phi_1(H(X,Y))+\int_{\alpha(X)}^\infty\int_{\beta(Y)}^\infty\sum_{i=1}^{l_1}\Big[b_i(s,t,X,Y)$$

$$+\int_s^\infty\int_t^\infty c_i(\xi,\eta,s,t)d\eta d\xi\Big]dtds, \tag{44}$$

which is rewritten as

$$\Psi_1(H(X,Y))\leqslant\int_{\alpha(X)}^\infty\int_{\beta(Y)}^\infty\sum_{i=1}^{l_1}\Big[b_i(s,t,X,Y)$$

$$+\int_s^\infty\int_t^\infty c_i(\xi,\eta,s,t)d\eta d\xi\Big]dtds, \tag{45}$$

where Ψ_1 is defined in (36). Since Ψ_1 is increasing, we obtain

$$H(X,Y)\leqslant\Psi_1^{-1}\Big(\int_{\alpha(X)}^\infty\int_{\beta(Y)}^\infty\sum_{i=1}^{l_1}\Big[b_i(s,t,X,Y)$$

$$+\int_s^\infty\int_t^\infty c_i(\xi,\eta,s,t)d\eta d\xi\Big]dtds\Big). \tag{46}$$

Combining (39), (42) and (46), we get the desired estimation (35). □

THEOREM 2. *Suppose that $w\in C(\Omega,\mathbf{R}_+)$, u, a, b_i, c_i, $i=1,2,\cdots,l_1$, d_i, e_i, $i=1,2,\cdots,l_2$, α, β, φ, ϕ are defined as in Theorem 1. Furthermore, assume $\varphi\circ\phi^{-1}$ is submultiplicative, that is, $\varphi(\phi^{-1}(xy))\leqslant\varphi(\phi^{-1}(x)\varphi(\phi^{-1}(y))$, $x,y\in\mathbb{R}_+$. If $u(x,y)$ satisfies*

$$\phi(u(x,y))\leqslant a(x,y)+\int_{\alpha(x)}^\infty w(s,y)\phi(u(s,y))ds$$

$$+\sum_{i=1}^{l_1}\int_{\alpha(x)}^\infty\int_{\beta(y)}^\infty\Big[b_i(s,t,x,y)\varphi(u(s,t))$$

$$+\int_s^\infty\int_t^\infty c_i(\xi,\eta,s,t)\varphi(u(\xi,\eta))d\eta d\xi\Big]dtds$$

$$+\sum_{i=1}^{l_2}\int_{\alpha(X)}^{\infty}\int_{\beta(y)}^{\infty}\Big[d_i(s,t,x,y)\phi(u(s,t)) + \int_s^{\infty}\int_t^{\infty} e_i(\xi,\eta,s,t)\phi(u(\xi,\eta))d\eta d\xi\Big]dtds, \quad (x,y)\in\Omega. \tag{47}$$

Then

$$u(x,y) \leqslant \phi^{-1}\Big\{\Phi_1^{-1}\Big\{\Phi_1\Big\{\Psi_2^{-1}\Big\{\int_{\alpha(X)}^{\infty}\int_{\beta(Y)}^{\infty}\sum_{i=1}^{l_1}\Big[\tilde{b}_i(s,t,X,Y) + \int_s^{\infty}\int_t^{\infty}\tilde{c}_i(\xi,\eta,s,t)d\eta d\xi\Big]dtds\Big\}\Big\} + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\sum_{i=1}^{l_1}\Big[\tilde{b}_i(s,t,x,y) + \int_s^{\infty}\int_t^{\infty}\tilde{c}_i(\xi,\eta,s,t)d\eta d\xi\Big]dtds\Big\}\exp\Big(\int_{\alpha(x)}^{\infty}w(s,y)ds\Big)\Big\}, \tag{48}$$

for all $(x,y)\in\Omega$*, provided that* Ψ_2 *is increasing, where* Φ_1 *is defined as in Lemma 2, and*

$$\Psi_2(u) = \Phi_1\Big(\frac{u-\mu_1}{\tilde{\mu}_2}\Big) - \Phi_1(u), \tag{49}$$

$$\mu_1 = a(X,Y), \tag{50}$$

$$\tilde{\mu}_2 = \sum_{i=1}^{l_2}\int_{\alpha(X)}^{\infty}\int_{\beta(Y)}^{\infty}\Big[\tilde{d}_i(s,t,X,Y) + \int_s^{\infty}\int_t^{\infty}\tilde{e}_i(\xi,\eta,s,t)d\eta d\xi\Big]dtds, \tag{51}$$

$$\tilde{b}_i(s,t,x,y) = b_i(s,t,x,y)\varphi\Big(\phi^{-1}\Big(\exp\Big(\int_{\alpha(s)}^{\infty}w(\tau,t)d\tau\Big)\Big)\Big), \tag{52}$$

$$\tilde{c}_i(\xi,\eta,s,t) = c_i(\xi,\eta,s,t)\varphi\Big(\phi^{-1}\Big(\exp\Big(\int_{\alpha(\xi)}^{\infty}w(\tau,\eta)d\tau\Big)\Big)\Big), \tag{53}$$

$$\tilde{d}_i(s,t,x,y) = d_i(s,t,x,y)\exp\Big(\int_{\alpha(s)}^{\infty}w(\tau,t)d\tau\Big), \tag{54}$$

$$\tilde{e}_i(\xi,\eta,s,t) = e_i(\xi,\eta,s,t)\exp\Big(\int_{\alpha(\xi)}^{\infty}w(\tau,\eta)d\tau\Big). \tag{55}$$

Proof. Let

$$z_4(x,y) = a(x,y) + \sum_{i=1}^{l_1}\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\Big[b_i(s,t,x,y)\varphi(u(s,t)) + \int_s^{\infty}\int_t^{\infty}c_i(\xi,\eta,s,t)\varphi(u(\xi,\eta))d\eta d\xi\Big]dtds + \sum_{i=1}^{l_2}\int_{\alpha(X)}^{\infty}\int_{\beta(y)}^{\infty}\Big[d_i(s,t,x,y)\phi(u(s,t)) + \int_s^{\infty}\int_t^{\infty}e_i(\xi,\eta,s,t)\phi(u(\xi,\eta))d\eta d\xi\Big]dtds, \tag{56}$$

for all $(x,y) \in [X,\infty) \times [Y,\infty)$. From (47), we have

$$\phi(u(x,y)) \leqslant z_4(x,y) + \int_{\alpha(x)}^{\infty} w(s,y)\phi(u(s,y))ds, \quad (x,y) \in [X,\infty) \times [Y,\infty). \tag{57}$$

Applying the result of Lemma 1 to (57), we have

$$\phi(u(x,y)) \leqslant z_4(x,y)\exp\Big(\int_{\alpha(x)}^{\infty} w(s,y)ds\Big), \quad (x,y) \in [X,\infty) \times [Y,\infty), \tag{58}$$

i.e.

$$u(x,y)) \leqslant \phi^{-1}\Big(z_4(x,y)\exp\Big(\int_{\alpha(x)}^{\infty} w(s,y)ds\Big)\Big), \quad (x,y) \in [X,\infty) \times [Y,\infty). \tag{59}$$

From (56) and (59), we have

$$\begin{aligned}
&z_4(x,y)\\
&\leqslant a(x,y) + \sum_{i=1}^{l_1}\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\Big[b_i(s,t,x,y)\varphi\Big(\phi^{-1}\Big(z_4(s,t)\exp\Big(\int_{\alpha(s)}^{\infty} w(\tau,t)d\tau\Big)\Big)\Big)\\
&\quad + \int_s^{\infty}\int_t^{\infty} c_i(\xi,\eta,s,t)\varphi\Big(\phi^{-1}\Big(z_4(\xi,\eta)\exp\Big(\int_{\alpha(\xi)}^{\infty} w(\tau,\eta)d\tau\Big)\Big)\Big)d\eta d\xi\Big]dtds\\
&\quad + \sum_{i=1}^{l_2}\int_{\alpha(X)}^{\infty}\int_{\beta(y)}^{\infty}\Big[d_i(s,t,x,y)z_4(s,t)\exp\Big(\int_{\alpha(s)}^{\infty} w(\tau,t)d\tau\Big)\\
&\quad + \int_s^{\infty}\int_t^{\infty} e_i(\xi,\eta,s,t)z_4(\xi,\eta)\exp\Big(\int_{\alpha(\xi)}^{\infty} w(\tau,\eta)d\tau\Big)d\eta d\xi\Big]dtds\\
&\leqslant a(x,y) + \sum_{i=1}^{l_1}\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\Big[\tilde{b}_i(s,t,x,y)\varphi\Big(\phi^{-1}\Big(z_4(s,t)\Big)\Big)\\
&\quad + \int_s^{\infty}\int_t^{\infty} \tilde{c}_i(\xi,\eta,s,t)\varphi\Big(\phi^{-1}\Big(z_4(\xi,\eta)d\tau\Big)\Big)d\eta d\xi\Big]dtds\\
&\quad + \sum_{i=1}^{l_2}\int_{\alpha(X)}^{\infty}\int_{\beta(y)}^{\infty}\Big[\tilde{d}_i(s,t,x,y)z_4(s,t)\\
&\quad + \int_s^{\infty}\int_t^{\infty} \tilde{e}_i(\xi,\eta,s,t)z_4(\xi,\eta)d\eta d\xi\Big]dtds
\end{aligned} \tag{60}$$

for all $(x,y) \in [X,\infty) \times [Y,\infty)$. Then similar to the process of (40)-(45), we obtain

$$\begin{aligned}
z_4(x,y) \leqslant \Phi_1^{-1}\Big\{\Phi_1\Big\{\Psi_2^{-1}\Big\{&\int_{\alpha(X)}^{\infty}\int_{\beta(Y)}^{\infty}\sum_{i=1}^{l_1}\Big[\tilde{b}_i(s,t,X,Y)\\
&+ \int_s^{\infty}\int_t^{\infty}\tilde{c}_i(\xi,\eta,s,t)d\eta d\xi\Big]dtds\Big\}\Big\} + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\sum_{i=1}^{l_1}\Big[\tilde{b}_i(s,t,x,y)\\
&+ \int_s^{\infty}\int_t^{\infty}\tilde{c}_i(\xi,\eta,s,t)d\eta d\xi\Big]dtds\Big\}
\end{aligned} \tag{61}$$

Combining (58) and (61), we get the desired estimation (48). □

3. Application

In this section, similar to the applications in [25, 26, 27, 28, 29], we apply our result in Theorem 1 to investigate a class of Volterra-Fredholm integral equation with infinity upper limit

$$\begin{aligned} u^p(x,y) = a(x,y) &+ \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\Big[F_1(s,t,x,y,u(s,t)) \\ &+ \int_s^{\infty}\int_t^{\infty}F_2(\xi,\eta,s,t,u(\xi,\eta))d\eta d\xi\Big]dtds \\ &+ \int_{\alpha(X)}^{\infty}\int_{\beta(Y)}^{\infty}\Big[F_3(s,t,x,y,u(s,t)) \\ &+ \int_s^{\infty}\int_t^{\infty}F_4(\xi,\eta,s,t,u(\xi,\eta))d\eta d\xi\Big]dtds, \quad (x,y)\in\Omega, \end{aligned} \tag{62}$$

where $u \in C(\Omega,\mathbf{R})$, $p>1$ is an odd number, $|a| \in C(\Omega,\mathbf{R}_+)$ is a nonincreasing function in every variable, $\alpha \in C^1([X,\infty),[X,\infty))$ $(\beta \in C^1([Y,\infty),[Y,\infty)))$ is nondecreasing such that $\alpha(x) \geqslant x$, $\alpha(X)=X$, $\alpha(\infty)=\infty$ ($\beta(y)\geqslant y$, $\beta(Y)=Y$, $\beta(\infty)=\infty$), and the functions $F_i \in C(\Omega^2\times\mathbf{R},\mathbf{R})$ $(i=1,2,3,4)$.

PROPOSITION 1. *Suppose that $u(x,y)$ is a solution of (62), and the functions $F_i \in C(\Omega^2\times\mathbf{R},\mathbf{R})(i=1,2,3,4)$ satisfy the following conditions*

$$|F_i(s,t,x,y,u(s,t))| \leqslant f_i(s,t,x,y)|u(s,t)|^{p/2}, \quad i=1,2, \tag{63}$$

$$|F_i(s,t,x,y,u(s,t))| \leqslant f_i(s,t,x,y)|u(s,t)|^{p}, \quad i=3,4, \tag{64}$$

where f_1,f_2,f_3,f_4 are nonincreasing in the third and fourth variables, and there is at least one function among f_3,f_4 not equivalent to zero, then we have

$$\begin{aligned} |u(x,y)| \leqslant \Big\{\Big\{\Psi_3^{-1}\Big\{\int_{\alpha(X)}^{\infty}\int_{\beta(Y)}^{\infty}\Big[f_1(s,t,X,Y)+\int_s^{\infty}\int_t^{\infty}f_2(\xi,\eta,s,t)d\eta d\xi\Big]dtds\Big\}\Big\}^{1/2} \\ + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\Big[f_1(s,t,x,y)+\int_s^{\infty}\int_t^{\infty}f_2(\xi,\eta,s,t)d\eta d\xi\Big]dtds\Big\}^{2/p}, \end{aligned} \tag{65}$$

provided that $\bar{\mu}_2<1$, where

$$\Psi_3(z) = 2\sqrt{\frac{z-\mu_1}{\bar{\mu}_2}} - 2\sqrt{z}, \tag{66}$$

$$\mu_1 = a(X,Y), \tag{67}$$

$$\bar{\mu}_2 = \int_{\alpha(X)}^{\infty}\int_{\beta(Y)}^{\infty}\Big[f_3(s,t,X,Y)+\int_s^{\infty}\int_t^{\infty}f_4(\xi,\eta,s,t)d\eta d\xi\Big]dtds. \tag{68}$$

Proof. By the conditions (63) and (64), from (62) we have

$$\begin{aligned} |u(x,y)|^p \leqslant |a(x,y)| &+ \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\Big[f_1(s,t,x,y)|u(s,t)|^{p/2} \\ &+ \int_s^{\infty}\int_t^{\infty}f_2(\xi,\eta,s,t)|u(\xi,\eta)|^{p/2}d\eta d\xi\Big]dtds \end{aligned}$$

$$+\int_{\alpha(X)}^{\infty}\int_{\beta(Y)}^{\infty}\Big[f_3(s,t,x,y)|u(s,t)|^p + \int_s^{\infty}\int_t^{\infty} f_4(\xi,\eta,s,t)|u(\xi,\eta)|^p d\eta d\xi\Big]dtds,\ \ (x,y)\in\Omega. \tag{69}$$

Define two functions by $\phi(u)=u^p, \varphi(u)=u^{p/2}$. Then $\phi^{-1}(u)=u^{1/p}$,

$$\Phi_1(z) = \int_c^z \frac{ds}{\varphi(\phi^{-1}(s))} = \int_c^z \frac{ds}{s^{1/2}} = 2\sqrt{z}-2\sqrt{c}, \quad z \geqslant c > 0, \tag{70}$$

$$\Phi_1^{-1}(z) = \Big(\frac{z+2\sqrt{c}}{2}\Big)^2, \quad z \geqslant c > 0, \tag{71}$$

$$\Psi_3(z) = \Phi_1\Big(\frac{z-\mu_1}{\bar{\mu}_2}\Big) - \Phi_1(z) = 2\sqrt{\frac{z-\mu_1}{\bar{\mu}_2}} - 2\sqrt{z}. \tag{72}$$

Differentiating $\Psi_3(u)$ with respect to u, we have

$$\Psi_3'(u) = \frac{1}{\sqrt{\bar{\mu}_2(u-\mu_1)}} - \frac{1}{\sqrt{u}} > 0. \tag{73}$$

therefore, Ψ_3 is strictly increasing, and a suitable application of Theorem 1 to (69) yields

$$\begin{aligned}
|u(x,y)| \leqslant{}& \Big\{\Phi_1^{-1}\Big\{\Phi_1\Big\{\Psi_3^{-1}\Big\{\int_{\alpha(X)}^{\infty}\int_{\beta(Y)}^{\infty}\Big[f_1(s,t,X,Y) \\
&+\int_s^{\infty}\int_t^{\infty} f_2(\xi,\eta,s,t)d\eta d\xi\Big]dtds\Big\}\Big\}+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\Big[f_1(s,t,x,y) \\
&+\int_s^{\infty}\int_t^{\infty} f_2(\xi,\eta,s,t)d\eta d\xi\Big]dtds\Big\}\Big\}^{1/p} \\
\leqslant{}& \Big\{\frac{1}{2}\Big\{2\Big\{\Psi_3^{-1}\Big\{\int_{\alpha(X)}^{\infty}\int_{\beta(Y)}^{\infty}\Big[f_1(s,t,X,Y) \\
&+\int_s^{\infty}\int_t^{\infty} f_2(\xi,\eta,s,t)d\eta d\xi\Big]dtds\Big\}\Big\}^{1/2} - 2\sqrt{c} + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\Big[f_1(s,t,x,y) \\
&+\int_s^{\infty}\int_t^{\infty} f_2(\xi,\eta,s,t)d\eta d\xi\Big]dtds\Big\} + \sqrt{c}\Big\}^{2/p} \\
\leqslant{}& \Big\{\Big\{\Psi_3^{-1}\Big\{\int_{\alpha(X)}^{\infty}\int_{\beta(Y)}^{\infty}\Big[f_1(s,t,X,Y) \\
&+\int_s^{\infty}\int_t^{\infty} f_2(\xi,\eta,s,t)d\eta d\xi\Big]dtds\Big\}\Big\}^{1/2} + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\Big[f_1(s,t,x,y) \\
&+\int_s^{\infty}\int_t^{\infty} f_2(\xi,\eta,s,t)d\eta d\xi\Big]dtds\Big\}^{2/p}.
\end{aligned} \tag{74}$$

That is the desired estimation (65). □

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REFERENCES

[1] T. H. GRONWALL, *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, Ann Math, **20** (1919) 292–296.

[2] R. BELLMAN, *The stability of solutions of linear differential equations*, Duke Math. J. **10** (1943) 643–647.

[3] I. A. BIHARI, *A generalization of a lemma of Bellman and its application to uniqueness problem of differential equation*, Acta Math. Acad. Sci. Hung. **7** (1956) 81–94.

[4] M. PINTO, *Integral inequalities of Bihari-type and applications*, Funkcial Ekvac. **33** (1990) 387–430.

[5] O. LIPOVAN, *A retarded Gronwall-like inequality and its applications*, J. Math. Anal. Appl. **252** (2000) 389–401.

[6] R. AGARWAL, S. DENG, AND W. ZHANG, *Generalization of a retarded Gronwall-like inequality and its applications*, Appl Math Comput. **165** (2005) 599–612.

[7] A. ABDELDAIM AND M. YAKOUT, *On some new integral inequalities of Gronwall-Bellman-Pachpatte type*, Appl. Math. Comput. **217** (2011) 7887–7899.

[8] D. BAINOV AND P. SIMEONOV, *Integral Inequalities and Applications*, Kluwer Acad. Publishers, Dordrecht, Boston, London, 1992.

[9] B. G. PACHPATTE, *A note on certain integral inequality*, Tamkang J. Math. **33** (2002) 353–358.

[10] B. G. PACHPATTE, *Explicit bound on a retarded integral inequality*, Math. Inequal. Appl. **7** (2004) 7–11.

[11] Q. H. MA, AND J. PEČARIĆ, *Estimates on solutions of some new nonlinear retarded Volterra-Fredholm type integral inequalities*, Nonlinear Anal., **69** (2008) 393–407.

[12] B. ZHENG AND B. FU, *Some Volterra-Fredholm type nonlinear discrete inequalities involving four iterated infinite sums*, Adv. Differ. Equ. **2012** (2012) Article ID 228.

[13] W. ZHANG, *Projected Gronwall's inequality*, in Chinese, J. Math. Res. Exp. **17** (1997) 257–260.

[14] W. ZHANG AND S. DENG, *Projected Gronwall-Bellman's inequality for integrable functions*, Math. Comput. Modell. **34** (2001) 394–402.

[15] L. ZHOU, K. LU, AND W. ZHANG, *Roughness of tempered exponential dichotomies for infinite-dimensional random difference equations*, J. Diff. Equ. **254** (9) (2013) 4024–4046.

[16] Q. H. MA, E. AND H. YANG, *Estimations on solutions of some weakly singular Volterra integral inequalities*, Acta Math. Appl. Sin. **25** (2002) 505–515.

[17] Q. H. MA AND J. PEČARIĆ, *Some new explicit bounds for weakly singular integral inequalities with applications to fractional differential and integral equations*, J. Math. Anal. Appl. **341** (2) (2008) 894–905.

[18] J. HENDERSON AND S. HRISTOVA, *Nonlinear integral inequalities involving maxima of unknown scalar functions*, Math. Compu. Model. **53** (5–6) (2011) 871–882.

[19] Y. YONG, *Nonlinear Gronwall-Bellman type integral inequalities with maxima*, Math. Inequ. Appl. **16** (3) (2013) 911–928.

[20] D. BAINOV AND P. SIMEONOV, *Integral Inequalities and Applications*, Kluwer Academic, Dordrecht, 1992.

[21] B. G. PACHPATTE, *Inequalities for Differential and Integral Equations*, Academic Press, London, 1998.

[22] W. S. CHEUNG, *Some new nonlinear inequalities and applications to boundary value problems*, Nonlinear Anal. **64** (2006) 2112–2128.

[23] R. P. AGARWAL, C. S. RYOO, AND Y. H. KIM, *New integral inequalities for iterated integrals with applications*, J. Inequ. Appl. **2007** (2007) Article ID 24385.

[24] W. S. WANG, *A generalized retarded Gronwall-like inequality in two variables and applications to BVPBVP*, Appl. Math. Comput. **191** (2007) 144–154.

[25] Q. H. MA, *Some new nonlinear Volterra–Fredholm-type discrete inequalities and their applications*, J. Comput. Appl. Math. **216** (2008) 451–466.

[26] W. S. CHEUNG Q. H. MA, AND J. PEČARIČ, *Some discrete nonlinear inequalities and applications to difference equations*, Acta Math. Sci., Ser. B. **28** (2008) 417–430.

[27] Q. H. MA, *Estimates on some power nonlinear Volterra-Fredholm type discrete inequalities and their applications*, J. Comput. Appl. Math. **233** (2010) 2170–2180.

[28] B. ZHENG, *Qualitative and quantitative analysis for solutions to a class of Volterra-Fredholm type difference equation*, Adv. Differ. Equ. **2011** (2011) Article ID 30.

[29] B. ZHENG AND Q. H. FENG, *Some new Volterra-Fredholm-type discrete inequalities and their applications in the theory of difference equations*, Abstr. Appl. Anal. **2011** (2011), Article ID 584951.

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