

A NOTE ON INTERPOLATION BETWEEN THE ARITHMETIC–GEOMETRIC MEAN AND CAUCHY–SCHWARZ MATRIX NORM INEQUALITIES

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Abstract. In this short note, we obtain an inequality for unitarily invariant norms which is a generalization of one shown by Audenaert [Oper. Matrices. 9 (2015) 475–479]. An application of our result is also given.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices. Let $||| \cdot |||$ be any unitarily invariant norm. Given a matrix $K \in M_n$, we denote by ReK the matrix $\frac{K + K^*}{2}$.

Recently, Audenaert [1] proved that if $A, B \in M_n$ and $q \in [0, 1]$, then

$$|||AB^*|||^2 \leq |||qA^*A + (1-q)B^*B||| \times |||(1-q)A^*A + qB^*B|||. \quad (1.1)$$

For $q = 0$ or $q = 1$, by inequality (1.1), we obtain the Cauchy-Schwarz inequality for unitarily invariant norms

$$|||AB^*|||^2 \leq |||A^*A||| \times |||B^*B|||,$$

which is due to Horn and Matthisa [7]. On the other hand, for $q = 1/2$, by inequality (1.1), we get the arithmetic-geometric mean inequality for unitarily invariant norms

$$|||AB^*||| \leq \frac{1}{2} |||A^*A + B^*B|||,$$

which is due to Bhatia and Kittaneh [5]. Thus, inequality (1.1) interpolates between the arithmetic-geometric mean and Cauchy-Schwarz matrix norm inequalities.

Let $A, X, B \in M_n$, Bhatia and Davis [4] proved that

$$|||AXB^*|||^2 \leq |||A^*AX||| \times |||XB^*B|||. \quad (1.2)$$

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These authors also proved in [3] that

$$|||AXB^*||| \leq \frac{1}{2} |||A^*AX + XB^*B|||. \tag{1.3}$$

The insertion of X is no idle generalization, a judicious choice can lead to powerful perturbation theorems [3-4]. Our aim here is to obtain a stronger version of inequality (1.1) in the same spirit.

Let $A, B \in M_n$ be positive semidefinite. Bhatia and Kittaneh [6] proved that

$$|||AB||| \leq \frac{1}{4} |||(A + B)^2|||. \tag{1.4}$$

In this short note, we obtain a generalization of inequality (1.1). As an application of our result, we present a generalization of inequality (1.4).

2. Main results

We begin this section with the following result, which is a generalization of inequality (1.1).

THEOREM 2.1. *Let $A, X, B \in M_n$ and $q \in [0, 1]$. Then*

$$|||AXB^*|||^2 \leq |||qA^*AX + (1 - q)XB^*B||| \times |||(1 - q)A^*AX + qXB^*B|||. \tag{2.1}$$

Proof. For $A, X, B \in M_n$, if X is positive semidefinite, then by inequality (1.1) we have

$$\begin{aligned} |||AXB^*|||^2 &= |||AX^{1/2}X^{1/2}B^*|||^2 \\ &\leq |||qX^{1/2}A^*AX^{1/2} + (1 - q)X^{1/2}B^*BX^{1/2}||| \\ &\quad \times |||(1 - q)X^{1/2}A^*AX^{1/2} + qX^{1/2}B^*BX^{1/2}||| \\ &= |||X^{1/2}(qA^*A + (1 - q)B^*B)X^{1/2}||| \\ &\quad \times |||X^{1/2}((1 - q)A^*A + qB^*B)X^{1/2}|||. \end{aligned} \tag{2.2}$$

By Proposition IX.1.2 in [2] if a product KH is Hermitian, then $|||KH||| \leq |||ReHK||| \leq |||HK|||$. Using this we obtain

$$\begin{aligned} |||X^{1/2}(qA^*A + (1 - q)B^*B)X^{1/2}||| &\leq \frac{1}{2} |||C(q)X + XC(q)||| \\ &= |||Re(qA^*AX + (1 - q)XB^*B)||| \\ &\leq |||qA^*AX + (1 - q)XB^*B|||, \end{aligned} \tag{2.3}$$

where

$$C(q) = qA^*A + (1 - q)B^*B.$$

It follows from (2.2) and (2.3) that

$$\begin{aligned} |||AXB^*|||^2 &\leq \frac{1}{4} |||C(q)X + XC(q)||| \times |||C(1-q)X + XC(1-q)||| \\ &= |||\operatorname{Re}(qA^*AX + (1-q)XB^*B)||| \times |||\operatorname{Re}((1-q)A^*AX + qXB^*B)||| \\ &\leq |||qA^*AX + (1-q)XB^*B||| \times |||(1-q)A^*AX + qXB^*B|||. \end{aligned} \tag{2.4}$$

Now we consider the general situation, when X is any matrix. Let $X = U\Sigma V^*$ be the singular value decomposition of X , from inequality (2.4) above, we have

$$\begin{aligned} |||AXB^*|||^2 &= |||AU\Sigma V^*B^*|||^2 \\ &\leq |||qU^*A^*AU\Sigma + (1-q)\Sigma V^*B^*BV||| \\ &\quad \times |||(1-q)U^*A^*AU\Sigma + q\Sigma V^*B^*BV||| \\ &= |||U^*(qA^*AU\Sigma V^* + (1-q)U\Sigma V^*B^*B)V||| \\ &\quad \times |||U^*((1-q)A^*AU\Sigma V^* + qU\Sigma V^*B^*B)V||| \\ &= |||qA^*AX + (1-q)XB^*B||| \times |||(1-q)A^*AX + qXB^*B|||. \end{aligned}$$

This completes the proof. \square

REMARK 2.2. For $q = 0$ or $q = 1$, by inequality (2.1), we obtain inequality (1.2). For $q = 1/2$, by inequality (2.1), we get inequality (1.3).

Next, as an application of inequality (2.1), we present a generalization of inequality (1.4).

THEOREM 2.3. *Let $A, B \in M_n$ be positive semidefinite and $q \in (0, 1)$. Then*

$$|||AB|||^2 \leq \frac{1}{4q(1-q)} |||(qA + (1-q)B)^2||| \times |||((1-q)A + qB)^2|||. \tag{2.5}$$

Proof. Since A, B are positive semidefinite, by inequality (2.1), we have

$$|||A^{1/2}XB^{1/2}|||^2 \leq |||qAX + (1-q)XB||| \times |||(1-q)AX + qXB|||.$$

Taking $X = A^{1/2}B^{1/2}$, we get

$$\begin{aligned} |||AB|||^2 &\leq |||qA^{3/2}B^{1/2} + (1-q)A^{1/2}B^{3/2}||| \times |||(1-q)A^{3/2}B^{1/2} + qA^{1/2}B^{3/2}||| \\ &= |||A^{1/2}(qA + (1-q)B)B^{1/2}||| \times |||A^{1/2}((1-q)A + qB)B^{1/2}|||. \end{aligned} \tag{2.6}$$

Bhatia and Kittaneh [2] have proved that

$$|||A^{1/2}(A+B)B^{1/2}||| \leq \frac{1}{2} |||(A+B)^2|||.$$

By this last inequality, we have

$$|||A^{1/2}(qA + (1-q)B)B^{1/2}||| \leq \frac{1}{2\sqrt{q(1-q)}} |||(qA + (1-q)B)^2||| \tag{2.7}$$

and

$$\| \|A^{1/2}((1-q)A + qB)B^{1/2}\| \| \leq \frac{1}{2\sqrt{q(1-q)}} \| \|((1-q)A + qB)^2\| \| . \quad (2.8)$$

It follows from (2.6), (2.7), and (2.8) that

$$\| \|AB\| \|^2 \leq \frac{1}{4q(1-q)} \| \| (qA + (1-q)B)^2 \| \| \times \| \| ((1-q)A + qB)^2 \| \| .$$

This completes the proof. \square

REMARK 2.4. For $q = 1/2$, by inequality (2.5), we obtain inequality (1.4).

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