Journal of
Mathematical
Inequalities
Volume 10, Number 4 (2016), 1137–1143 doi:10.7153/jmi-10-90

THE GEOMETRIC PROOF TO A SHARP VERSION OF BLUNDON'S INEQUALITIES

DORIN ANDRICA, CĂTĂLIN BARBU AND LAURIAN IOAN PIŞCORAN

(*Communicated by J. Pečarić*)

Abstract. A geometric approach to the improvement of Blundon's inequalites given in [11] is presented. If $\phi = \min\{|A-B|,|B-C|,|C-A|\}$, then we proved the inequality $-\cos\phi \leqslant \cos\widehat{ION} \leqslant \cos\phi$, where O is the circumcenter, I is the incenter, and N is the Nagel point of triangle ABC. As a direct consequence, we obtain a sharp version to Gerretsen's inequalities [7].

1. Introduction

Given a triangle ABC, denote by O the circumcenter, I the incenter, N the Nagel point, s the semiperimeter, R the circumradius, and r the inradius of ABC. W. J. Blundon [5] has proved in 1965 that the following inequalities hold

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr} \leqslant s^2 \leqslant 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}. \tag{1}$$

The inequalities (1) are fundamental in triangle geometry because they represent necessary and sufficient conditions (see [6]) for the existence of a triangle with given elements R, r and s. The original proof obtained by W. J. Blundon [4] is based on the following algebraic property of the roots of a cubic equation: The roots x_1, x_2, x_3 to the equation

$$x^3+a_1x^2+a_2x+a_3=0$$

are the side lengths of a (nondegenerate) triangle if and only if the following three conditions are verified: i) $18a_1a_2a_3+a_1^2a_2^2-27a_3^3-4a_2^3-4a_1^3a_3>0$; ii) $-a_1>0$, $a_2>0$, $-a_3>0$; iii) $a_1^3-4a_1a_2+8a_3>0$. For more details we refer to the monograph of D. Mitrinović, J. Pečarić, V. Volenec [7], and to the papers of C. Niculescu [8], [9], and R. A. Satnoianu [10]. Recall that G. Dospinescu, M. Lascu, C. Pohoaţă, M. Tetiva [6] have proposed an algebraic proof to the weaker Blundon's inequality

$$s \leqslant 2R+(3\sqrt{3}-4)r.$$

This inequality is a direct consequence of the right-hand side of (1). In fact, all these approaches illustrate the algebraic character of inequalities (1).

Mathematics subject classification (2010): 26D05, 26D15, 51N35.
Keywords and phrases: Blundon's inequalities, law of cosines, circumcenter, incenter, Nagel point of a triangle, Gerretsen's inequalities.

We mention that D. Andrica, C. Barbu [2] (see also [1, Section 4.6.5, pp. 125–127]) give a direct geometric proof to Blundon's inequalities by using the Law of Cosines in triangle ION. They have obtained the formula

$$\cos\widehat{ION} = \frac{2R^2+10Rr-r^2-s^2}{2(R-2r)\sqrt{R^2-2Rr}}. \tag{2}$$

Because $-1 \leqslant \cos\widehat{ION} \leqslant 1$, obviously it follows that (2) implies (1), showing the geometric character of (1). In the paper [3] other Blundon's type inequalities are obtained using the same idea and different points instead of points I, O, N. S. Wu [11] gives a sharp version of the Blundon's inequalities by introducing the parameter ϕ of the triangle defined by $\phi = \min\{|A-B|,|B-C|,|C-A|\}$, and proving the following inequality:

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}\cos\phi \leqslant s^2 \leqslant 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}\cos\phi. \tag{3}$$

The original proof to (3) given by S. Wu in the paper [11] considers various cases for the angles of the triangle and it uses many algebraic and trigonometric computations. Because the formula (2) is exact, it is natural to expect to obtain a direct proof to inequalities (3) based on it. In this short note we explore this idea and we present a geometric proof to (3).

2. The main result

It is well-known that distance between points O and N is given by

$$ON = R-2r \tag{4}$$

The relation (4) reflects geometrically the difference between the quantities involved in the Euler's inequality $R \geqslant 2r$. In the book of T. Andreescu and D. Andrica [1, Theorem 1, pp. 122–123] is given a proof to the relation (4) using complex numbers. In the paper [4] similar relations involving the circumradius and the exradii of the triangle are proved and discussed.

Denote by $\mathscr{T}(R,r)$ the family of all triangles having the circumradius R and the inradius r. Let us observe that the inequalities (1) give in terms of R and r the exact interval containing the semiperimeter s for triangles in family $\mathscr{T}(R,r)$. More exactly, we have

$$s_{\min}^2 = 2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}$$

and

$$s_{\max}^2 = 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}.$$

The triangles in the family $\mathscr{T}(R,r)$ are "between" two extremal triangles $A_{\min}B_{\min}C_{\min}$ and $A_{\max}B_{\max}C_{\max}$ determined by $s_{\min}$ and $s_{\max}$. These triangles are isosceles. Indeed, according to formula (2), the triangle in the family $\mathscr{T}(R,r)$ with minimal semiperimeter corresponds to the equality case $\cos\widehat{ION}=1$, i.e. the points I, O, N are collinear and I and N belong to the same ray with the origin O. Let G and H be the centroid and the orthocenter of triangle. Taking in to account the well-known property that points O, G, H belong to Euler's line of triangle, this implies that O, I, G must be collinear, hence in this case triangle ABC is isosceles. In similar way, the triangle in the family $\mathscr{T}(R,r)$ with maximal semiperimeter corresponds to the equality case $\cos\widehat{ION}=-1$, i.e. the points I, O, N are collinear and O is situated between I and N. Using again the Euler's line of the triangle, it follows that triangle ABC is isosceles. Note that we have $B_{\min}C_{\min} \geqslant B_{\max}C_{\max}$.

Denote by $N_{\min}$ and $N_{\max}$ the Nagel's points of the triangles $A_{\min}B_{\min}C_{\min}$ and $A_{\max}B_{\max}C_{\max}$, respectively.

Obviously, because the distance ON is constant, the Nagel's point N moves on the circle of diameter $N_{\min}N_{\max}$, and the angle $\widehat{ION}$ varies from 0 to 180°.

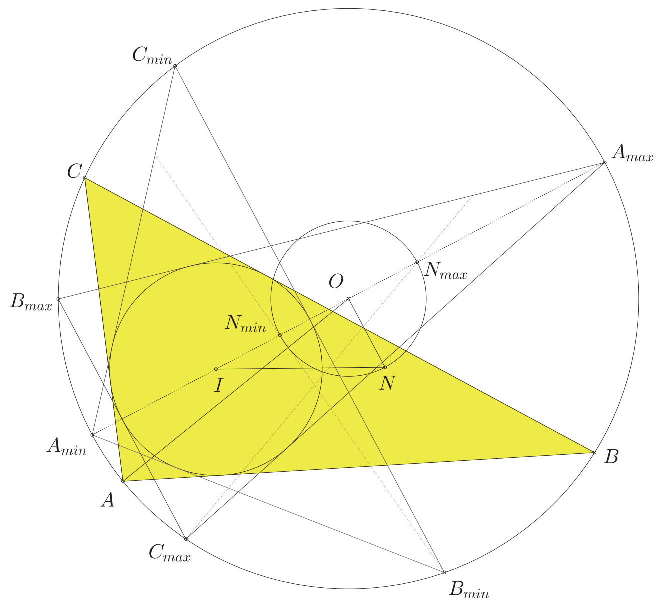

Figure 1: *Nagel's point N moves on the circle of diameter $N_{\min}N_{\max}$.*

We will give a geometric proof to the following result.

THEOREM. *For any triangle ABC, the following inequalities hold*

$$-\cos\phi \leqslant \cos\widehat{ION} \leqslant \cos\phi, \tag{5}$$

where $\phi=\min\{|A-B|,|B-C|,|C-A|\}$. Both equalities in (5) hold if and only if the triangle is equilateral.

Clearly, combining relations (2) and (5) we obtain the stronger inequalities (3).

Firstly, we prove the right-hand side of inequality (5), that is

$$\cos\widehat{ION} \leqslant \cos\phi.$$

Let us note that the vertices of the two extremal triangles $A_{\min}B_{\min}C_{\min}$ and $A_{\max}B_{\max}C_{\max}$ give a partition of the circumcircle into six arcs, each of them corresponding to an order of the angles of triangle ABC. Therefore, without loss of generality, we can assume that $A > C > B$ (i.e. $a > c > b$, where a,b,c are the sidelengths of triangle ABC), and the vertices of triangle ABC move in trigonometric sense on the circumcircle. In this case A is between $A_{\min}$ and $C_{\max}$, B is between $B_{\min}$ and $A_{\max}$, and C is between $C_{\min}$ and $B_{\max}$ (Figure 1). Clearly, we have

$$\phi = \min\{|A-B|,|B-C|,|C-A|\} = C-B.$$

Now, we refer to the configuration in Figure 2. Let D be the intersection point of the line AI with the circumcircle of the triangle ABC. Denote by E and F the points of intersection of the Nagel's line NI with the lines DO and AO, respectively. The triangle AOD is isosceles, then we have $\widehat{ADO} = \widehat{DAO}$. It follows $\widehat{AOC} = 2\widehat{B}$ and $\widehat{COD} = \widehat{A}$.

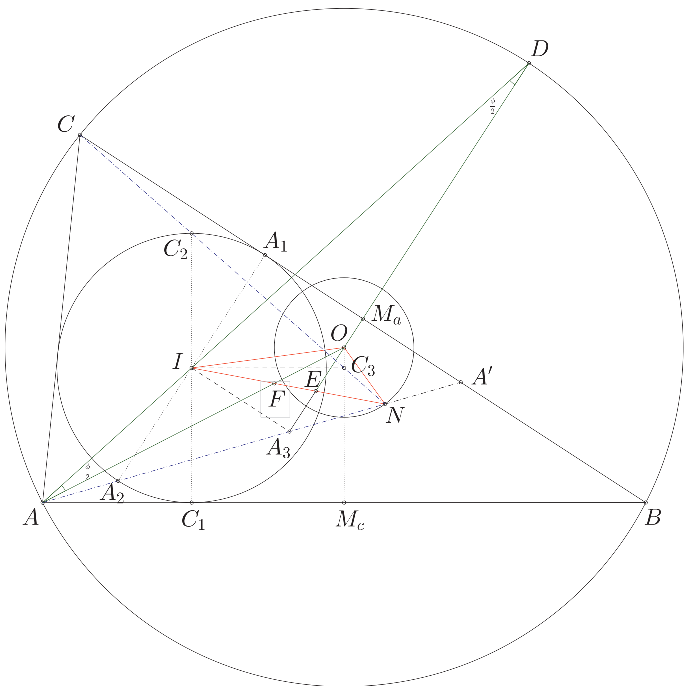

Figure 2: *The geometric illustration of the parameter* ϕ

Therefore

$$\widehat{ADO} = \frac{180^\circ - (\widehat{AOC} + \widehat{COD})}{2} = \frac{180^\circ - (2\widehat{B} + \widehat{A})}{2} = \frac{\widehat{C} - \widehat{B}}{2}$$

that is

$$\phi = 2\widehat{ADO}. \tag{6}$$

Let A_1, B_1, C_1 be the projections of the incenter I on the sides of triangle ABC, and let A_2, C_2 be the antipodal points to A_1 and C_1 in the incircle of triangle ABC. Consider $\{A'\} = AN \cap BC$, I_a the excenter and r_a the exradius corresponding to the side BC. Because IA_1 and $A'I_a$ are perpendicular to BC, we have $IA_1 \parallel A'I_a$. The incircle and the a-excircle are homothetic by the homothecy of center A and ratio $\frac{r_a}{r}$, and we have $\frac{A'I_a}{IA_2} = \frac{r_a}{r}$. It follows that the points A' and A_2 correspond by this homothecy, hence $A_2 \in AN$.

Let $\{A_3\} = AN \cap OM_a$ and $\{C_3\} = CN \cap OM_c$, where M_a and M_c are the midpoints of the sides BC and AC, respectively. Because $BA' = CA_1 = s - c$, it follows that M_a is the midpoint of the segment $A'A_1$, hence M_aA_3 is midline in triangle A_1A_2A'. From $IA_1 = IA_2$ we obtain that IA_3 is midline in triangle A_1A_2A'. From $IA_3 \parallel M_aA_1, M_aA_3 \parallel IA_1$ and $\widehat{IA_1M_a} = 90^\circ$, it follows that the quadrilateral $IA_1M_aA_3$ is a rectangle. Similarly, we prove that the quadrilateral $IC_1M_cC_3$ is a rectangle. Considering the position of the point N with respect to the perpendicular bisector OM_a of the side BC, we have the following three possibilities:

1) If N and B are in the same halfplane, then $\widehat{ION} > \widehat{IOA_3} = \widehat{IOE} > \widehat{FOE}$.

2) If $N = A_3 = E$, then $\widehat{ION} = \widehat{IOE} \geqslant \widehat{FOE}$.

3) If N and A are in the same halfplane, then $N \in [AA_3]$, not possible since $N \in CC_3$ and $C_3 \in OM_c$.

In all possible situations considered above we have obtained the inequality

$$\widehat{FOE} \leqslant \widehat{ION}. \tag{7}$$

Remark that

$$2\widehat{ADO} = \widehat{AOE} = \widehat{FOE}, \tag{8}$$

and by relations (6) - (8), it follows

$$\phi \leqslant \widehat{ION}. \tag{9}$$

Because the function cos is strictly decreasing on $(0, 180^\circ)$, it follows

$$\cos\widehat{ION} \leqslant \cos\phi, \tag{10}$$

and we are done. Now, let us prove the left-hand side inequality in (5), that is

$$-\cos\phi \leqslant \cos\widehat{ION} \tag{11}$$

If $\widehat{ION} \leqslant 90^\circ$, then the inequality (11) is trivial, because the numbers $\cos\phi$ and $\cos\widehat{ION}$ are non-negative. If $\widehat{ION} > 90^\circ$, then the inequality (11) is equivalent to

$$-2\cos\frac{\alpha+\phi}{2}\cos\frac{\alpha-\phi}{2} \leqslant 0,$$

that is

$$\cos\frac{\alpha+\phi}{2}\cos\frac{\alpha-\phi}{2} \geqslant 0. \tag{12}$$

where we note $\alpha = \widehat{ION}$. The inequality (12) is true because we have $\frac{\alpha+\phi}{2}, \frac{\alpha-\phi}{2} \in (0, 90^\circ)$.

REMARK. In fact, the parameter ϕ divides the triangles of family $\mathscr{T}(R,r)$ according to the position of the point A on the circumcircle of triangle ABC.

Recall the Gerretsen's inequalities [7]

$$16Rr - 5r^2 \leqslant s^2 \leqslant 4R^2 + 4Rr + 3r^2. \tag{13}$$

A simple computation shows that the inequalities (13) can be written in the equivalent form

$$|s^2 - 2R^2 - 10Rr + r^2| \leqslant 2(R^2 - 3Rr + 2r^2). \tag{14}$$

Using the inequalities (5) proved in our main result, we have

$$|\cos\widehat{ION}| \leqslant \cos\phi \leqslant \frac{R-r}{\sqrt{R^2-2Rr}}\cos\phi,$$

since clearly $\cos\phi \geqslant 0$ and the right hand side inequality is reducing to $r^2 \geqslant 0$. Now, from formula (2), we get

$$\frac{|s^2 - 2R^2 - 10Rr + r^2|}{2(R-2r)\sqrt{R^2-2Rr}} \leqslant \frac{R-r}{\sqrt{R^2-2Rr}}\cos\phi.$$

After easy computation, we obtain the following sharp version to Gerretsen's inequalities involving the parameter ϕ of the triangle:

COROLLARY. *For every triangle ABC, the following inequality holds*

$$|s^2 - 2R^2 - 10Rr + r^2| \leqslant 2(R^2 - 3Rr + 2r^2)\cos\phi. \tag{15}$$

Acknowledgements. The authors express their thanks to the anonymous referee for his very useful remarks and suggestions improving the level of the paper.

REFERENCES

[1] T. ANDREESCU AND D. ANDRICA, *Complex Number from A to Z*, Second Edition, Birkhäuser, 2014.

[2] D. ANDRICA AND C. BARBU, *A geometric proof of Blundon's Inequalities*, Math. Inequal. Appl. **15** 2 (2012) 361–370.

[3] D. ANDRICA, C. BARBU, N. MINCULETE, *A geometric way to generate Blundon type inequalities*, Acta Universitatis Apulensis **31** (2012), 93–106.

[4] D. ANDRICA AND K. L. NGUYEN, *A note on the Nagel and Gergonne points*, Creative Math.& Inf. **17** (2008) 127–136.

[5] W. J. BLUNDON, *Inequalities associated with the triangle*, Canad. Math. Bull. **8** (1965) 615–626.

[6] G. DOSPINESCU, M. LASCU, C. POHOAŢĂ, AND M. TETIVA, *An elementary proof of Blundon's Inequality*, J. Inequal. Pure Appl. Math. **9** (2008), A 100.

[7] D. S. MITRINOVIĆ, J. E. PEČARIĆ, AND V. VOLENEC, *Recent advances in geometric inequalities*, Kluwer Acad. Publ., Amsterdam, 1989.

[8] C. P. NICULESCU, *A new look at Newton's inequality*, J. Inequal. Pure Appl. Math. **1** (2000), A 17.

[9] C. P. NICULESCU, *On the algebraic character of Blundon's inequality*, Inequality Theory and Applications, Edited by Y. J. Cho, S. S. Dragomir and J. Kim, Vol. 3, Nova Science Publishers, New York, 2003, 139–144.

[10] R. A. SATNOIANU, *General power inequalities between the sides and the circumscribed and inscribed radii related to the fundamental triangle inequality*, Math. Inequal. Appl. **5** 4 (2002) 745–751.

[11] S. WU, *A sharpened version of the fundamental triangle inequality*, Math. Inequalities Appl. **11** 3 (2008) 477–482.

(Received September 10, 2015)

Dorin Andrica
"Babeş-Bolyai" University
Faculty of Mathematics and Computer Sciences
400084 Cluj-Napoca, Romania
e-mail: dandrica@math.ubbcluj.ro

Cătălin Barbu
"Vasile Alecsandri" National College
Department of Mathematics
600011 Bacău, Romania
e-mail: kafka_mate@yahoo.com

Laurian Ioan Pişcoran
Technical University of Cluj Napoca
North University Center of Baia Mare
Department of Mathematics and Computer Science
Victoriei 76, 430122 Baia Mare, Romania
e-mail: plaurian@yahoo.com