THE GEOMETRIC PROOF TO A SHARP VERSION OF BLUNDON’S INEQUALITIES

DORIN ANDRICA, CĂTĂLIN BARBU AND LAURIAN IOAN PIȘCORNAN

(Communicated by J. Pečarić)

Abstract. A geometric approach to the improvement of Blundon’s inequalities given in [11] is presented. If
\( \phi = \min\{|A - B|, |B - C|, |C - A|\} \), then we proved the inequality
\[-\cos \phi \leq \cos \hat{\text{I}} \leq \cos \phi, \]
where \( O \) is the circumcenter, \( I \) is the incenter, and \( N \) is the Nagel point of triangle \( ABC \). As a direct consequence, we obtain a sharp version to Gerretsen’s inequalities [7].

1. Introduction

Given a triangle \( ABC \), denote by \( O \) the circumcenter, \( I \) the incenter, \( N \) the Nagel point, \( s \) the semiperimeter, \( R \) the circumradius, and \( r \) the inradius of \( ABC \). W. J. Blundon [5] has proved in 1965 that the following inequalities hold

\[
2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr} \leq s^2 \leq 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr}.
\]

(1)

The inequalities (1) are fundamental in triangle geometry because they represent necessary and sufficient conditions (see [6]) for the existence of a triangle with given elements \( R, r \) and \( s \). The original proof obtained by W. J. Blundon [4] is based on the following algebraic property of the roots of a cubic equation: The roots \( x_1, x_2, x_3 \) to the equation

\[
x^3 + a_1x^2 + a_2x + a_3 = 0
\]

are the side lengths of a (nondegenerate) triangle if and only if the following three conditions are verified: i) \( 18a_1a_2a_3 + a_1^2a_2^2 - 27a_3^3 - 4a_2^3 - 4a_1^3a_3 > 0 \); ii) \( -a_1 > 0, a_2 > 0, -a_3 > 0 \); iii) \( a_1^3 - 4a_1a_2 + 8a_3 > 0 \). For more details we refer to the monograph of D. Mitrinović, J. Pečarić, V. Volenec [7], and to the papers of C. Niculescu [8], [9], and R. A. Satnoianu [10]. Recall that G. Dospinescu, M. Lascu, C. Pohoat¸, M. Tetiva [6] have proposed an algebraic proof to the weaker Blundon’s inequality

\[
s \leq 2R + (3\sqrt{3} - 4)r.
\]

This inequality is a direct consequence of the right-hand side of (1). In fact, all these approaches illustrate the algebraic character of inequalities (1).


Keywords and phrases: Blundon’s inequalities, law of cosines, circumcenter, incenter, Nagel point of a triangle, Gerretsen’s inequalities.
We mention that D. Andrica, C. Barbu [2] (see also [1, Section 4.6.5, pp. 125–127]) give a direct geometric proof to Blundon’s inequalities by using the Law of Cosines in triangle ION. They have obtained the formula

\[ \cos \widehat{ION} = \frac{2R^2 + 10Rr - r^2 - s^2}{2(R - 2r)\sqrt{R^2 - 2Rr}}. \] (2)

Because \(-1 \leq \cos \widehat{ION} \leq 1\), obviously it follows that (2) implies (1), showing the geometric character of (1). In the paper [3] other Blundon’s type inequalities are obtained using the same idea and different points instead of points I, O, N. S. Wu [11] gives a sharp version of the Blundon’s inequalities by introducing the parameter \( \phi \) of the triangle defined by

\[ \phi = \min \{|A - B|, |B - C|, |C - A|\}, \]

and proving the following inequality:

\[ 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr \cos \phi} \leq s^2 \leq 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr \cos \phi}. \] (3)

The original proof to (3) given by S. Wu in the paper [11] considers various cases for the angles of the triangle and it uses many algebraic and trigonometric computations. Because the formula (2) is exact, it is natural to expect to obtain a direct proof to inequalities (3) based on it. In this short note we explore this idea and we present a geometric proof to (3).

2. The main result

It is well-known that distance between points O and N is given by

\[ ON = R - 2r \] (4)

The relation (4) reflects geometrically the difference between the quantities involved in the Euler’s inequality \( R \geq 2r \). In the book of T. Andreescu and D. Andrica [1, Theorem 1, pp. 122–123] is given a proof to the relation (4) using complex numbers. In the paper [4] similar relations involving the circumradius and the exradii of the triangle are proved and discussed.

Denote by \( \mathcal{T}(R, r) \) the family of all triangles having the circumradius \( R \) and the inradius \( r \). Let us observe that the inequalities (1) give in terms of \( R \) and \( r \) the exact interval containing the semiperimeter \( s \) for triangles in family \( \mathcal{T}(R, r) \). More exactly, we have

\[ s_{\min}^2 = 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr} \]

and

\[ s_{\max}^2 = 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr}. \]

The triangles in the family \( \mathcal{T}(R, r) \) are “between” two extremal triangles \( A_{\min}B_{\min}C_{\min} \) and \( A_{\max}B_{\max}C_{\max} \) determined by \( s_{\min} \) and \( s_{\max} \). These triangles are isosceles. Indeed,
according to formula (2), the triangle in the family $T(R, r)$ with minimal semiperimeter corresponds to the equality case $\cos \widehat{ION} = 1$, i.e. the points $I$, $O$, $N$ are collinear and $I$ and $N$ belong to the same ray with the origin $O$. Let $G$ and $H$ be the centroid and the orthocenter of triangle. Taking into account the well-known property that points $O$, $G$, $H$ belong to Euler’s line of triangle, this implies that $O$, $I$, $G$ must be collinear, hence in this case triangle $ABC$ is isosceles. In similar way, the triangle in the family $T(R, r)$ with maximal semiperimeter corresponds to the equality case $\cos \widehat{ION} = -1$, i.e. the points $I$, $O$, $N$ are collinear and $O$ is situated between $I$ and $N$. Using again the Euler’s line of the triangle, it follows that triangle $ABC$ is isosceles. Note that we have $B_{\text{min}}C_{\text{min}} \geq B_{\text{max}}C_{\text{max}}$.

Denote by $N_{\text{min}}$ and $N_{\text{max}}$ the Nagel’s points of the triangles $A_{\text{min}}B_{\text{min}}C_{\text{min}}$ and $A_{\text{max}}B_{\text{max}}C_{\text{max}}$, respectively.

Obviously, because the distance $ON$ is constant, the Nagel’s point $N$ moves on the circle of diameter $N_{\text{min}}N_{\text{max}}$, and the angle $\widehat{ION}$ varies from 0 to $180^\circ$.

![Diagram](https://via.placeholder.com/150)

Figure 1: Nagel’s point $N$ moves on the circle of diameter $N_{\text{min}}N_{\text{max}}$.

We will give a geometric proof to the following result.

**Theorem.** For any triangle $ABC$, the following inequalities hold

$$-\cos \phi \leq \cos \widehat{ION} \leq \cos \phi,$$

where $\phi = \min\{|A - B|, |B - C|, |C - A|\}$. Both equalities in (5) hold if and only if the triangle is equilateral.
Clearly, combining relations (2) and (5) we obtain the stronger inequalities (3). Firstly, we prove the right-hand side of inequality (5), that is

\[ \cos IO\!N \leq \cos \phi. \]

Let us note that the vertices of the two extremal triangles \(A_{\min}B_{\min}C_{\min}\) and \(A_{\max}B_{\max}C_{\max}\) give a partition of the circumcircle into six arcs, each of them corresponding to an order of the angles of triangle \(ABC\). Therefore, without loss of generality, we can assume that \(A > C > B\) (i.e. \(a > c > b\), where \(a, b, c\) are the sidelengths of triangle \(ABC\)), and the vertices of triangle \(ABC\) move in trigonometric sense on the circumcircle. In this case \(A\) is between \(A_{\min}\) and \(C_{\max}\), \(B\) is between \(B_{\min}\) and \(A_{\max}\), and \(C\) is between \(C_{\min}\) and \(B_{\max}\) (Figure 1). Clearly, we have

\[ \phi = \min\{|A - B|, |B - C|, |C - A|\} = C - B. \]

Now, we refer to the configuration in Figure 2. Let \(D\) be the intersection point of the line \(AI\) with the circumcircle of the triangle \(ABC\). Denote by \(E\) and \(F\) the points of intersection of the Nagel’s line \(NI\) with the lines \(DO\) and \(AO\), respectively. The triangle \(AOD\) is isosceles, then we have \(\widehat{ADO} = \widehat{DAO}\). It follows \(\widehat{AO\!C} = 2\widehat{B}\) and \(\widehat{COD} = \widehat{A}\).

![Figure 2: The geometric illustration of the parameter \(\phi\)](image-url)
Therefore
\[
\widehat{ADO} = \frac{180^\circ - (\widehat{OC} + \widehat{OD})}{2} = \frac{180^\circ - (2\widehat{B} + \widehat{A})}{2} = \frac{\widehat{C} - \widehat{B}}{2}
\]
that is
\[
\phi = 2\widehat{ADO}.
\]

Let \(A_1, B_1, C_1\) be the projections of the incenter \(I\) on the sides of triangle \(ABC\), and let \(A_2, C_2\) be the antipodal points to \(A_1\) and \(C_1\) in the incircle of triangle \(ABC\). Consider \(\{A'\} = AN \cap BC, I_a\) the excenter and \(r_a\) the exradius corresponding to the side \(BC\). Because \(IA_1 = IA = s - c\), it follows that \(M_a\) is the midpoint of the segment \(A'A_1\), hence \(M_aA_3\) is midline in triangle \(A_1A_2A'\). From \(IA_1 = IA_2\) we obtain that \(IA_3\) is midline in triangle \(A_1A_2A'\). From \(IA_3 \parallel M_aA_1, M_aA_3 \parallel IA_1\) and \(IA_1M_a = 90^\circ\), it follows that the quadrilateral \(IA_1M_aA_3\) is a rectangle. Similarly, we prove that the quadrilateral \(IC_1M_cC_3\) is a rectangle. Considering the position of the point \(N\) with respect to the perpendicular bisector \(OM_a\) of the side \(BC\), we have the following three possibilities:

1) If \(N\) and \(B\) are in the same halfplane, then \(\widehat{ION} > \widehat{IOA_3} = \widehat{IOE} > \widehat{FOE}\).

2) If \(N = A_3 = E\), then \(\widehat{ION} = \widehat{IOE} = \widehat{FOE}\).

3) If \(N\) and \(A\) are in the same halfplane, then \(N \in [AA_3]\), not possible since \(N \in CC_3\) and \(C_3 \in OM_c\).

In all possible situations considered above we have obtained the inequality
\[
\widehat{FOE} \leq \widehat{ION}.
\]

Remark that
\[
2\widehat{ADO} = \widehat{AOE} = \widehat{FOE},
\]
and by relations (6) - (8), it follows
\[
\phi \leq \widehat{ION}.
\]

Because the function \(\cos\) is strictly decreasing on \((0, 180^\circ)\), it follows
\[
\cos\widehat{ION} \leq \cos\phi,
\]
and we are done. Now, let us prove the left-hand side inequality in (5), that is
\[
-\cos\phi \leq \cos\widehat{ION}
\]
If $\widehat{ION} \leq 90^\circ$, then the inequality (11) is trivial, because the numbers $\cos \phi$ and $\cos \widehat{ION}$ are non-negative. If $\widehat{ION} > 90^\circ$, then the inequality (11) is equivalent to

$$-2 \cos \frac{\alpha + \phi}{2} \cos \frac{\alpha - \phi}{2} \leq 0,$$

that is

$$\cos \frac{\alpha + \phi}{2} \cos \frac{\alpha - \phi}{2} \geq 0. \quad (12)$$

where we note $\alpha = \widehat{ION}$. The inequality (12) is true because we have $\frac{\alpha + \phi}{2}, \frac{\alpha - \phi}{2} \in (0, 90^\circ)$.

**Remark.** In fact, the parameter $\phi$ divides the triangles of family $\mathcal{T}(R, r)$ according to the position of the point $A$ on the circumcircle of triangle $ABC$.

Recall the Gerretsen’s inequalities [7]

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2. \quad (13)$$

A simple computation shows that the inequalities (13) can be written in the equivalent form

$$|s^2 - 2R^2 - 10Rr + r^2| \leq 2(R^2 - 3Rr + 2r^2). \quad (14)$$

Using the inequalities (5) proved in our main result, we have

$$|\cos \widehat{ION}| \leq \cos \phi \leq \frac{R - r}{\sqrt{R^2 - 2Rr}} \cos \phi,$$

since clearly $\cos \phi \geq 0$ and the right hand side inequality is reducing to $r^2 \geq 0$. Now, from formula (2), we get

$$\frac{|s^2 - 2R^2 - 10Rr + r^2|}{2(R - 2r) \sqrt{R^2 - 2Rr}} \leq \frac{R - r}{\sqrt{R^2 - 2Rr}} \cos \phi.$$

After easy computation, we obtain the following sharp version to Gerretsen’s inequalities involving the parameter $\phi$ of the triangle:

**Corollary.** For every triangle $ABC$, the following inequality holds

$$|s^2 - 2R^2 - 10Rr + r^2| \leq 2(R^2 - 3Rr + 2r^2) \cos \phi. \quad (15)$$

**Acknowledgements.** The authors express their thanks to the anonymous referee for his very useful remarks and suggestions improving the level of the paper.
REFERENCES


(Received September 10, 2015)

Dorin Andrica
“Babeș-Bolyai” University
Faculty of Mathematics and Computer Sciences
400084 Cluj-Napoca, Romania
e-mail: dandrica@math.ubbcluj.ro

Cătălin Barbu
“Vasile Alecsandri” National College
Department of Mathematics
600011 Bacău, Romania
e-mail: kafka_mate@yahoo.com

Laurian Ioan Pischean
Technical University of Cluj Napoca
North University Center of Baia Mare
Department of Mathematics and Computer Science
Victoriei 76, 430122 Baia Mare, Romania
e-mail: plaurian@yahoo.com

Journal of Mathematical Inequalities
www.ele-math.com
jmi@ele-math.com