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NONCOMMUTATIVE ORLICZ MODULAR INEQUALITIES RELATED TO PARALLELOGRAM LAW

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Abstract. In this paper, we generalize the related parallelogram low inequalities to noncommutative Orlicz modular case. Besides, we give the k sets of operators of related noncommutative parallelogram law inequalities.

1. Introduction

Let $\mathscr{B}(\mathscr{H})$ be the space of all bounded linear operators on a separable complex Hilbert space $\mathscr{H}$. Let $A \in \mathscr{B}(\mathscr{H})$ be a compact operator. The Schatten p-norm for $p > 0$ is defined by $||A||_p = (tr|A|^p)^{\frac{1}{p}}$, where tr is the usual trace functional and $|A| = (A^*A)^{\frac{1}{2}}$. Usually, the Schatten p-class is denoted by $\mathscr{C}_p$ for $p > 0$. Since C_2 is a Hilbert space under the inner product $\langle A, B\rangle = tr(B^*A)$, [14, Corollary 2.7] implies the following parallelogram low equality for $A_1,...,A_n,B_1,...,B_n \in \mathscr{C}_2$.

$$\sum_{i,j=1}^{n} \|A_i - A_j\|_2^2 + \sum_{i,j=1}^{n} \|B_i - B_j\|_2^2 = 2\sum_{i,j=1}^{n} \|A_i - B_j\|_2^2 - 2\|\sum_{i=1}^{n}(A_i - B_i)\|_2^2. \tag{1.1}$$

Moslehian-Tominaga-Saito [5] proved the following generalization of the equality 1.1 for Schatten p-norms orderly and gave a problem for further research.

Let $A = \{A_1,A_2,...,A_n\}$, $B = \{B_1,B_2,...,B_n\}$, $C = \{C_1,C_2,...,C_n\} \subset \mathscr{C}_p$ for $p > 0$. Then

$$\begin{aligned}
&\sum_{i,j=1}^{n} \|A_i - A_j\|_p^p + \sum_{i,j=1}^{n} \|B_i - B_j\|_p^p + \sum_{i,j=1}^{n} \|C_i - C_j\|_p^p \\
&+ \|\sum_{i=1}^{n}(A_i - B_i)\|_p^p + \|\sum_{i=1}^{n}(B_i - C_i)\|_p^p + \|\sum_{i=1}^{n}(C_i - A_i)\|_p^p \qquad (1.2)\\
&\geqslant \mathscr{D}_{A-B}^{\frac{p-2}{2}} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^p + \mathscr{D}_{B-C}^{\frac{p-2}{2}} \sum_{i,j=1}^{n} \|B_i - C_j\|_p^p + \mathscr{D}_{C-A}^{\frac{p-2}{2}} \sum_{i,j=1}^{n} \|C_i - A_j\|_p^p
\end{aligned}$$

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and

$$
\begin{aligned}
&2n^{p-2}\sum_{i,j=1}^{n}\|A_i-B_j\|_p^p\\
&\leqslant \sum_{i,j=1}^{n}\|A_i-A_j\|_p^p+\sum_{i,j=1}^{n}\|B_i-B_j\|_p^p+2\|\sum_{i=1}^{n}(A_i-B_i)\|_p^p \qquad (1.3)\\
&\leqslant 2(n^2-n+1)^{\frac{2-p}{2}}\sum_{i,j=1}^{n}\|A_i-B_j\|_p^p
\end{aligned}
$$

for $0<p\leqslant 2$ and the reverse inequality holds for $2\leqslant p<\infty$.

PROBLEM. What the form of the identity is for the general case of k sets of operators?

Let $A=\{A_1,A_2,\ldots,A_n\}$, $B=\{B_1,B_2,\ldots,B_n\}\subset\mathscr{C}_p$ for $p>0$, such that $\sum_{i,j=1}^{n}A_i^*B_j=0$. Hirzallah-Kittaneh-Moslehian [6] proved the following Schatten p-norm inequalities related to a characterization of inner product spaces.

$$
2^{\frac{p}{2}-1}n^{p-1}\left(\sum_{i=1}^{n}\|A_i\|_p^p+\sum_{i=1}^{n}\|B_i\|_p^p\right)\leqslant\sum_{i,j=1}^{n}\|A_i\pm B_j\|_p^p \tag{1.4}
$$

for $0<p\leqslant 2$ and the reverse inequality holds for $2\leqslant p<\infty$.

Let Φ be a growth function. Then Φ is a continuous and nondecreasing function from $[0,\infty)$ onto itself and $\Phi_0(t)=t\Phi'(t)$ is also a growth function. If $a=\sup\{t:\Phi(t)=0\}$, then $a<\infty$ and $\Phi(t)=0$ for all $t\in[0,\ a]$. Hence we may assume that $\Phi(t)>0$ for all $t>0$ (otherwise replace Φ by $\Phi(a+\cdot)$). The quantitative indices of growth function defined by:

$$
p_\Phi=\inf_{t>0}\frac{t\Phi'(t)}{\Phi(t)},\quad q_\Phi=\sup_{t>0}\frac{t\Phi'(t)}{\Phi(t)}.
$$

Then we have

$$
s^{p_\Phi}\Phi(t)\geqslant\Phi(st)\geqslant s^{q_\Phi}\Phi(t) \tag{1.5}
$$

for $0\leqslant s\leqslant 1$ and reverse inequalities holds for $s\geqslant 1$.

For more information on growth function, we refer the interested readers to [8] and [9].

Let $\mathscr{M}$ be a semi-finite von Neumann algebra equipped with a normal faithful semi-finite trace τ. Let $L_0(\mathscr{M})$ denote the topological $*$-algebra of measurable operators with respect to $(\mathscr{M},\tau)$. The topology of $L_0(\mathscr{M})$ is determined by the convergence in measure. For given a growth function Φ we denoted by $L^{\rho_\Phi}(\mathscr{M})$ the noncommutative Orlicz modular space equipped with a noncommutative Orlicz modular $\rho_\Phi(x)=\tau(\Phi(|x|))=\int_0^\infty\Phi(\mu_t(x))dt$ (see [8]). The absolute value of an operator $x\in L_0(\mathscr{M})$ is defined as $|x|=(x^*x)^{\frac{1}{2}}$.

In this note, first we prove the parallelogram low equalities are cyclic by giving the two, three and k sets cases of extended noncommutative parallelogram low equalities. Next, we generalize the inequalities (1.2), (1.3) and (1.4) to noncommutative Orlicz modular case.

2. Related parallelogram low inequalities

Following results are repeatedly used in our later study. Here we list them for convenience.

DEFINITION A. ([9]) (i) If there is a constant $K > 1$ such that $\Phi(2t) \leqslant K\Phi(t)$, then we say a growth function Φ obeys Δ_2-condition for all $t > 0$ and denoted by $\Phi \in \Delta_2$.

(ii) If there is a constant $0 < K < 1$ such that $\Phi(\frac{t}{2}) \leqslant K\Phi(t)$, then we say a growth function Φ satisfies the $\Delta_{\frac{1}{2}}$-condition for all $t > 0$ and denoted by $\Phi \in \Delta_{\frac{1}{2}}$.

Set $\Phi^{(\alpha)}(t) = \Phi(t^{\alpha})$, where α is a positive real number.

LEMMA B. ([8]) *Let Φ be a growth function and $\Phi \in \Delta_{\frac{1}{2}} \cap \Delta_2$. Set $\Phi_0(t) = t\Phi'(t)$.*

(i) *If $\alpha q_{\Phi_0} \leqslant 1$, then $\Phi^{(\alpha)}$ is a concave growth function.*

(ii) *If $\alpha p_{\Phi_0} \geqslant 1$, then $\Phi^{(\alpha)}$ is a convex growth function.*

Then $\Phi^{(\frac{1}{2})}$ is a concave growth function for $q_{\Phi_0} \leqslant 2$ and $\Phi^{(\frac{1}{2})}$ is a convex growth function for $p_{\Phi_0} \geqslant 2$.

LEMMA C. ([4]) *Let $x_1, x_2, \ldots, x_n$ be positive τ-measurable operators and Φ is a growth function.*

(i) *If Φ is concave growth function, then*

$$n^{-1}\sum_{i=1}^{n}\rho_\Phi(nx_i) \leqslant \rho_\Phi(\sum_{i=1}^{n}x_i) \leqslant \sum_{i=1}^{n}\rho_\Phi(x_i). \tag{2.1}$$

(ii) *If Φ is convex growth function, then*

$$n^{-1}\sum_{i=1}^{n}\rho_\Phi(nx_i) \geqslant \rho_\Phi(\sum_{i=1}^{n}x_i) \geqslant \sum_{i=1}^{n}\rho_\Phi(x_i). \tag{2.2}$$

Let us define a constant $\mathscr{D}_x$ for a set of operators $x = \{x_1, x_2, \ldots, x_n\}$ as follows:

$$\mathscr{D}_x := \sum_{i=1}^{n}\delta(x_i)$$

where $\delta(x_i) = 1$ if $x_i \neq 0$ and $\delta(x_i) = 0$ if $x_i = 0$. If there exists $1 \leqslant i \leqslant n$ with $x_i = 0$, then the Lemma 2 is refined as follows:

$$\mathscr{D}_x^{-1}\sum_{i=1}^{n}\rho_\Phi(\mathscr{D}_x x_i) \leqslant \rho_\Phi(\sum_{i=1}^{n}x_i) \leqslant \sum_{i=1}^{n}\rho_\Phi(x_i) \tag{2.3}$$

for concave growth functions and the reverse inequalities holds for convex growth functions.

We also put

$$x-y := \{x_i - y_j \ : \ 1 \leqslant i,j \leqslant n\}$$
$$(x-y) := \textstyle\sum_{i=1}^n x_i - y_i$$

for sets of operators $x = \{x_1,x_2,\dots,x_n\}$ and $y = \{y_1,y_2,\dots,y_n\}$. Then we have $0 \leqslant \mathscr{D}_{x-y} \leqslant n^2$, $0 \leqslant \mathscr{D}_{x-x} \leqslant n^2 - n$ and $0 \leqslant \mathscr{D}_{(x-y)} \leqslant 1$.

Now we present the noncommutative version of two, three and k sets of operators of parallelogram equalities.

THEOREM 2.1. ([11]) *Let $x = \{x_1,x_2,\dots,x_n\}$ and $y = \{y_1,y_2,\dots,y_n\}$ be two sets of τ-measurable operators. Then*

$$\sum_{i,j=1}^{n} |x_i - x_j|^2 + \sum_{i,j=1}^{n} |y_i - y_j|^2 = \sum_{i,j=1}^{n} |x_i - y_j|^2 + \sum_{i,j=1}^{n} |y_i - x_j|^2 - |\sum_{i=1}^{n}(x_i - y_i)|^2 - |\sum_{i=1}^{n}(y_i - x_i)|^2, \tag{2.4}$$

equivalently

$$\sum_{1\leqslant i<j\leqslant n} |x_i - x_j|^2 + \sum_{1\leqslant i<j\leqslant n} |y_i - y_j|^2 + |\sum_{i=1}^{n}(x_i - y_i)|^2 = \sum_{i,j=1}^{n} |x_i - y_j|^2. \tag{2.5}$$

THEOREM 2.2. *Let $x = \{x_1,x_2,\dots,x_n\}$, $y = \{y_1,y_2,\dots,y_n\}$, and $z = \{z_1,z_2,\dots,z_n\}$ be three sets of τ-measurable operators. Then*

$$\begin{aligned}
&\sum_{i,j=1}^{n} |x_i - x_j|^2 + \sum_{i,j=1}^{n} |y_i - y_j|^2 + \sum_{i,j=1}^{n} |z_i - z_j|^2 \\
&\quad + |\sum_{i=1}^{n}(x_i - y_i)|^2 + |\sum_{i=1}^{n}(y_i - z_i)|^2 + |\sum_{i=1}^{n}(z_i - x_i)|^2 \\
&= \sum_{i,j=1}^{n} |x_i - y_j|^2 + \sum_{i,j=1}^{n} |y_i - z_j|^2 + \sum_{i,j=1}^{n} |z_i - x_j|^2.
\end{aligned}$$

Proof. By the Theorem 2.1,we have

$$\begin{aligned}
&\sum_{i,j=1}^{n} |x_i - x_j|^2 + \sum_{i,j=1}^{n} |y_i - y_j|^2 + \sum_{i,j=1}^{n} |z_i - z_j|^2 \\
&\quad + |\sum_{i=1}^{n}(x_i - y_i)|^2 + |\sum_{i=1}^{n}(y_i - z_i)|^2 + |\sum_{i=1}^{n}(z_i - x_i)|^2 \\
&= 2\left(\sum_{1\leqslant i<j\leqslant n} |x_i - x_j|^2 + \sum_{1\leqslant i<j\leqslant n} |y_i - y_j|^2 + \sum_{1\leqslant i<j\leqslant n} |z_i - z_j|^2\right) \\
&\quad + |\sum_{i=1}^{n}(x_i - y_i)|^2 + |\sum_{i=1}^{n}(y_i - z_i)|^2 + |\sum_{i=1}^{n}(z_i - x_i)|^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{1\leqslant i<j\leqslant n} |x_i - x_j|^2 + \sum_{1\leqslant i<j\leqslant n} |y_i - y_j|^2 + |\sum_{i=1}^{n}(x_i - y_i)|^2 \\
&\quad + \sum_{1\leqslant i<j\leqslant n} |y_i - y_j|^2 + \sum_{1\leqslant i<j\leqslant n} |z_i - z_j|^2 + |\sum_{i=1}^{n}(y_i - z_i)|^2 \\
&\quad + \sum_{1\leqslant i<j\leqslant n} |x_i - x_j|^2 + \sum_{1\leqslant i<j\leqslant n} |z_i - z_j|^2 + |\sum_{i=1}^{n}(z_i - x_i)|^2 \\
&= \sum_{i,j=1}^{n} |x_i - y_j|^2 + \sum_{i,j=1}^{n} |y_i - z_j|^2 + \sum_{i,j=1}^{n} |z_i - x_j|^2. \quad \square
\end{aligned}
$$

COROLLARY 2.3. *Let* $x_1 = \{x_{11}, x_{12}, \dots, x_{1n}\}$, $x_2 = \{x_{21}, x_{22}, \dots, x_{2n}\}$, ..., $x_k = \{x_{k1}, x_{k2}, \dots, x_{kn}\}$ *be* k *sets of* τ*-measurable operators. Then*

$$
\begin{aligned}
&\sum_{i,j=1}^{n} |x_{1i} - x_{1j}|^2 + \sum_{i,j=1}^{n} |x_{2i} - x_{2j}|^2 + \dots + \sum_{i,j=1}^{n} |x_{ki} - x_{kj}|^2 \\
&\quad + |\sum_{i=1}^{n}(x_{1i} - x_{2i})|^2 + |\sum_{i=1}^{n}(x_{2i} - x_{3i})|^2 + \dots + |\sum_{i=1}^{n}(x_{ki} - x_{1i})|^2 \\
&= \sum_{i,j=1}^{n} |x_{1i} - x_{2j}|^2 + \sum_{i,j=1}^{n} |x_{2i} - x_{3j}|^2 + \dots + \sum_{i,j=1}^{n} |x_{ki} - x_{1j}|^2.
\end{aligned}
$$

Next we prove the generalization of (1.3) and (1.2) for noncommutative Orlicz modular. In what follows, we always assume any two sets are not completely equivalent to each other. Otherwise we denote them by one set.

THEOREM 2.4. *Let a growth function* $\Phi \in \Delta_{\frac{1}{2}} \cap \Delta_2$ *and* $x = \{x_1, x_2, \dots, x_n\}$, $y = \{y_1, y_2, \dots, y_n\} \subset L^{\rho_\Phi}(\mathscr{M})$. *Then*

$$
\begin{aligned}
&2\mathscr{D}_{x-y}^{-1} \sum_{i,j=1}^{n} \rho_\Phi(\mathscr{D}_{x-y}^{\frac{1}{2}}(x_i - y_j)) \\
&\leqslant \sum_{i,j=1}^{n} \rho_\Phi(x_i - x_j) + \sum_{i,j=1}^{n} \rho_\Phi(y_i - y_j) + 2\rho_\Phi(\sum_{i=1}^{n}(x_i - y_i)) \\
&\leqslant 2\left(\frac{\mathscr{D}_{x-x} + \mathscr{D}_{y-y}}{2} + \mathscr{D}_{(x-y)}\right) \sum_{i,j=1}^{n} \rho_\Phi\left(\frac{(x_i - y_j)}{\sqrt{\frac{\mathscr{D}_{x-x} + \mathscr{D}_{y-y}}{2} + \mathscr{D}_{(x-y)}}}\right).
\end{aligned}
$$

for $q_{\Phi_0} \leqslant 2$ *and the reverse inequalities holds for* $p_{\Phi_0} \geqslant 2$.

Proof. We only prove the case $q_{\Phi_0} \leqslant 2$. The another case can be proved by similar argument. The first inequality can be obtained from [11, Theorem 4.1] by putting $\alpha_i = \alpha_j = D_{x-y}^{\frac{1}{2}}$ and using concavity of $\Phi^{(\frac{1}{2})}$. Now we prove the second inequality.

$$
\begin{aligned}
&\sum_{i,j=1}^{n} \rho_\Phi(x_i - x_j) + \sum_{i,j=1}^{n} \rho_\Phi(y_i - y_j) + 2\rho_\Phi(\sum_{i=1}^{n}(x_i - y_i)) \\
&= \sum_{i,j=1}^{n} \tau(\Phi^{(\frac{1}{2})}(|x_i - x_j|^2)) + \sum_{i,j=1}^{n} \tau(\Phi^{(\frac{1}{2})}(|y_i - y_j|^2)) + 2\tau(\Phi^{(\frac{1}{2})}(|\sum_{i=1}^{n}(x_i - y_i)|^2)) \\
&= \sum_{1\leqslant i<j\leqslant n} \tau(\Phi^{(\frac{1}{2})}(|x_i - x_j|^2)) + \sum_{1\leqslant i<j\leqslant n} \tau(\Phi^{(\frac{1}{2})}(|y_i - y_j|^2)) + \tau(\Phi^{(\frac{1}{2})}(|\sum_{i=1}^{n}(x_i - y_i)|^2))\Big) \\
&\quad + \sum_{1\leqslant i<j\leqslant n} \tau(\Phi^{(\frac{1}{2})}(|y_i - y_j|^2)) + \sum_{1\leqslant i<j\leqslant n} \tau(\Phi^{(\frac{1}{2})}(|x_i - x_j|^2)) + \tau(\Phi^{(\frac{1}{2})}(|\sum_{i=1}^{n}(y_i - x_i)|^2))
\end{aligned}
$$

by the first inequality of (2.3) and equality (2.5), we obtain

$$
\begin{aligned}
&\leqslant (\frac{\mathscr{D}_{x-x} + \mathscr{D}_{y-y}}{2} + \mathscr{D}_{(x-y)})\tau(\Phi^{(\frac{1}{2})}(\frac{\sum_{i,j=1}^{n}|x_i - y_j|^2}{\frac{\mathscr{D}_{x-x}+\mathscr{D}_{y-y}}{2} + \mathscr{D}_{(x-y)}})) \\
&\quad + (\frac{\mathscr{D}_{y-y} + \mathscr{D}_{x-x}}{2} + \mathscr{D}_{(y-x)})\tau(\Phi^{(\frac{1}{2})}(\frac{\sum_{i,j=1}^{n}|y_i - x_j|^2}{\frac{\mathscr{D}_{y-y}+\mathscr{D}_{x-x}}{2} + \mathscr{D}_{(y-x)}})) \\
&\leqslant 2\left(\frac{\mathscr{D}_{x-x} + \mathscr{D}_{y-y}}{2} + \mathscr{D}_{(x-y)}\right)\sum_{i,j=1}^{n}\rho_\Phi\left(\frac{(x_i - y_j)}{\sqrt{\frac{\mathscr{D}_{x-x}+\mathscr{D}_{y-y}}{2} + \mathscr{D}_{(x-y)}}}\right). \quad \square
\end{aligned}
$$

When $\Phi(t) = t^P$ for $p > 0$, by the definition of $\mathscr{D}_{x-y}$, $\mathscr{D}_{x-x}$ and $\mathscr{D}_{(x-y)}$ we obtain noncommutative version of (1.3).

Besides, by the inequality of (1.1), we have

$$
\begin{aligned}
2n^{p_\Phi - 2}\sum_{i,j=1}^{n}\rho_\Phi(x_i - y_j) &\leqslant \sum_{i,j=1}^{n} \rho_\Phi(x_i - x_j) + \sum_{i,j=1}^{n} \rho_\Phi(y_i - y_j) + 2\rho_\Phi(\sum_{i=1}^{n}(x_i - y_i)) \\
&\leqslant 2(n^2 - n + 1)^{\frac{2-p_\Phi}{2}}\sum_{i,j=1}^{n}\rho_\Phi(x_i - y_j)
\end{aligned}
$$

for $q_{\Phi_0} \leqslant 2$ and following inequalities holds for $p_{\Phi_0} \geqslant 2$.

$$
\begin{aligned}
2n^{q_\Phi - 2}\sum_{i,j=1}^{n}\rho_\Phi(x_i - y_j) &\geqslant \sum_{i,j=1}^{n} \rho_\Phi(x_i - x_j) + \sum_{i,j=1}^{n} \rho_\Phi(y_i - y_j) + 2\rho_\Phi(\sum_{i=1}^{n}(x_i - y_i)) \\
&\geqslant 2(n^2 - n + 1)^{\frac{2-q_\Phi}{2}}\sum_{i,j=1}^{n}\rho_\Phi(x_i - y_j).
\end{aligned}
$$

THEOREM 2.5. *Let a growth function* $\Phi \in \Delta_{\frac{1}{2}} \cap \Delta_2$ *and* $x = \{x_1, x_2, ..., x_n\}$, $y = \{y_1, y_2, ..., y_n\}$, $z = \{z_1, z_2, ..., z_n\} \subset L^{\rho_\Phi}(\mathscr{M})$. *Then*

$$
\begin{aligned}
&\mathscr{D}_{x-y}^{-1} \sum_{i,j=1}^{n} \rho_\Phi(\mathscr{D}_{x-y}^{\frac{1}{2}}(x_i - y_j)) + \mathscr{D}_{y-z}^{-1} \sum_{i,j=1}^{n} \rho_\Phi(\mathscr{D}_{y-z}^{\frac{1}{2}}(y_i - z_j)) + \mathscr{D}_{z-x}^{-1} \sum_{i,j=1}^{n} \rho_\Phi(\mathscr{D}_{z-x}^{\frac{1}{2}}(z_i - x_j)) \\
&\leqslant \sum_{i,j=1}^{n} \rho_\Phi(x_i - x_j) + \sum_{i,j=1}^{n} \rho_\Phi(y_i - y_j) + \sum_{i,j=1}^{n} \rho_\Phi(z_i - z_j) \\
&\quad + \rho_\Phi(\sum_{i=1}^{n}(x_i - y_i)) + \rho_\Phi(\sum_{i=1}^{n}(y_i - z_i)) + \rho_\Phi(\sum_{i=1}^{n}(z_i - x_i)) \\
&\leqslant \left(\frac{\mathscr{D}_{x-x} + \mathscr{D}_{y-y}}{2} + \mathscr{D}_{(x-y)}\right) \sum_{i,j=1}^{n} \rho_\Phi\left(\frac{(x_i - y_j)}{\sqrt{\frac{\mathscr{D}_{x-x}+\mathscr{D}_{y-y}}{2} + \mathscr{D}_{(x-y)}}}\right) \\
&\quad + \left(\frac{\mathscr{D}_{y-y} + \mathscr{D}_{z-z}}{2} + \mathscr{D}_{(y-z)}\right) \sum_{i,j=1}^{n} \rho_\Phi\left(\frac{(y_i - z_j)}{\sqrt{\frac{\mathscr{D}_{y-y}+\mathscr{D}_{z-z}}{2} + \mathscr{D}_{(y-z)}}}\right) \\
&\quad + \left(\frac{\mathscr{D}_{z-z} + \mathscr{D}_{x-x}}{2} + \mathscr{D}_{(z-x)}\right) \sum_{i,j=1}^{n} \rho_\Phi\left(\frac{(z_i - x_j)}{\sqrt{\frac{\mathscr{D}_{z-z}+\mathscr{D}_{x-x}}{2} + \mathscr{D}_{(z-x)}}}\right)
\end{aligned}
$$

for $q_{\Phi_0} \leqslant 2$ *and the reverse inequalities holds for* $p_{\Phi_0} \geqslant 2$.

Proof. We only prove the case $q_{\Phi_0} \leqslant 2$. The other case can be proved in similar way. Now we prove the first inequality.

$$
\begin{aligned}
&\sum_{i,j=1}^{n} \rho_\Phi(x_i - x_j) + \sum_{i,j=1}^{n} \rho_\Phi(y_i - y_j) + \sum_{i,j=1}^{n} \rho_\Phi(z_i - z_j) \\
&\quad + \rho_\Phi(\sum_{i=1}^{n}(x_i - y_i)) + \rho_\Phi(\sum_{i=1}^{n}(y_i - z_i)) + \rho_\Phi(\sum_{i=1}^{n}(z_i - x_i)) \\
&= 2\left(\sum_{1\leqslant i<j\leqslant n} \rho_\Phi(x_i - x_j) + \sum_{1\leqslant i<j\leqslant n} \rho_\Phi(y_i - y_j) + \sum_{1\leqslant i<j\leqslant n} \rho_\Phi(z_i - z_j)\right) \\
&\quad + \rho_\Phi(\sum_{i=1}^{n}(x_i - y_i)) + \rho_\Phi(\sum_{i=1}^{n}(y_i - z_i)) + \rho_\Phi(\sum_{i=1}^{n}(z_i - x_i)) \\
&= 2\left(\sum_{1\leqslant i<j\leqslant n} \tau(\Phi^{(\frac{1}{2})}(|x_i - x_j|^2)) + \sum_{1\leqslant i<j\leqslant n} \tau(\Phi^{(\frac{1}{2})}(|y_i - y_j|^2)) + \sum_{1\leqslant i<j\leqslant n} \tau(\Phi^{(\frac{1}{2})}(|z_i - z_j|^2))\right) \\
&\quad + \tau(\Phi^{(\frac{1}{2})}(|\sum_{i=1}^{n}(x_i - y_i)|^2)) + \tau(\Phi^{(\frac{1}{2})}(|\sum_{i=1}^{n}(y_i - z_i)|^2)) + \tau(\Phi^{(\frac{1}{2})}(|\sum_{i=1}^{n}(z_i - x_i)|^2))
\end{aligned}
$$

by the second inequality of (2.1),

$$
\geqslant \tau(\Phi^{(\frac{1}{2})}(\sum_{1\leqslant i<j\leqslant n} |x_i - x_j|^2 + \sum_{1\leqslant i<j\leqslant n} |y_i - y_j|^2 + |\sum_{i=1}^{n}(x_i - y_i)|^2))
$$

$$
\begin{aligned}
&+\tau(\Phi^{(\frac{1}{2})}(\sum_{1\leqslant i<j\leqslant n}|y_i-y_j|^2+\sum_{1\leqslant i<j\leqslant n}|z_i-z_j|^2+|\sum_{i=1}^{n}(y_i-z_i)|^2))\\
&+\tau(\Phi^{(\frac{1}{2})}(\sum_{1\leqslant i<j\leqslant n}|z_i-z_j|^2+\sum_{1\leqslant i<j\leqslant n}|x_i-x_j|^2+|\sum_{i=1}^{n}(z_i-x_i)|^2))\\
&=\tau(\Phi^{(\frac{1}{2})}(\sum_{i,j=1}^{n}|x_i-y_j|^2))+\tau(\Phi^{(\frac{1}{2})}(\sum_{i,j=1}^{n}|y_i-z_j|^2))+\tau(\Phi^{(\frac{1}{2})}(\sum_{i,j=1}^{n}|z_i-x_j|^2))\quad(\text{by}(2.5))
\end{aligned}
$$

by the first inequality of (2.3)

$$
\begin{aligned}
&\geqslant \mathscr{D}_{x-y}^{-1}\sum_{i,j=1}^{n}\tau(\Phi^{(\frac{1}{2})}(\mathscr{D}_{x-y}|x_i-y_j|^2))+\mathscr{D}_{y-z}^{-1}\sum_{i,j=1}^{n}\tau(\Phi^{(\frac{1}{2})}(\mathscr{D}_{y-z}|y_i-z_j|^2))\\
&\quad+\mathscr{D}_{z-x}^{-1}\sum_{i,j=1}^{n}\tau(\Phi^{(\frac{1}{2})}(\mathscr{D}_{z-x}|z_i-x_j|^2))\\
&=\mathscr{D}_{x-y}^{-1}\sum_{i,j=1}^{n}\rho_\Phi(\mathscr{D}_{x-y}^{\frac{1}{2}}(x_i-y_j))+\mathscr{D}_{y-z}^{-1}\sum_{i,j=1}^{n}\rho_\Phi(\mathscr{D}_{y-z}^{\frac{1}{2}}(y_i-z_j))\\
&\quad+\mathscr{D}_{z-x}^{-1}\sum_{i,j=1}^{n}\rho_\Phi(\mathscr{D}_{z-x}^{\frac{1}{2}}(z_i-x_j)).
\end{aligned}
$$

Next, we prove the second inequality.

$$
\begin{aligned}
&\sum_{i,j=1}^{n}\rho_\Phi(x_i-x_j)+\sum_{i,j=1}^{n}\rho_\Phi(y_i-y_j)+\sum_{i,j=1}^{n}\rho_\Phi(z_i-z_j)\\
&\quad+\rho_\Phi(\sum_{i=1}^{n}(x_i-y_i))+\rho_\Phi(\sum_{i=1}^{n}(y_i-z_i))+\rho_\Phi(\sum_{i=1}^{n}(z_i-x_i))\\
&=2\sum_{1\leqslant i<j\leqslant n}\tau(\Phi^{(\frac{1}{2})}(|x_i-x_j|^2))+2\sum_{1\leqslant i<j\leqslant n}\tau(\Phi^{(\frac{1}{2})}(|y_i-y_j|^2))+2\sum_{1\leqslant i<j\leqslant n}\tau(\Phi^{(\frac{1}{2})}(|z_i-z_j|^2))\\
&\quad+\tau(\Phi^{(\frac{1}{2})}(|\sum_{i=1}^{n}(x_i-y_i)|^2))+\tau(\Phi^{(\frac{1}{2})}(|\sum_{i=1}^{n}(y_i-z_i)|^2))+\tau(\Phi^{(\frac{1}{2})}(|\sum_{i=1}^{n}(z_i-x_i)|^2))\\
&=\sum_{1\leqslant i<j\leqslant n}\tau(\Phi^{(\frac{1}{2})}(|x_i-x_j|^2))+\sum_{1\leqslant i<j\leqslant n}\tau(\Phi^{(\frac{1}{2})}(|y_i-y_j|^2))+\tau(\Phi^{(\frac{1}{2})}(|\sum_{i=1}^{n}(x_i-y_i)|^2))\\
&\quad+\sum_{1\leqslant i<j\leqslant n}\tau(\Phi^{(\frac{1}{2})}(|y_i-y_j|^2))+\sum_{1\leqslant i<j\leqslant n}\tau(\Phi^{(\frac{1}{2})}(|z_i-z_j|^2))+\tau(\Phi^{(\frac{1}{2})}(|\sum_{i=1}^{n}(y_i-z_i)|^2))\\
&\quad+\sum_{1\leqslant i<j\leqslant n}\tau(\Phi^{(\frac{1}{2})}(|z_i-z_j|^2))+\sum_{1\leqslant i<j\leqslant n}\tau(\Phi^{(\frac{1}{2})}(|x_i-x_j|^2))+\tau(\Phi^{(\frac{1}{2})}(|\sum_{i=1}^{n}(z_i-x_i)|^2))
\end{aligned}
$$

by the first inequality of (2.3) and equality (2.5), we obtain

$$
\leqslant\left(\frac{\mathscr{D}_{x-x}+\mathscr{D}_{y-y}}{2}+\mathscr{D}_{(x-y)}\right)\tau\left(\Phi^{(\frac{1}{2})}\left(\frac{\sum_{i,j=1}^{n}|x_i-y_j|^2}{\frac{\mathscr{D}_{x-x}+\mathscr{D}_{y-y}}{2}+\mathscr{D}_{(x-y)}}\right)\right)
$$

$$
\begin{aligned}
&+\left(\frac{\mathscr{D}_{y-y}+\mathscr{D}_{z-z}}{2}+\mathscr{D}_{(y-z)}\right)\tau\left(\Phi^{(\frac{1}{2})}\left(\frac{\sum_{i,j=1}^{n}|y_i-z_j|^2}{\frac{\mathscr{D}_{y-y}+\mathscr{D}_{z-z}}{2}+\mathscr{D}_{(y-z)}}\right)\right)\\
&+\left(\frac{\mathscr{D}_{z-z}+\mathscr{D}_{x-x}}{2}+\mathscr{D}_{(z-x)}\right)\tau\left(\Phi^{(\frac{1}{2})}\left(\frac{\sum_{i,j=1}^{n}|z_i-x_j|^2}{\frac{\mathscr{D}_{z-z}+\mathscr{D}_{x-x}}{2}+\mathscr{D}_{(z-x)}}\right)\right)\\
\leqslant&\left(\frac{\mathscr{D}_{x-x}+\mathscr{D}_{y-y}}{2}+\mathscr{D}_{(x-y)}\right)\sum_{i,j=1}^{n}\rho_\Phi\left(\frac{(x_i-y_j)}{\sqrt{\frac{\mathscr{D}_{x-x}+\mathscr{D}_{y-y}}{2}+\mathscr{D}_{(x-y)}}}\right)\\
&+\left(\frac{\mathscr{D}_{y-y}+\mathscr{D}_{z-z}}{2}+\mathscr{D}_{(y-z)}\right)\sum_{i,j=1}^{n}\rho_\Phi\left(\frac{(y_i-z_j)}{\sqrt{\frac{\mathscr{D}_{y-y}+\mathscr{D}_{z-z}}{2}+\mathscr{D}_{(y-z)}}}\right)\\
&+\left(\frac{\mathscr{D}_{z-z}+\mathscr{D}_{x-x}}{2}+\mathscr{D}_{(z-x)}\right)\sum_{i,j=1}^{n}\rho_\Phi\left(\frac{(z_i-x_j)}{\sqrt{\frac{\mathscr{D}_{z-z}+\mathscr{D}_{x-x}}{2}+\mathscr{D}_{(z-x)}}}\right). \qquad \square
\end{aligned}
$$

When $\Phi(t)=t^P$ for $p>0$, the first inequality is noncommutative form of (1.2).

COROLLARY 2.6. *Let a growth function $\Phi\in\Delta_{\frac{1}{2}}\cap\Delta_2$ and $x_1=\{x_{11},x_{12},\ldots,x_{1n}\}$, $x_2=\{x_{21},\ x_{22},\ldots,x_{2n}\}$, ..., $x_k=\{x_{k1},x_{k2},\ldots,x_{kn}\}\subset L^{\rho_\Phi}(\mathscr{M})$. Then*

$$
\begin{aligned}
&\mathscr{D}_{x_1-x_2}^{-1}\sum_{i,j=1}^{n}\rho_\Phi(\mathscr{D}_{x_1-x_2}^{\frac{1}{2}}(x_{1i}-x_{2j}))+\mathscr{D}_{x_2-x_3}^{-1}\sum_{i,j=1}^{n}\rho_\Phi(\mathscr{D}_{x_2-x_3}^{\frac{1}{2}}(x_{2i}-x_{3j}))\\
&\quad+\ldots+\mathscr{D}_{x_k-x_1}^{-1}\sum_{i,j=1}^{n}\rho_\Phi(\mathscr{D}_{x_k-x_1}^{\frac{1}{2}}(x_{ki}-x_{1j}))\\
&\leqslant\sum_{i,j=1}^{n}\rho_\Phi(x_{1i}-x_{1j})+\sum_{i,j=1}^{n}\rho_\Phi(x_{2i}-x_{2j})+\ldots+\sum_{i,j=1}^{n}\rho_\Phi(x_{ki}-x_{kj})\\
&\quad+\rho_\Phi(\sum_{i=1}^{n}(x_{1i}-x_{2i}))+\rho_\Phi(\sum_{i=1}^{n}(x_{2i}-x_{3i}))+\ldots+\rho_\Phi(\sum_{i=1}^{n}(x_{ki}-x_{1i}))\\
&\leqslant\left(\frac{\mathscr{D}_{x_1-x_1}+\mathscr{D}_{x_2-x_2}}{2}+\mathscr{D}_{(x_1-x_2)}\right)\sum_{i,j=1}^{n}\rho_\Phi\left(\frac{(x_{1i}-x_{2j})}{\sqrt{\frac{\mathscr{D}_{x_1-x_1}+\mathscr{D}_{x_2-x_2}}{2}+\mathscr{D}_{(x_1-x_2)}}}\right)\\
&\quad+\left(\frac{\mathscr{D}_{x_2-x_2}+\mathscr{D}_{x_3-x_3}}{2}+\mathscr{D}_{(x_2-x_3)}\right)\sum_{i,j=1}^{n}\rho_\Phi\left(\frac{(x_{2i}-x_{3j})}{\sqrt{\frac{\mathscr{D}_{x_2-x_2}+\mathscr{D}_{x_3-x_3}}{2}+\mathscr{D}_{(x_2-x_3)}}}\right)\\
&\quad+\ldots\\
&\quad+\left(\frac{\mathscr{D}_{x_k-x_k}+\mathscr{D}_{x_1-x_1}}{2}+\mathscr{D}_{(x_k-x_1)}\right)\sum_{i,j=1}^{n}\rho_\Phi\left(\frac{(x_{ki}-x_{1j})}{\sqrt{\frac{\mathscr{D}_{x_k-x_k}+\mathscr{D}_{x_1-x_1}}{2}+\mathscr{D}_{(x_k-x_1)}}}\right)
\end{aligned}
$$

for $q_{\Phi_0}\leqslant 2$ and the reverse inequalities holds for $p_{\Phi_0}\geqslant 2$.

Besides, by the inequalities of (1.1), we get

$$n^{p_\Phi-2}\Big(\sum_{i,j=1}^{n}\rho_\Phi(x_{1i}-x_{2j})+\sum_{i,j=1}^{n}\rho_\Phi(x_{2i}-x_{3j})+\dots+\sum_{i,j=1}^{n}\rho_\Phi(x_{ki}-x_{1j})\Big)$$
$$\leqslant \sum_{i,j=1}^{n}\rho_\Phi(x_{1i}-x_{1j})+\sum_{i,j=1}^{n}\rho_\Phi(x_{2i}-x_{2j})+\dots+\sum_{i,j=1}^{n}\rho_\Phi(x_{ki}-x_{kj})$$
$$+\rho_\Phi(\sum_{i=1}^{n}(x_{1i}-x_{2i}))+\rho_\Phi(\sum_{i=1}^{n}(x_{2i}-x_{3i}))+\dots+\rho_\Phi(\sum_{i=1}^{n}(x_{ki}-x_{1i}))$$
$$\leqslant (n^2-n+1)^{\frac{2-p_\Phi}{2}}\Big(\sum_{i,j=1}^{n}\rho_\Phi(x_{1i}-x_{2j})+\sum_{i,j=1}^{n}\rho_\Phi(x_{2i}-x_{3j})+\dots+\sum_{i,j=1}^{n}\rho_\Phi(x_{ki}-x_{1j})\Big)$$

for $q_{\Phi_0}\leqslant 2$ and the reverse analogous inequalities holds for $p_{\Phi_0}\geqslant 2$.

COROLLARY 2.7. *Let a growth function $\Phi\in\Delta_{\frac{1}{2}}\cap\Delta_2$ and $x_1=\{x_{11},x_{12},\dots,x_{1n}\}$, $x_2=\{x_{21},\ x_{22},\dots,x_{2n}\}$, ..., $x_k=\{x_{k1},x_{k2},\dots,x_{kn}\}\subset L^{\rho_\Phi}(\mathscr{M})$. If $\sum_{i=1}^{n}x_{1i}=\sum_{i=1}^{n}x_{2i}=\dots=\sum_{i=1}^{n}x_{ki}$, then*

$$\sum_{i,j=1}^{n}\rho_\Phi(x_{1i}-x_{1j})+\sum_{i,j=1}^{n}\rho_\Phi(x_{2i}-x_{2j})+\dots+\sum_{i,j=1}^{n}\rho_\Phi(x_{ki}-x_{kj})$$
$$\geqslant \mathscr{D}_{x_1-x_2}^{-1}\sum_{i,j=1}^{n}\rho_\Phi(\mathscr{D}_{x_1-x_2}^{\frac{1}{2}}(x_{1i}-x_{2j}))+\mathscr{D}_{x_2-x_3}^{-1}\sum_{i,j=1}^{n}\rho_\Phi(\mathscr{D}_{x_2-x_3}^{\frac{1}{2}}(x_{2i}-x_{3j}))$$
$$+\dots+\mathscr{D}_{x_k-x_1}^{-1}\sum_{i,j=1}^{n}\rho_\Phi(\mathscr{D}_{x_k-x_1}^{\frac{1}{2}}(x_{ki}-x_{1j}))$$

for $q_{\Phi_0}\leqslant 2$ and the reverse inequality holds for $p_{\Phi_0}\geqslant 2$.

THEOREM 2.8. *Let a growth function $\Phi\in\Delta_{\frac{1}{2}}\cap\Delta_2$ and $x=\{x_1,x_2,\dots,x_n\}$, $y=\{y_1,y_2,\dots,y_n\}\subset L^{\rho_\Phi}(\mathscr{M})$. If $\sum_{i,j=1}^{n}x_i^*y_j=0$, then*

$$\sum_{i,j=1}^{n}\rho_\Phi(x_i\pm y_j)\geqslant\Big(\mathscr{D}_x+\mathscr{D}_y\Big)^{-1}\Big(\sum_{i=1}^{n}\rho_\Phi(\sqrt{n(\mathscr{D}_x+\mathscr{D}_y)}x_i)+\sum_{i=1}^{n}\rho_\Phi(\sqrt{n(\mathscr{D}_x+\mathscr{D}_y)}y_i)\Big)$$

for $q_{\Phi_0}\leqslant 2$ and the reverse inequality holds for $p_{\Phi_0}\geqslant 2$.

Proof. By the second inequality of (2.2), we obtain

$$\sum_{i,j=1}^{n}\rho_\Phi(x_i\pm y_j)=\sum_{i,j=1}^{n}\rho_{\Phi(\frac{1}{2})}(|x_i\pm y_j|^2)\geqslant\rho_{\Phi(\frac{1}{2})}(\sum_{i,j=1}^{n}|x_i\pm y_j|^2)$$
$$=\rho_{\Phi(\frac{1}{2})}(\sum_{i,j=1}^{n}(|x_i|^2+|y_j|^2\pm x_i^*y_j\pm y_j^*x_i))=\rho_{\Phi(\frac{1}{2})}(\sum_{i=1}^{n}n|x_i|^2+\sum_{i=1}^{n}n|y_i|^2)$$

by the first inequality of (2.3)

$$\geqslant \left(\mathscr{D}_x+\mathscr{D}_y\right)^{-1}\left(\sum_{i=1}^{n}\rho_{\Phi^{(\frac{1}{2})}}(n(\mathscr{D}_x+\mathscr{D}_y)|x_i|^2)+\sum_{i=1}^{n}\rho_{\Phi^{(\frac{1}{2})}}(n(\mathscr{D}_x+\mathscr{D}_y)|y_i|^2)\right)$$
$$=\left(\mathscr{D}_x+\mathscr{D}_y\right)^{-1}\left(\sum_{i=1}^{n}\rho_{\Phi}(\sqrt{n(\mathscr{D}_x+\mathscr{D}_y)}x_i)+\sum_{i=1}^{n}\rho_{\Phi}(\sqrt{n(\mathscr{D}_x+\mathscr{D}_y)}y_i)\right).$$

By the same argument we can get the reverse inequality for $p_{\Phi_0}\geqslant 2$. $\square$

If $\Phi(t)=t^p$ for $p>0$, then we get the extended form of (1.4).

COROLLARY 2.9. *Let a growth function* $\Phi\in\Delta_{\frac{1}{2}}\cap\Delta_2$ *and* $x_1=\{x_{11},x_{12},\dots,x_{1n}\}$, $x_2=\{x_{21},x_{22},\dots,x_{2n}\}$, ..., $x_k=\{x_{k1},x_{k2},\dots,x_{kn}\}\subset L^{\rho_\Phi}(\mathscr{M})$. *If*

$$\sum_{Ni,Mi=1}^{n} x_{Ni}^{*}x_{Mj}=0$$

where $1\leqslant N,M\leqslant k$, *then*

$$\sum_{1i,2i,\dots,ki=1}^{n}\rho_{\Phi}(x_{1i}\pm x_{2i}\pm\dots\pm x_{ki})$$
$$\geqslant\left(\mathscr{D}_{x_1}+\mathscr{D}_{x_2}+\dots+\mathscr{D}_{x_k}\right)^{-1}\left(\sum_{i=1}^{n}\rho_{\Phi}(\sqrt{n^{k-1}(\mathscr{D}_{x_1}+\mathscr{D}_{x_2}+\dots+\mathscr{D}_{x_k})}x_{1i})\right.$$
$$+\sum_{i=1}^{n}\rho_{\Phi}(\sqrt{n^{k-1}(\mathscr{D}_{x_1}+\mathscr{D}_{x_2}+\dots+\mathscr{D}_{x_k})}x_{2i})+\dots$$
$$\left.+\sum_{i=1}^{n}\rho_{\Phi}(\sqrt{n^{k-1}(\mathscr{D}_{x_1}+\mathscr{D}_{x_2}+\dots+\mathscr{D}_{x_k})}x_{ki})\right)$$

for $q_{\Phi_0}\leqslant 2$ *and the reverse inequality holds for* $p_{\Phi_0}\geqslant 2$.

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