Journal of Mathematical Inequalities
Volume 10, Number 4 (2016), 1157–1167
doi:10.7153/jmi-10-92

L^r CONVERGENCE FOR WEIGHTED SUMS OF EXTENDED NEGATIVELY DEPENDENT RANDOM VARIABLES

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(*Communicated by J. Pečarić*)

Abstract. In this paper, we establish some results on L^r convergence for weighted sums of extended negatively dependent random variables under r-th uniform integrability. The results obtained in the paper generalize some corresponding ones for independent random variables and some dependent random variables.

1. Introduction

In many statistical applications, we may assume that the variables are independent. But in real studies, this assumption is not plausible. So, many statisticians extended this condition to the dependent cases. One of these dependent structures is extended negatively dependence. The definition of extended negatively dependence is as follows.

DEFINITION 1.1. We call random variables $\{X_n, n \geqslant 1\}$ extended negatively dependent (END, in short) if there exists a constant $M > 0$ such that both

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leqslant M \prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P(X_1 \leqslant x_1, X_2 \leqslant x_2, \ldots, X_n \leqslant x_n) \leqslant M \prod_{i=1}^{n} P(X_i \leqslant x_i)$$

hold for each $n \geqslant 1$ and all real numbers $x_1, x_2, \ldots, x_n$.

The concept of END sequence was introduced by Liu [1]. In the case $M = 1$, the notion of END reduces to negative dependence (ND, in short) which was introduced by Lehmann [2] and carefully studied by Joag-Dev and Proschan [3]. As it is mentioned in Liu [1], the END structure can reflect not only a negatively dependent structure but also a positive one (inequalities from the definition of ND random variables hold both in reverse direction), to some extend. The interested readers can refer to Example

Mathematics subject classification (2010): 60F05, 60F25.

Keywords and phrases: Extended negatively dependent random variables, L^r convergence, weighted sums, uniform integrability.

Supported by the National Natural Science Foundation of China (11201001, 11171001, 11501004, 11501005), the Natural Science Foundation of Anhui Province (1508085J06), the Research Teaching Model Curriculum of Anhui University (xjyjkc1407) and the Students Innovative Training Project of Anhui University (ZLTS2015138).

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4.1 in Liu [1], where END random variables can be taken as negatively or positively dependent. Also, Joag-Dev and Proschan [3] pointed out that negatively associated (NA, in short) random variables are ND and thus NA random variables are END.

Some applications for END sequence have been found. For example, Liu [1] obtained the precise large deviations for dependent random variables with heavy tails, Liu [4] studied the sufficient and necessary conditions of moderate deviations for dependent random variables with heavy tails, Chen et al. [5] established the strong law of large numbers for END random variables and showed applications to risk theory and renewal theory, Shen [6, 7] presented some probability inequalities for END random variables and gave some applications; Wu and Guan [8] presented some convergence properties for the partial sums of END random variables; Wang and Wang [9] investigated a more general precise large deviation result for random sums of END real-valued random variables in the presence of consistent variation; Qiu et al. [10] and Wang et al. [11–13] provided some results on complete convergence for END random variables, Wang et al. [14] established the complete consistency for the estimator of nonparametric regression models based on END errors, and so forth. Since the assumption of END is much weaker than independence, negative association and negative dependence, a study on a limiting behavior of END sequences is of interest.

Ryke and Root [15] once proved the following result on L^r convergence for independent and identically distributed random variables $\{X, X_n, n \geqslant 1\}$: if $0 < r < 2$, $E|X|^r < \infty$, then

$$\lim_{n\to\infty} E\left| n^{-1/r}\left(\sum_{i=1}^{n} X_i - nb\right)\right|^r = 0, \tag{1.1}$$

where $b = 0$ if $0 < r < 1$, and $b = EX$ if $1 \leqslant r < 2$. The result has been extended to different cases. See for example, when $1 \leqslant r < 2$, it has been generalized to independent but not identically distributed case and martingale difference case; when $0 < r < 1$, it has been generalized to any cases of random variables, of course the condition $E|X|^r < \infty$ is replaced by a more general one, i.e. r-th Cesàro uniform integrability (see Chen et al. [16]). Recently, Chen [17] not only extended (1.1) for independent and identically distributed random variables to the case of pairwise NQD sequence, but also to a more general case, that is not necessarily identical distribution but r-th Cesàro uniform integrability.

In this paper, we will study L^r convergence for weighted sums $S_n = \sum_{i=-\infty}^{\infty} a_{ni} X_i$, where $\{X_i, i \geqslant 1\}$ is a sequence of r-th uniformly integrable END random variables, $\{a_{ni}, -\infty < i < \infty, n \geqslant 1\}$ is an array of constants satisfying $\sup_{n\geqslant 1} n^{-1} \sum_{i=-\infty}^{\infty} |a_{ni}|^s < \infty$ for each $s \geqslant 1$.

Throughout this paper, let $I(A)$ be the indicator function of the set A. Let M and C be positive constants which may be different in various places.

2. Main results and their proofs

First, let us present some important lemmas, which will be used to prove the main results of the paper. The first one is the Rosenthal-type inequality for END random variables, which was established by Shen [6].

LEMMA 2.1. *Let* $p \geqslant 2$, $\{X_n, n \geqslant 1\}$ *be a sequence of END random variables with* $EX_i = 0$, *and* $E|X_i|^p < \infty$ *for each* $i \geqslant 1$. *Then, there exists a positive constant* C_p *depending only on* p *such that*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leqslant C_p \left[\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} E|X_i|^2\right)^{p/2}\right].$$

The next one is a basic property for END random variables. We refer the readers to Liu [4] for instance.

LEMMA 2.2. *Let random variables* $X_1, X_2, \cdots, X_n$ *be END. If* $f_1, f_2, \cdots, f_n$ *are all nondecreasing (or nonincreasing) functions, then random variables* $f_1(X_1), f_2(X_2), \cdots, f_n(X_n)$ *are END.*

Based on the two lemmas above, we can establish the following results.

THEOREM 2.1. *Let* $1 < r < 2$ *and* $\{X_i, -\infty < i < \infty\}$ *be a sequence of END random variables with* $EX_i = 0$, $-\infty < i < \infty$. *Let* $\{a_{ni}, -\infty < i < \infty, n \geqslant 1\}$ *be an array of constants such that for any* $s \geqslant 1$,

$$\sup_{n \geqslant 1} n^{-1} \sum_{i=-\infty}^{\infty} |a_{ni}|^s < \infty. \tag{2.1}$$

Denote $S_n = \sum_{i=-\infty}^{\infty} a_{ni} X_i$.

(1) If $\lim_{x \to +\infty} x^r \sup_{-\infty < i < \infty} P(|X_i| > x) = 0$, *then for any* $\varepsilon > 0$,

$$\lim_{n \to \infty} P\left(|S_n| > \varepsilon n^{1/r}\right) = 0. \tag{2.2}$$

(2) If $\lim_{x \to +\infty} x^r \sup_{-\infty < i < \infty} P(|X_i| > x) = 0$ *and* $\sup_{-\infty < i < \infty} E|X_i|^r < \infty$, *then for any* $p \in (0, r)$,

$$\lim_{n \to \infty} E|n^{-1/r} S_n|^p = 0. \tag{2.3}$$

(3) If $\lim_{x \to +\infty} \sup_{-\infty < i < \infty} E|X_i|^r I(|X_i| > x) = 0$, *that is* $\{X_i, -\infty < i < \infty\}$ *is* r*-th uniformly integrable, then*

$$\lim_{n \to \infty} E|n^{-1/r} S_n|^r = 0. \tag{2.4}$$

Proof. For fixed $n \geqslant 1$, denote

$$X'_{ni} = -n^{-1/r} I(X_i < -n^{1/r}) + X_i I(|X_i| \leqslant n^{1/r}) + n^{1/r} I(X_i > n^{1/r}),$$

$$\begin{aligned}
X''_{ni} &= X_i - X'_{ni} = (X_i + n^{1/r})I(X_i < -n^{1/r}) + (X_i - n^{1/r})I(X_i > n^{1/r}),\\
X'''_{ni} &= -s^{-1/r}I(X_i < -s^{1/r}) + X_iI(|X_i| \leqslant s^{1/r}) + s^{1/r}I(X_i > s^{1/r}),\\
X''''_{ni} &= X_i - X'''_{ni} = (X_i + s^{1/r})I(X_i < -s^{1/r}) + (X_i - s^{1/r})I(X_i > s^{1/r}).
\end{aligned}$$

(1) For any $\varepsilon > 0$,

$$\begin{aligned}
P\left(|S_n| > \varepsilon n^{1/r}\right) &= P\left(\left|\sum_{i=-\infty}^{\infty} a_{ni}X_i\right| > \varepsilon n^{1/r}\right)\\
&= P\left(\left|\sum_{i=-\infty}^{\infty} a_{ni}(X'_{ni} + X''_{ni})\right| > \varepsilon n^{1/r}\right)\\
&= P\left(\left|\sum_{i=-\infty}^{\infty} a_{ni}(X'_{ni} - EX'_{ni}) + \sum_{i=-\infty}^{\infty} a_{ni}(X''_{ni} - EX''_{ni})\right| > \varepsilon n^{1/r}\right)\\
&\leqslant P\left(\left|\sum_{i=-\infty}^{\infty} a_{ni}(X'_{ni} - EX'_{ni})\right| > \frac{\varepsilon n^{1/r}}{2}\right)\\
&\quad + P\left(\left|\sum_{i=-\infty}^{\infty} a_{ni}(X''_{ni} - EX''_{ni})\right| > \frac{\varepsilon n^{1/r}}{2}\right)\\
&\doteq I_1 + I_2.
\end{aligned}$$

For I_1, it follows by Markov's inequality, Lemma 2.1 and condition (2.1) that,

$$\begin{aligned}
I_1 &= P\left(\left|\sum_{i=-\infty}^{\infty} a_{ni}(X'_{ni} - EX'_{ni})\right| > \frac{\varepsilon n^{1/r}}{2}\right)\\
&\leqslant Cn^{-2/r}E\left|\sum_{i=-\infty}^{\infty} a_{ni}(X'_{ni} - EX'_{ni})\right|^2\\
&\leqslant Cn^{-2/r}\sum_{i=-\infty}^{\infty} a_{ni}^2E|X'_{ni}|^2\\
&= Cn^{-2/r}\sum_{i=-\infty}^{\infty} a_{ni}^2E|-n^{1/r}I(X_i < -n^{1/r}) + X_iI(|X_i| \leqslant n^{1/r}) + n^{1/r}I(X_i > n^{1/r})|^2\\
&\leqslant Cn^{-2/r}\sum_{i=-\infty}^{\infty} |a_{ni}|^2\left[EX_i^2I(|X_i| \leqslant n^{1/r}) + n^{2/r}P(|X_i| > n^{1/r})\right]\\
&= Cn^{-2/r}\sum_{i=-\infty}^{\infty} a_{ni}^2EX_i^2I(|X_i| \leqslant n^{1/r}) + C\sum_{i=-\infty}^{\infty} a_{ni}^2P(|X_i| > n^{1/r})\\
&= I_{11} + I_{12}.
\end{aligned}$$

For I_{12}, we have

$$I_{12} \leqslant Cn \sup_{-\infty<i<\infty} P\left(|X_i| > n^{1/r}\right) \to 0, \text{ as } n \to \infty.$$

For I_{11}, we have by dominated convergence theorem that

$$
\begin{aligned}
I_{11} &= Cn^{-2/r}\sum_{i=-\infty}^{\infty} a_{ni}^2 \int_0^{n^{2/r}} P(|X_i|^2 I(|X_i| \leqslant n^{1/r}) > x)dx \\
&= Cn^{-2/r}\sum_{i=-\infty}^{\infty} a_{ni}^2 \int_0^{n^{1/r}} 2xP(|X_i| I(|X_i| \leqslant n^{1/r}) > x)dx \\
&\leqslant Cn^{-2/r}\sum_{i=-\infty}^{\infty} a_{ni}^2 \int_0^{n^{1/r}} xP(|X_i| > x)dx \quad (x = yn^{1/r}) \\
&= C\sum_{i=-\infty}^{\infty} a_{ni}^2 \int_0^{1} yP(|X_i| > yn^{1/r})dy \\
&\leqslant C\int_0^1 yn \sup_{-\infty<i<\infty} P(|X_i| > yn^{1/r})dy \\
&\to 0, \text{ as } n\to\infty.
\end{aligned}
$$

Thus, $I_1 \to 0$ as $n\to\infty$. For I_2, it follows by Markov's inequality and dominated convergence theorem that

$$
\begin{aligned}
I_2 &\leqslant Cn^{-1/r}E\left|\sum_{i=-\infty}^{\infty} a_{ni}(X''_{ni} - EX''_{ni})\right| \\
&\leqslant Cn^{-1/r}\sum_{i=-\infty}^{\infty} |a_{ni}| E|X''_{ni}| \\
&= Cn^{-1/r}\sum_{i=-\infty}^{\infty} |a_{ni}| E|(X_i + n^{1/r})I(X_i < -n^{1/r}) + (X_i - n^{1/r})I(X_i > n^{1/r})| \\
&\leqslant Cn^{-1/r}\sum_{i=-\infty}^{\infty} |a_{ni}| E|X_i| I(|X_i| > n^{1/r}) \\
&= Cn^{-1/r}\sum_{i=-\infty}^{\infty} |a_{ni}| \int_0^{\infty} P\left(|X_i| I(|X_i| > n^{1/r}) > x\right)dx \\
&\leqslant Cn^{-1/r}\sum_{i=-\infty}^{\infty} |a_{ni}| \left(\int_0^{n^{1/r}} P(|X_i| > n^{1/r})dx + \int_{n^{1/r}}^{\infty} P(|X_i| > x)dx\right) \quad (x = yn^{1/r}) \\
&\leqslant Cn \sup_{-\infty<i<\infty} P\left(|X_i| > n^{1/r}\right) + C\int_1^{\infty} n \sup_{-\infty<i<\infty} P(|X_i| > yn^{1/r})dy \\
&\to 0, \text{ as } n\to\infty.
\end{aligned}
$$

Hence, (2.2) is obtained.

(2) By (1), we can see that in order to prove (2.3), we only need to show that $\{\left|n^{-1/r}S_n\right|^p, n \geqslant 1\}$ is uniformly integrable. Noting that $r/p > 1$, to prove (2.3), it

suffices to show $\sup_{n\geqslant 1} E(|n^{-1/r}S_n|^p)^{r/p} < \infty$. It is easily checked that

$$\begin{aligned}E(|n^{-1/r}S_n|^p)^{r/p} &\leqslant 1+n^{-1}\int_n^\infty P(|S_n|>s^{1/r})ds\\&\leqslant 1+n^{-1}\int_n^\infty P\left(\left|\sum_{i=-\infty}^{\infty}a_{ni}(X'''_{ni}-EX'''_{ni})\right|>\frac{s^{1/r}}{2}\right)ds\\&\quad+n^{-1}\int_n^\infty P\left(\left|\sum_{i=-\infty}^{\infty}a_{ni}(X''''_{ni}-EX''''_{ni})\right|>\frac{s^{1/r}}{2}\right)ds.\end{aligned}$$

Thus, to prove (2.3), we only need to show

$$I_3=\sup_{n\geqslant 1}n^{-1}\int_n^\infty P\left(\left|\sum_{i=-\infty}^{\infty}a_{ni}(X'''_{ni}-EX'''_{ni})\right|>\frac{s^{1/r}}{2}\right)ds<\infty$$

and

$$I_4=\sup_{n\geqslant 1}n^{-1}\int_n^\infty P\left(\left|\sum_{i=-\infty}^{\infty}a_{ni}(X''''_{ni}-EX''''_{ni})\right|>\frac{s^{1/r}}{2}\right)ds<\infty.$$

Noting that

$$\begin{aligned}&\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}\int_{n^{1/r}}^\infty t^{r-1}P(|X_i|>t)dt\\&=\frac{1}{r}\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}\int_{n^{1/r}}^\infty P(|X_i|>t)dt^r\\&=\frac{1}{r}\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}\left(P(|X_i|>t)t^r|_{n^{1/r}}^\infty-\int_{n^{1/r}}^\infty t^r dP(|X_i|>t)\right)\\&=\frac{1}{r}\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}\left(-P(|X_i|>n^{1/r})n+\int_{n^{1/r}}^\infty t^r dP(|X_i|\leqslant t)\right)\\&=\frac{1}{r}\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}\left(-P(|X_i|>n^{1/r})n+E|X_i|^rI(|X_i|>n^{1/r})\right)\\&\leqslant C\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}E|X_i|^rI(|X_i|>n^{1/r})\\&\leqslant C\sup_{-\infty<i<\infty}E|X_i|^r,\end{aligned}\tag{2.5}$$

we have by Markov's inequality and Lemma 2.1 that

$$\begin{aligned}I_3&\leqslant C\sup_{n\geqslant 1}n^{-1}\int_n^\infty s^{-2/r}E\left|\sum_{i=-\infty}^{\infty}a_{ni}(X'''_{ni}-EX'''_{ni})\right|^2ds\\&\leqslant C\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}\int_n^\infty s^{-2/r}E|X'''_{ni}|^2ds\\&\leqslant C\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}\int_n^\infty s^{-2/r}\left[EX_i^2I(|X_i|\leqslant s^{1/r})+s^{2/r}P(|X_i|>s^{1/r})\right]ds\end{aligned}$$

$$
\begin{aligned}
&= C\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}\int_n^\infty s^{-2/r}EX_i^2 I(|X_i|\leqslant s^{1/r})ds + C\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}\int_n^\infty P(|X_i|>s^{1/r})ds \\
&\leqslant C\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}\int_n^\infty s^{-2/r}ds\int_0^{s^{1/r}} tP(|X_i|>t)dt + C\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}E|X_i|^r I(|X_i|>n^{1/r}) \\
&\leqslant C\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}\int_0^\infty tP(|X_i|>t)dt\int_{\max(n,t^r)}^\infty s^{-2/r}ds + C\sup_{-\infty<i<\infty}E|X_i|^r \\
&= C\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}\left(\int_0^{n^{1/r}} n^{1-2/r}tP(|X_i|>t)dt + \int_{n^{1/r}}^\infty t^{r-1}P(|X_i|>t)dt\right) \\
&\quad + C\sup_{-\infty<i<\infty}E|X_i|^r \\
&\leqslant C\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}\int_0^1 ynP(|X_i|>yn^{1/r})dy + C\sup_{-\infty<i<\infty}E|X_i|^r \quad (t=yn^{1/r}) \\
&\leqslant C\sup_{-\infty<i<\infty}E|X_i|^r \\
&< \infty.
\end{aligned}
$$

For I_4, we have by Markov's inequality and (2.5) again that

$$
\begin{aligned}
I_4 &\leqslant C\sup_{n\geqslant 1} n^{-1}\int_n^\infty s^{-1/r}\sum_{i=-\infty}^\infty |a_{ni}|E|X_{ni}''''|ds \\
&= C\sup_{n\geqslant 1} n^{-1}\int_n^\infty s^{-1/r}\sum_{i=-\infty}^\infty |a_{ni}|E|(X_i+s^{1/r})I(X_i<-s^{1/r}) + (X_i-s^{1/r})I(X_i>s^{1/r})|ds \\
&\leqslant C\sup_{n\geqslant 1} n^{-1}\int_n^\infty s^{-1/r}\sum_{i=-\infty}^\infty |a_{ni}|E|X_i|I(|X_i|>s^{1/r})ds \\
&\leqslant C\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}\int_n^\infty s^{-1/r}\left(s^{1/r}P(|X_i|>s^{1/r}) + \int_{s^{1/r}}^\infty P(|X_i|>t)dt\right)ds \\
&\leqslant C\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}\left(\int_n^\infty P(|X_i|>s^{1/r})ds + \int_n^\infty s^{-1/r}ds\int_{s^{1/r}}^\infty P(|X_i|>t)dt\right) \\
&\leqslant C\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}E|X_i|^r I(|X_i|>n^{1/r}) + C\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}\int_{n^{1/r}}^\infty P(|X_i|>t)dt\int_n^{t^r} s^{-1/r}ds \\
&\leqslant C\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}E|X_i|^r I(|X_i|>n^{1/r}) + C\sup_{n\geqslant 1}\sup_{-\infty<i<\infty}\int_{n^{1/r}}^\infty t^{r-1}P(|X_i|>t)dt \\
&\leqslant C\sup_{-\infty<i<\infty}E|X_i|^r \\
&< \infty.
\end{aligned}
$$

Thus, (2.3) is obtained.

(3) For any fixed $\varepsilon>0$, it can be checked that

$$
\begin{aligned}
E|n^{-1/r}S_n|^r &= n^{-1}\int_0^\infty P(|S_n|>s^{1/r})ds \\
&\leqslant \varepsilon + n^{-1}\int_{n\varepsilon}^\infty P(|S_n|>s^{1/r})ds
\end{aligned}
$$

$$\leqslant \varepsilon + n^{-1}\int_{n\varepsilon}^{\infty} P\left(\left|\sum_{i=-\infty}^{\infty} a_{ni}(X_{ni}''' - EX_{ni}''')\right| > \frac{s^{1/r}}{2}\right) ds$$
$$+n^{-1}\int_{n\varepsilon}^{\infty} P\left(\left|\sum_{i=-\infty}^{\infty} a_{ni}(X_{ni}'''' - EX_{ni}'''')\right| > \frac{s^{1/r}}{2}\right) ds.$$

Hence, to prove (2.4), it suffices to show that as $n \to \infty$,

$$I_5 \doteq n^{-1}\int_{n\varepsilon}^{\infty} P\left(\left|\sum_{i=-\infty}^{\infty} a_{ni}(X_{ni}''' - EX_{ni}''')\right| > \frac{s^{1/r}}{2}\right) ds \to 0$$

and

$$I_6 \doteq n^{-1}\int_{n\varepsilon}^{\infty} P\left(\left|\sum_{i=-\infty}^{\infty} a_{ni}(X_{ni}'''' - EX_{ni}'''')\right| > \frac{s^{1/r}}{2}\right) ds \to 0.$$

For I_5, it follows by Markov's inequality, Lemma 2.1, (2.5) and dominated convergence theorem that

$$\begin{aligned}
I_5 &\leqslant Cn^{-1}\int_{n\varepsilon}^{\infty} s^{-2/r}\sum_{i=-\infty}^{\infty} a_{ni}^2\left[EX_i^2 I(|X_i| \leqslant s^{1/r}) + s^{2/r}P(|X_i| > s^{1/r})\right] ds \\
&\leqslant C\sup_{-\infty<i<\infty}\int_{n\varepsilon}^{\infty} s^{-2/r}EX_i^2 I(|X_i| \leqslant s^{1/r})ds + C\sup_{-\infty<i<\infty}\int_{n\varepsilon}^{\infty} P(|X_i| > s^{1/r})ds \\
&\leqslant C\sup_{-\infty<i<\infty}\int_{n\varepsilon}^{\infty} s^{-2/r}ds\int_0^{s^{1/r}} tP(|X_i| > t)dt + C\sup_{-\infty<i<\infty} E|X_i|^r I(|X_i|^r > n\varepsilon) \\
&\leqslant C\sup_{-\infty<i<\infty}\int_0^{\infty} tP(|X_i| > t)dt\int_{\max(n\varepsilon,t^r)}^{\infty} s^{-2/r}ds + C\sup_{-\infty<i<\infty} E|X_i|^r I(|X_i|^r > n\varepsilon) \\
&= C\sup_{-\infty<i<\infty}\left((n\varepsilon)^{1-2/r}\int_0^{(n\varepsilon)^{1/r}} tP(|X_i| > t)dt + \int_{(n\varepsilon)^{1/r}}^{\infty} t^{r-1}P(|X_i| > t)dt\right) \\
&\quad + C\sup_{-\infty<i<\infty} E|X_i|^r I(|X_i|^r > n\varepsilon) \\
&\leqslant C\sup_{-\infty<i<\infty}\int_0^1 (n\varepsilon)yP(|X_i| > (n\varepsilon)^{1/r}y)dy + C\sup_{-\infty<i<\infty} E|X_i|^r I(|X_i|^r > n\varepsilon) \\
&\qquad\qquad (t = (n\varepsilon)^{1/r}y) \\
&\to 0, \text{ as } n \to \infty.
\end{aligned}$$

For I_6, we have by Markov's inequality and (2.5) again that

$$\begin{aligned}
I_6 &\leqslant Cn^{-1}\int_{n\varepsilon}^{\infty} s^{-1/r}E\left|\sum_{i=-\infty}^{\infty} a_{ni}(X_{ni}'''' - EX_{ni}'''')\right| ds \\
&\leqslant Cn^{-1}\int_{n\varepsilon}^{\infty} s^{-1/r}\sum_{i=-\infty}^{\infty} |a_{ni}|E|X_{ni}''''|ds \leqslant C\sup_{-\infty<i<\infty}\int_{n\varepsilon}^{\infty} s^{-1/r}E|X_{ni}''''|ds
\end{aligned}$$

$$
\begin{aligned}
&= C \sup_{-\infty<i<\infty} \int_{n\varepsilon}^{\infty} s^{-1/r} E|(X_i + s^{1/r}) I(X_i < -s^{1/r}) + (X_i - s^{1/r}) I(X_i > s^{1/r})| ds \\
&\leqslant C \sup_{-\infty<i<\infty} \int_{n\varepsilon}^{\infty} s^{-1/r} E|X_i| I(|X_i| > s^{1/r}) ds \\
&\leqslant C \sup_{-\infty<i<\infty} \int_{n\varepsilon}^{\infty} P(|X_i| > s^{1/r}) ds + C \sup_{-\infty<i<\infty} \int_{n\varepsilon}^{\infty} s^{-1/r} ds \int_{s^{1/r}}^{\infty} P(|X_i| > t) dt \\
&\leqslant C \sup_{-\infty<i<\infty} E|X_i|^r I(|X_i|^r > n\varepsilon) + C \sup_{-\infty<i<\infty} \int_{(n\varepsilon)^{1/\varepsilon}}^{\infty} P(|X_i| > t) dt \int_{n\varepsilon}^{t^r} s^{-1/r} ds \\
&\leqslant C \sup_{-\infty<i<\infty} E|X_i|^r I(|X_i|^r > n\varepsilon) + C \sup_{-\infty<i<\infty} \int_{(n\varepsilon)^{1/\varepsilon}}^{\infty} t^{r-1} P(|X_i| > t) dt \\
&\leqslant C \sup_{-\infty<i<\infty} E|X_i|^r I(|X_i|^r > n\varepsilon) \\
&\to 0, \text{ as } n \to \infty.
\end{aligned}
$$

Thus, (2.4) is obtained. This completes the proof of the theorem. □

By Theorem 2.1, we can have the following corollary immediately.

COROLLARY 2.1. *Let* $1 < r < 2$ *and* $\{X_i, -\infty < i < \infty\}$ *be a sequence of END random variables with* $EX_i = 0$, $-\infty < i < \infty$. *Let* $\{a_i, -\infty < i < \infty\}$ *be a sequence of absolutely summable constants. Denote*

$$Y_k = \sum_{i=-\infty}^{\infty} a_i X_{i+k}, \quad k \geqslant 1,$$

and $S_n = \sum_{k=1}^{n} Y_k$.

(1) If $\lim_{x\to+\infty} x^r \sup_{-\infty<i<\infty} P(|X_i| > x) = 0$*, then for any* $\varepsilon > 0$*,*

$$\lim_{n\to\infty} P\left(|S_n| > \varepsilon n^{1/r}\right) = 0.$$

(2) If $\lim_{x\to+\infty} x^r \sup_{-\infty<i<\infty} P(|X_i| > x) = 0$ *and* $\sup_{-\infty<i<\infty} E|X_i|^r < \infty$*, then for any* $p \in (0, r)$*,*

$$\lim_{n\to\infty} E|n^{-1/r} S_n|^p = 0.$$

(3) If $\lim_{x\to+\infty} \sup_{-\infty<i<\infty} E|X_i|^r I(|X_i| > x) = 0$*, that is* $\{X_i, -\infty < i < \infty\}$ *is* r*-th uniformly integrable, then*

$$\lim_{n\to\infty} E|n^{-1/r} S_n|^r = 0.$$

Proof. Note that

$$S_n = \sum_{k=1}^{n} Y_k = \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_i X_{i+k} = \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_{i-k} X_i = \sum_{i=-\infty}^{\infty} \left(\sum_{k=1}^{n} a_{i-k} \right) X_i.$$

Denote $a_{ni} = \sum_{k=1}^{n} a_{i-k}$ for $-\infty < i < \infty$ and $n \geqslant 1$. It is obvious that $\{a_{ni}, -\infty < i < \infty, n \geqslant 1\}$ satisfies the conditions of Theorem 2.1. Hence, by Theorem 2.1, we can get the desired results. The proof is completed. $\square$

REMARK 2.1. Obviously, the main results of the paper still hold for NA, ND and independent random variables, which are special cases of END.

REMARK 2.2. The result of Corollary 2.1 generalizes the corresponding one of Sung [18].

If we take $a_{ni} = 1$ for $1 \leqslant i \leqslant n$ and $n \geqslant 1$, and $a_{ni} = 0$ otherwise in Theorem 2.1, then we can get the following result.

COROLLARY 2.2. *Let* $1 < r < 2$ *and* $\{X_n, n \geqslant 1\}$ *be a sequence of END random variables with* $EX_i = 0$, $i \geqslant 1$. *Denote* $S_n = \sum_{i=1}^{n} X_i$.

(1) If $\lim_{x \to +\infty} x^r \sup_{n \geqslant 1} P(|X_n| > x) = 0$, *then for any* $\varepsilon > 0$,

$$\lim_{n \to \infty} P\left(|S_n| > \varepsilon n^{1/r}\right) = 0.$$

(2) If $\lim_{x \to +\infty} x^r \sup_{n \geqslant 1} P(|X_n| > x) = 0$ *and* $\sup_{n \geqslant 1} E|X_n|^r < \infty$, *then for any* $p \in (0, r)$,

$$\lim_{n \to \infty} E|n^{-1/r} S_n|^p = 0.$$

(3) If $\lim_{x \to +\infty} \sup_{n \geqslant 1} E|X_n|^r I(|X_n| > x) = 0$, *that is* $\{X_n, n \geqslant 1\}$ *is* r*-th uniformly integrable, then*

$$\lim_{n \to \infty} E|n^{-1/r} S_n|^r = 0. \tag{2.6}$$

Especially, if $\{X_n, n \geqslant 1\}$ *is a sequence of END random variables with identical distribution and* $E|X|^r < \infty$, *then (2.6) still holds.*

REMARK 2.3. The result of Corollary 2.2 generalizes the corresponding one of Pyke and Root [15].

Acknowledgements. The authors are most grateful to the Editor Tomislav Buric and anonymous referees for careful reading of the manuscript and valuable suggestions which helped significantly improving an earlier version of this paper.

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(Received September 16, 2015)

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