ON THE CYCLIC HOMOGENEOUS POLYNOMIAL INEQUALITIES OF DEGREE FOUR OF THREE NONNEGATIVE REAL VARIABLES

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(Communicated by T. Burić)

Abstract. Let \( f(x, y, z) \) is a cyclic homogeneous polynomial of degree four of three nonnegative real variables satisfying the condition \( f(1, 1, 1) = 0 \). We find necessary and sufficient condition to be true the inequality \( f(x, y, z) \geq 0 \), for this aim we introduce a characteristic polynomial \( J_f(t) \) and by its root \( t_0 > 0 \) we formulate the condition.

1. Introduction

Inequalities of cyclic or symmetric homogeneous polynomials of three variables are explored in numerous articles [1]–[9]. Tetsuya Ando solves this problem by finding necessary and sufficient conditions of degree three ([1], [2] and [3], Theorem 1.2). We could reformulate the condition in [3], Theorem 1.2 as we introduce an auxiliary function \( h(t) = t^2 - 2t^{-1} \), \( t \in (0, +\infty) \). This function is continuous, monotone increasing and takes values in the whole real line.

Theorem. Let \( a, b, c \) are real constants and let \( t_0 > 0 \) is a root of the characteristic equation \( h(t) = t^2 - 2t^{-1} = a \). For arbitrary nonnegative numbers \( x, y \) and \( z \) it is true the inequality

\[
x^3 + y^3 + z^3 + a(xy^2 + yz^2 + zx^2) + b(x^2y + y^2z + z^2x) + cxyz \geq 0,
\]

if and only if when \( 3 + 3a + 3b + c \geq 0 \) and \( b \geq h(t_0^{-1}) = t_0^{-2} - 2t_0 \).

Let \( a, b, c \) are real constants and

\[
f(x, y, z) = x^4 + y^4 + z^4 + a(x^3y + y^3z + z^3x) + b(xy^3 + yz^3 + zx^3) + c(x^2y^2 + y^2z^2 + z^2x^2) - (1 + a + b + c)xyz(x + y + z)
\]

is a cyclic homogeneous polynomial of degree four, \( f(1, 1, 1) = 0 \).

When \( x, y \) and \( z \) are real variables Vasile Cirtoaje proves that the necessary and sufficient condition to be true \( f(x, y, z) \geq 0 \) is \( 3(c + 1) \geq a^2 + ab + b^2 \) ([4], Theorem 2.1). Vasile Cirtoaje finds also the necessary and sufficient condition when \( f \) is a symmetric polynomial of nonnegative real variables ([4], Theorem 2.6).


Keywords and phrases: Cyclic, homogeneous, polynomial, inequalities, degree four.
When \( f \) is a cyclic polynomial of nonnegative real variables Tetsuya Ando proves that to be true the inequality \( f(x, y, z) \geq 0 \) is necessary and sufficient to be verified at least one of six conditions ([3], Theorem 1.3).

We introduce the characteristic polynomial \( J(t) = 2t^4 + at^3 - bt - 2 \) and by its root \( t_0 > 0 \) we formulate the necessary and sufficient condition to be true \( f(x, y, z) \geq 0 \) (Theorem 1).

Necessary and sufficient conditions for cyclic homogeneous polynomial inequalities of degree four for which \( f(1, 1, 1) \geq 0 \) are pointed by Cirtoaje and Zhou for real variables [5] and for nonnegative real variables ([9], Theorem 2.1 and Theorem 2.2). T. Ando explore symmetric cyclic homogeneous polynomial inequalities of degree five ([3], Theorem 1.4) while V. Cirtoaje of degree six [6], [7], [8].

2. Main results

For brevity we set

\[
\begin{align*}
\nu_1 &= x^4 + y^4 + z^4 - xyz(x + y + z), & \nu_2 &= x^3 y + y^3 z + z^3 x - xyz(x + y + z), \\
\nu_3 &= x^2 y^2 + y^2 z^2 + z^2 x^2 - xyz(x + y + z) & \text{and} & \nu_4 &= x y^3 + y z^3 + z x^3 - xyz(x + y + z).
\end{align*}
\]

**Remark.** For arbitrary real numbers \( x, y, z \) the following inequalities hold \( \nu_4 \geq \nu_3 \geq \nu_1 \) and \( \nu_4 \geq \nu_2 \geq 0 \), and for arbitrary nonnegative numbers \( x, y, z \) the following inequalities hold \( \nu_1 \geq 0 \), \( \nu_3 \geq 0 \) and \( \nu_3 \nu_1 \geq (\nu_2)^2 \) ([4], inequality 5.2). The last inequality follows from the identity

\[ \nu_3 \nu_1 - (\nu_2)^2 = xyz(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)^2 \geq 0. \]

Let \( a, b \) and \( c \) be real constants. For arbitrary nonnegative numbers \( x, y, z \) we will explore the inequality \( f(x, y, z) = \nu_4 + aw_3 + bw_1 + cw_2 \geq 0 \).

The equation \( J(t) = 2t^4 + at^3 - bt - 2 = 0 \) has at least one root \( t_0 \in (0, +\infty) \), because \( J(0) = -2 \) and \( \lim_{t \to +\infty} J(t) = +\infty \). Let us set

\[ a_0 = t_0^{-1} - 2t_0, \quad b_0 = t_0 - 2t_0^{-1} \quad \text{and} \quad k = (a - a_0)t_0. \quad (1) \]

From \( J(t_0) = 0 \) we find \( b = b_0 + kt_0 \), i.e.

\[ a = a_0 + kt_0^{-1} = (k + 1)t_0^{-1} - 2t_0, \quad \text{and} \quad b = b_0 + kt_0 = (k + 1)t_0 - 2t_0^{-1}. \quad (2) \]

**Remark.** When \( k \geq -1 \) the equation \( J(t) = 0 \) has exactly one root \( t_0 \in (0, +\infty) \), because

\[ J(t) = 2t^4 + [(k + 1)t_0^{-1} - 2t_0]t^3 - [(k + 1)t_0 - 2t_0^{-1}]t - 2 = (t - t_0)J_1(t), \]

where

\[ J_1(t) = 2t^3 + (k + 1)t_0^{-1}t^2 + (k + 1)t + 2t_0^{-1} > 0. \]
THEOREM 1. Let $a$, $b$ and $c$ be real constants and $t_0 > 0$ is a root of the characteristic equation $J(t) = 2t^4 + at^3 - bt - 2 = 0$, $k = at_0 + 2t_0^{-1}$ and $c_0 = (t_0 - t_0^{-1})^2$. For arbitrary nonnegative numbers $x$, $y$ and $z$, the inequality

$$f(x, y, z) = w_4 + aw_3 + bw_1 + cw_2 \geq 0,$$

holds if and only if

$$c \geq \begin{cases} c_0 - 2k & \text{when } k \geq 0 \\ r & \text{when } k < 0, \text{ where } r = \frac{a^2 + ab + b^2}{3} - 1. \end{cases}$$

REMARK. It is always true that $r \geq c_0 - 2k$ as $3(r - c_0 + 2k) = k^2(t_0^2 + t_0^{-2} + 1) \geq 0$.

When $k \geq 0$ and $c = c_0 - 2k$ the equality holds if and only if $x = y = z$ or $\{z = 0, x = t_0y\}$, or any cyclic permutation thereof.

When $k \geq 0$ and $c > c_0 - 2k$ the equality holds if and only if $x = y = z$.

When $k < 0$ and $c > r$ the equality holds if and only if $x = y = z$.

When $k < 0$ and $c = r$ the equality holds when $x = y = z$ and at least for one triplet $(x_0, y_0, 1)$ such that $x_0 > 0$, $x_0 \neq 1$ and $y_0 > 0$, or any cyclic permutation thereof. The only exception is the case $a = b = -2 \Rightarrow c = r = 3$, the equality holds if and only if $x = y = z$.

For completeness we will also add the following theorem.

THEOREM 2. Let $x$, $y$, $z$ be arbitrary nonnegative numbers and $a$, $b$, $c$, $d$ be real constants. A necessary and sufficient condition the inequality

$$f(x, y, z) = aw_3 + bw_1 + cw_2 + dxyz(x + y + z) \geq 0$$

to be true is $a \geq 0$, $b \geq 0$, $d \geq 0$ and $c + 2\sqrt{ab} \geq 0$.

Proof Theorem 1. We will divide the proof into several lemmas.

We set $p = \frac{1}{3}(2a + b)$, $q = \frac{1}{3}(a + 2b)$,

$$u = x^2 - z^2 - pxy + (p - q)xz + qyz \quad \text{and} \quad v = y^2 - z^2 - qxy + (q - p)yq + pxz.$$

The following identity holds ([4], 3.2)

$$f = w_4 + aw_3 + bw_1 + cw_2 = u^2 - uv + v^2 + (c - r)w_2. \quad \Box \quad (3)$$

LEMMA 1.1. ([4], 3.2) If $c \geq r$ then for arbitrary real numbers $x$, $y$, $z$ the following inequality holds

$$f(x, y, z) = w_4 + aw_3 + bw_1 + cw_2 \geq 0.$$

Proof. According to identity (3) $f = u^2 - uv + v^2 + (c - r)w_2 \geq 0$, because

$$2(u^2 - uv + v^2) = u^2 + v^2 + (u - v)^2 \geq 0 \quad \text{and} \quad w_2 \geq 0. \quad \Box$$
Let $x$, $y$, $z$ be arbitrary nonnegative numbers. If $a \geq 0$ and $b \geq 0$, then the inequality $aw_3 + bw_1 \geq 2\sqrt{abw_2}$ holds.

The equality holds if and only if $x = y = z$ or $x = y = 0$, or $x : y : z = 0 : \sqrt{a} : \sqrt{b}$, or any cyclic permutation thereof.

Proof. As $a \geq 0$, $b \geq 0$, $w_3 \geq 0$, $w_1 \geq 0$ and $w_2 \geq 0$, then from $w_3w_1 \geq (w_2)^2$ it follows the inequality $aw_3 + bw_1 \geq 2\sqrt{abw_3w_1} \geq 2\sqrt{abw_2}$.

Lemma 1.3. When $p > 0$, $q > 0$, $pq > 1$ and $(p, q) \neq (2, 2)$ the system

$$\begin{cases}
    u = x^2 - z^2 - pxz + (p - q)xz + qyz = 0 \\
    v = y^2 - z^2 - qxy + (q - p)yxy + pxz = 0
\end{cases}$$

has at least one solution $(x_0, y_0, 1)$ such that $x_0 > 0$, $x_0 \neq 1$ and $y_0 > 0$.

Proof. (Similarly to lemma 3.1 from [4]).

If $p = q \neq 2$ then from $p = q > 0$ and $pq > 1$ it follows that $p > 1$, i.e. $(x_0, y_0, z_0) = (p - 1, 1, 1) \neq (1, 1, 1)$ is a solution of the system (4).

Let $p \neq q$ and $z = 1$. From the first equation $u = 0$ we find

$$y_0 = [x^2 + (p - q)x - 1](px - q)^{-1}$$

and we substitute into the second equation $(px - q)^2 v(x, y_0, 1) = (x - 1)l(x)$ where

$$l(x) = (pq - 1)(1 - x^3) - [(p - 1)^2 + (p^2 + 2)q - (p + 1)^2 + q^3]x + [(q - 1)^2 + (q^2 + 2)p - (q + 1)^2 + p^3]x^2.$$

We find $l(1) = (p - q)[(p - 2)^2 + (q - 2)^2 - (p - 2)(q - 2)]$.

Case 1. Let $p > q$. From $l(1) > 0$ and $\lim_{x \to +\infty} l(x) = -\infty$ follows that it exists a root $x_0 \in (1; +\infty)$.

From $x_0 > 1 \Rightarrow px_0 - q > p - q > 0$ and $x_0^2 + (p - q)x_0 - 1 > 1 + (p - q) - 1 > 0$, i.e. $y_0 = [x_0^2 + (p - q)x_0 - 1](px_0 - q)^{-1} > 0$.

Case 2. Let $p < q$. From $l(1) < 0$ and $l(0) = pq - 1 > 0$ follows that it exists a root $x_0 \in (0; 1)$.

From $x_0 < 1 \Rightarrow px_0 - q < p - q < 0$ and from $0 < x_0 < 1 \Rightarrow x_0^2 + (p - q)x_0 - 1 = (x_0^2 - 1) + (p - q)x_0 < 0$, i.e. again $y_0 > 0$. □

Let us proceed to the proof of the theorem.

When $k = 0$ then we have $3c_0 = a_0^2 + ab_0 + b_0^2 - 3$ and according to Lemma 1.1 it follows that $w_4 + a_0w_3 + b_0w_1 + c_0w_2 \geq 0$. In this case $p = t_0$, $q = t_0^{-1}$ and the solution of system (4) are: \(\{x = y = z\}, \{x = 0, x = t_0z\}, \{y = 0, z = t_0x\}\) and \(\{z = 0, x = t_0y\}\), i.e. the equality holds if only if $x = y = z$ or $\{z = 0, x = t_0y\}$, or any cyclic permutation thereof.

When $t > 0$ according to Lemma 1.2. we have $t^{-1}w_3 + tw_1 \geq 2w_2$. The equality holds if and only if $x = y = z$ or $x = y = 0$, or $\{z = 0, x = ty\}$, or any cyclic permutation thereof.
Let \( k \geq 0 \) and \( c \geq c_0 - 2k \). Then
\[
f = w_4 + aw_3 + bw_1 + cw_2 \\
= w_4 + (a_0 + kt_0^{-1})w_3 + (b_0 + kt_0)w_1 + (c_0 + c - c_0)w_2 \\
= w_4 + a_0w_3 + b_0w_1 + c_0w_2 + k(t_0^{-1}w_3 + t_0w_1) + (c - c_0)w_2 \\
\geq 0 + 2kw_2 + (c - c_0)w_2 = (c - c_0 + 2k)w_2 \geq 0 \tag{5}
\]

When \( k \geq 0 \) and \( c = c_0 - 2k \) the equality holds if only if \( x = y = z \) or \( \{ z = 0, x = t_0y \} \), or any cyclic permutation thereof.

When \( k \geq 0 \) and \( c > c_0 - 2k \) the equality holds if only if \( x = y = z \).

When \( k \geq 0 \) and \( c < c_0 - 2k \) the inequality is not true. From the expression \( f = w_4 + aw_3 + bw_1 + (c - 2k)w_2 + (c - c_0 + 2k)w_2 \) we obtain \( f(t_0,1,0) = 0 + (c - c_0 + 2k)t_0^2 < 0 \).

When \( k < 0 \) and \( c \geq r \) the inequality holds according to Lemma 1.1.

Let \( k < 0 \) and \( c < r \). From \( t_0 > 0 \) \( \Rightarrow p = -\frac{1}{3}(2a + b) = t_0 - \frac{k}{3}(t_0 + 2t_0^{-1}) > t_0 > 0 \),
\[
q = -\frac{1}{3}(a + 2b) = t_0^{-1} - \frac{k}{3}(t_0^{-1} + 2t_0) > t_0^{-1} > 0 \quad \text{and} \quad pq > t_0t_0^{-1} = 1.
\]

According to Lemma 1.3 when \((p,q) \neq (2,2)\) it exists positive numbers \((x_0,y_0,1)\) \(\neq (1,1,1)\) such that \(u(x_0,y_0,1) = 0\) and \(v(x_0,y_0,1) = 0\). According to the identity (3) we have \( f = u^2 - uv + v^2 + (c - r)w_2 \) and \( f(x_0,y_0,1) = 0 + (c - r)w_2(x_0,y_0,1) < 0 \) because \( c - r < 0 \) and \( w_2(x_0,y_0,1) > 0 \).

When \( p = q = 2 \) then \( u(x,1,1) = (x - 1)^2\), \( v(x,1,1) = 0\), \( w_2(x,1,1) = (x - 1)^2 \) and for \( x_1 = 1 + \sqrt{0.5(r - c)} \) then
\[
f(x_1,1,1) = [0.5(r - c)]^2 + (c - r)[0.5(r - c)] = -0.25(r - c)^2 < 0.
\]

When \( k < 0 \) and \( c > r \) the equality holds if only if \( x = y = z \). When \( k < 0 \) and \( c = r \) the equality holds if only when \( x = y = z \) and at least for one triple \((x_0,y_0,1)\) such that \( x_0 > 0 \), \( x_0 \neq 1 \) and \( y_0 > 0 \), or any cyclic permutation thereof, with the only exception of the case \( a = b = -2 \Rightarrow c = r = 3 \). When \( a = b = -2 \) the equality holds if and only if \( x = y = z \). \( \square \)

**Proof of Theorem 2. Necessity.** From \( 0 \leq f(1,1,1) = 3d \Rightarrow d \geq 0 \). Let \( n \) be an arbitrary natural number. From \( 0 \leq f(n,n^{-3},0) = a + bn^{-8} + cn^{-4} \) follows that \( \lim_{n \to +\infty} (a + bn^{-8} + cn^{-4}) = a \geq 0 \). Analogously, from \( f(n^{-3},n,0) \geq 0 \) follows that \( b \geq 0 \). From \( f(\sqrt{b},\sqrt{a},0) = ab(c + 2\sqrt{ab}) \geq 0 \). If \( a > 0 \), \( b > 0 \Rightarrow c + 2\sqrt{ab} \geq 0 \). If \( a = 0 \), then from \( 0 \leq f(n,n^{-1},0) = c + bn^{-2} \) and \( \lim_{n \to +\infty} (c + bn^{-2}) = c \) it follows that \( c \geq 0 \). Analogously, from \( b = 0 \Rightarrow c \geq 0 \), i.e. always \( c + 2\sqrt{ab} \geq 0 \).

**Sufficiency.** We apply Lemma 1.2 and we obtain
\[
f = aw_3 + bw_1 + cw_2 + dxyz(x + y + z) \geq (2\sqrt{ab} + c)w_2 + 0 \geq 0. \square
\]

**Remark.** All identities are verified via the Maplesoft platform.
REFERENCES


(Received October 14, 2015)

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