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BOUNDEDNESS FOR THE GENERALIZED COMMUTATOR OF SJÖLIN TYPE OPERATORS

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Abstract. In this paper, we study the generalized commutators of Sjölin type operator $T_{\alpha,A}^{a,m}$ defined by

$$T_{\alpha,A}^{a,m}f(x)=\int_{\mathbb{R}^n}K_\alpha^a(x-y)\frac{R_m(A;x,y)}{|x-y|^{m-1}}f(y)dy=\int_{\mathbb{R}^n}\frac{e^{i|x-y|^a}}{|x-y|^\alpha}\frac{R_m(A;x,y)}{|x-y|^{m-1}}f(y)dy,$$

where $R_m(A;x,y)=A(x)-\sum\limits_{|\alpha|<m}\frac{1}{\alpha!}D^\alpha A(y)(x-y)^\alpha$ with $m\in\mathbb{Z}^+$.

By using the scale changing method, we prove that if $D^\gamma A\in\dot{\Lambda}_\beta(0<\beta<1)$ with $|\gamma|=m-1$, $m\geqslant 2$ or $A\in\dot{\Lambda}_\beta$ $(0<\beta<1)$ when $m=1$, $T_{\alpha,A}^{a,m}$ is bounded on $L^p(\mathbb{R}^n)$ for certain range of p.

1. Introduction

In 1976, Janson [4] studied the commutator T_b generated by the Lipschitz function and the singular integrals as follows.

$$T_bf(x)=bT(f)(x)-T(bf)(x), \tag{1.1}$$

where T is the classical C-Z singular integral operator and Janson [4] proved that T_b is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $1/p-1/q=\beta/n$ if and only if $b\in\dot{\Lambda}_\beta(0<\beta<1)$. Here $\dot{\Lambda}_\beta$ is the homogeneous Lipschitz space with its definition defined by

$$\|f\|_{\dot{\Lambda}_\beta}=\sup_{x,h\in\mathbb{R}^n,h\neq 0}\frac{\Delta_h^{[\beta]+1}f(x)}{|h|^\beta}<\infty, \tag{1.2}$$

where $\Delta_h^1f(x)=f(x+h)-f(x)$ and $\Delta_h^{k+1}f(x)=\Delta_h^kf(x+h)-\Delta_h^kf(x)(k\in\mathbb{Z}^+)$. Obviously, when $0<\beta<1$ and $f\in\dot{\Lambda}_\beta$, we have $|f(x)-f(y)|\leqslant|x-y|^\beta\|f\|_{\dot{\Lambda}_\beta}$ for $\forall x,y\in\mathbb{R}^n$.

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In 2003, Lu, Wu and Zhang [9] studied the following generalized commutator T_A^m defined as

$$T_A^m f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A;x,y) f(y)dy, \tag{1.3}$$

where $R_m(A;x,y) = A(x) - \sum_{|\gamma|<m} \frac{1}{\gamma!} D^\gamma A(y)(x-y)^\gamma$, the m-th remainder of Taylor series of the function A at y about x. Lu, Wu and Zhang [9] proved that if A has derivatives of order $m-1 (m \geqslant 2)$ in $\dot{\Lambda}_\beta$ $(0<\beta<1)$, then T_A^m is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $1/p - 1/q = \beta/n$ and $1 < p < \frac{n}{\beta}$.

Here we would like to point out that the operator T_A^m was first studied by Cohen and Gossenlin [2]. In [2], Cohen and Gosselin proved that if A has derivatives of order $m-1$ in BMO($\mathbb{R}^n$) where BMO($\mathbb{R}^n$) denotes the bounded mean oscillation space, then the operator T_A^m is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. For the study of the Cohen-Gosselin type operators, one may see [10, 15, 19] et. al. for more details. In 2012, Wang and Zhang [16] gave a new and simpler proof of Wu's theorem in [17] by using some results about the generalized commutator of Cohen-Gosselin type proved by Yan [18] (or see [10]).

In 1980, Sjölin [12] introduced the convolution operator T_α^a with oscillating kernels defined by

$$T_\alpha^a f(x) = \int_{\mathbb{R}^n} K_\alpha^a(x-y) f(y)dy = \int_{\mathbb{R}^n} \frac{e^{i|x-y|^a}}{|x-y|^\alpha} f(y)dy, \tag{1.4}$$

where $a > 0, a \neq 1$ and $\alpha < n$.

Sjölin [12] proved the following theorem.

THEOREM A. ([12]) *If $\alpha \geqslant n(1-a/2)$, then the Sjölin type operator T_α^a is bounded on $L^p(\mathbb{R}^n)$ if and only if $p_0 \leqslant p \leqslant p_0'$ with $p_0 = \frac{na}{na-n+\alpha}$. If $\alpha < n(1-\frac{a}{2})$, then T_α^a is not bounded on any $L^p(\mathbb{R}^n)(1 \leqslant p \leqslant \infty)$.*

The operator T_α^a was also studied by many other authors, one may see [5] or [13] for more details. Especially in [5], Li proved the boundedness of T_α^a by using the scale changing method which was proposed by Carleson and Sjölin [1]. Moreover, Li [5] also studied the following generalized commutator of the Sjölin type operator $T_{\alpha,A}^{a,m}$ with its definition defined by

$$T_{\alpha,A}^{a,m} f(x) = \int_{\mathbb{R}^n} K_\alpha^a(x-y) \frac{R_m(A;x,y)}{|x-y|^{m-1}} f(y)dy = \int_{\mathbb{R}^n} \frac{e^{i|x-y|^a}}{|x-y|^\alpha} \frac{R_m(A;x,y)}{|x-y|^{m-1}} f(y)dy. \tag{1.5}$$

Li [5] proved that if A has derivatives of order $m-1$ in BMO($\mathbb{R}^n$) with $m \geqslant 1$, then the operator $T_{\alpha,A}^{a,m}$ is bounded on the $L^p(\mathbb{R}^n)$ for some $p > 1$.

Motivated by the above background, it is natural to ask whether we can prove the L^p boundedness of $T_{\alpha,A}^{a,m}$ if $D^\gamma A \in \dot{\Lambda}_\beta (|\gamma| = m-1)$ with $m \in \mathbb{Z}^+$? In this paper, we will give a positive answer to this question. Moreover, when $m = 1$, we simply denote $T_{A,\alpha}^{a,1}$ by $T_{A,\alpha}^a$, that is

$$T_{A,\alpha}^a f(x) := T_{A,\alpha}^{a,1} f(x) = A(x) T_\alpha^a f(x) - T_\alpha^a(Af)(x). \tag{1.6}$$

Our results can be stated as follows.

THEOREM 1.1. *Suppose that* $a>0$, $a\neq 1$ *and* $an<2$. *If* $\beta+n-an<\alpha<n+\beta-\frac{an}{2}$ *and* $A\in\dot{\Lambda}_\beta$ *with* $\frac{an}{2}<\beta<1$, *then* $T^a_{A,\alpha}$ *is bounded on* $L^p(\mathbb{R}^n)$ *with* $p_0<p<p_0'$ *where* $1<p_0=\frac{an}{\beta+n-\alpha}<2$.

THEOREM 1.2. *Suppose that* $a>0$, $a\neq 1$ *and* $an<2$. *If* $\beta+n-an<\alpha<n+\beta-\frac{an}{2}$ *and* $D^\gamma A\in\dot{\Lambda}_\beta(|\gamma|=m-1)$ *with* $\frac{an}{2}<\beta<1$ *and* $m\geqslant 2$, *then* $T^{a,m}_{A,\alpha}$ *is bounded on* $L^p(\mathbb{R}^n)$ *with* $p_0<p<p_0'$ *where* $1<p_0=\frac{an}{\beta+n-\alpha}<2$.

2. Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1. The basic idea of proving Theorem 1.1 is the scale changing method which was introduced by Carleson and Sjölin in [1]. In [1], Carleson and Sjölin used this method to prove the $L^p(\mathbb{R}^2)$ boundedness of the Bochner-Riesz operators below the critical index. Later, the scale changing method was used to study the boundedness of strongly singular integral and its commutators, please see [6, 7, 8] for details.

Before giving the proof of Theorem 1, we introduce some lemmas and notations. Let Ψ be a smooth function of compact support in x and ξ, and Φ be real valued and smooth. We assume that the support of Ψ, the Hessian determinant of Φ is nonvanishing, i.e.

$$\det\left(\frac{\partial^2\Phi(x,\xi)}{\partial x_i\partial\xi_j}\right)\neq 0. \tag{2.1}$$

Moreover, we have the following lemma.

LEMMA 2.1. ([7] or [14]) *Denote* $T_\lambda f(\xi)=\int_{\mathbb{R}^n}e^{i\lambda\Phi(x,\xi)}\Psi(x,\xi)f(x)dx$, *where* Φ *and* Ψ *satisfy (2.1). Then we have*

$$\|T_\lambda f\|_{L^2}\leqslant C\lambda^{-\frac{n}{2}}\|f\|_{L^2}.$$

Obviously, we also have $\|T_\lambda f\|_{L^\infty}\leqslant C\|f\|_{L^\infty}$ *and* $\|T_\lambda f\|_{L^1}\leqslant C\|f\|_{L^1}$. *Then using interpolation, we obtain (see* [7]*)*

$$\|T_\lambda f\|_{L^p}\leqslant C\lambda^{-\frac{n}{p}}\|f\|_{L^p},\quad 2\leqslant p<\infty. \tag{2.2}$$

$$\|T_\lambda f\|_{L^p}\leqslant C\lambda^{-\frac{n}{p'}}\|f\|_{L^p},\quad 1\leqslant p<2. \tag{2.3}$$

Proof of Theorem 1.1. We only need to prove that for any fixed $N\in\mathbb{Z}^+$, there exists a positive constant C independent of f and A, such that

$$\int_{[0,N]^n}\left|\int_{[0,N]^n}\frac{e^{i|x-y|^a}}{|x-y|^\alpha}(A(x)-A(y))f(y)dy\right|^p dx\leqslant C\|A\|^p_{\dot{\Lambda}_\beta}\int_{[0,N]^n}|f(x)|^p dx. \tag{2.4}$$

By the scale changing method, (2.4) is equivalent to

$$\int_I\left|N^{n-\alpha}\int_I(A(Nx)-A(Ny))\frac{e^{iN^a|x-y|^a}}{|x-y|^\alpha}f(Ny)dy\right|^p dx\leqslant CN^{\beta p}\|A\|^p_{\dot{\Lambda}_\beta}\int_I|f(Nx)|^p dx. \tag{2.5}$$

where $I = [0,1]^n$ is the unit cube in $\mathbb{R}^n$.

Now, we take some notations from [7].

Let Ω_μ $(\mu = 0,1,\cdots)$ be the set of all dyadic cubes in $(-2,2)^n$ with side length $2^{-\mu}$, and let Ω^* be the set of all cubes which satisfy the union of 2^n cubes in Ω_μ. For $x \in I$ and x does not belong to the boundary of any dyadic cubes, we denote that $\omega_\mu^*(x)$ is the unique element of Ω_μ^* satisfying $x \in \frac{1}{2}\omega_\mu^*(x)$. Moreover, we set $\omega_{-1}^*(x) = (-2,2)^n$. Then for a measurable set $D \subset I$, we denote

$$E_A^{a,\alpha}(x,D) = N^{n-\alpha}\int_D (A(Nx)-A(Ny))\frac{e^{iN^a|x-y|^a}}{|x-y|^\alpha} f(Ny)dy, \qquad x \in I,$$

and

$$E_\mu(x) = E_A^{a,\alpha}(x, \omega_{\mu-1}^*(x) \setminus \omega_\mu^*(x) \cap I), \qquad \mu \geqslant 0.$$

Furthermore, we may denote

$$S_{N,A}^{a,\alpha} f(Nx) = N^{n-\alpha}\int_I \frac{e^{iN^a|x-y|^a}}{|x-y|^\alpha}(A(Nx)-A(Ny))f(Ny)dy.$$

Thus, we conclude that Theorem 1.1 reduces to prove the following inequality,

$$\|S_{N,A}^{a,\alpha} f(N\cdot)\|_{L^p(I)} \leqslant CN^\beta \|A\|_{\dot{\Lambda}_\beta} \|f(N\cdot)\|_{L^p(I)}. \tag{2.6}$$

To prove (2.6), by the definition of E_μ, we can decompose $S_{N,A}^{a,\alpha}$ as

$$S_{N,A}^{a,\alpha} f(Nx) \leqslant \sum_{\mu=0}^{\mu_N} E_\mu(x) + \sum_{\mu=\mu_N+1}^{\infty} E_\mu(x),$$

where μ_N belong to $\mathbb{Z}^+$ and satisfies $2^{-\mu_N-1} < N^{-1} \leqslant 2^{-\mu_N}$. By the fact that when μ is big enough, there is $\omega_{\mu-1}^* \setminus \omega_\mu^* \subset I$.

Thus, we may assume that $E_\mu(x) = E_A^{a,\alpha}(x, \omega_{\mu-1}^*(x) \setminus \omega_\mu^*(x))$, $\mu \geqslant 0$. By the construction of $\omega_{\mu-1}^*(x) \setminus \omega_\mu^*(x)$, we have $E_\mu(x) = \sum_{\omega \in \Omega_\mu} E_A^{a,\alpha}(x,\omega)\chi_{F(\omega)}(x)$ where $\chi_{F(\omega)}(x)$ is the characteristic function of $F(\omega)$. As $\sum_{\omega \in \Omega_\mu} \chi_{F(\omega)}(x) \leqslant 6^n - 2^n$ and $F(\omega) = 6\omega \setminus 2\omega$ is the union of cubes in $\Omega_{\mu+1}$, with the property that the distance from each cube to ω is approximately $2^{-\mu}$. Then by the Hölder inequality, we conclude that $|E_\mu(x)|^p \leqslant \sum_{\omega \in \Omega_\mu} |E_A^{a,\alpha}(x,\omega)|^p \chi_{F(\omega)}(x)$. Thus, we get

$$\int_I |E_\mu(x)|^p dx \leqslant C \sum_{\omega \in \Omega_\mu} \int_{F(\omega)} |E_A^{a,\alpha}(x,\omega)|^p dx.$$

For any fixed $\omega \in \Omega_\mu$, we denote $x_\omega = (x_1, x_2, \cdots, x_n)$ be a point in ω, such that for any $y = (y_1, y_2, \cdots, y_m) \in \omega$, $x_i \leqslant y_i$, $i = 1,2,\cdots,n$. Recall the fact that the side length of ω is $2^{-\mu}$. Then for any $\mu < \mu_N$, we have

$$\begin{aligned}
&\int_{F(\omega)} |E_A^{a,\alpha}(x,\omega)|^p dx \\
&= \int_{F(\omega)} \left| \int_\omega N^{n-\alpha} \frac{e^{iN^a|x-y|^a}(A(Nx)-A(Ny))f(Ny)}{|x-y|^\alpha} dy \right|^p dx \\
&= \int_{F(\omega)-x_\omega} \left| \int_{[0,2^{-\mu}]^n} N^{n-\alpha} \frac{e^{iN^a|x-y|^a}}{|x-y|^\alpha} [A(Nx+x_\omega)-A(Ny+x_\omega)] f(Ny) dy \right|^p dx \\
&= \int_{F(I)} 2^{-\mu n} \left| \int_I N^{n-\alpha} 2^{\mu\alpha} 2^{-\mu n} \frac{e^{i(N2^{-\mu})^a|x-y|^a}}{|x-y|^\alpha} [A(2^{-\mu}Nx+x_\omega) \right. \\
&\left. -A(2^{-\mu}Ny+x_\omega)] f(2^{-\mu}Ny+x_\omega) dy \right|^p dx.
\end{aligned}$$

First, we consider the case $\mu \leqslant \mu_N$. As $x \in F(I)$, $y \in I$, we obtain

$$|x-y| \leqslant |x-y_0| + |y-y_0| \leqslant C|y-x_0| \leqslant C2^{-kn},$$

Thus, we get

$$\begin{aligned}
S_{N,A}^{a,\alpha} f(Nx) &\leqslant C\|A\|_{\dot{\Lambda}_\beta} N^\beta N^{n-\alpha} \int_I |x-y|^{\beta-\alpha} |f(Ny)| dy \\
&\leqslant C\|A\|_{\dot{\Lambda}_\beta} N^{n+\beta-\alpha} M(f(N\cdot)\chi_I)(x).
\end{aligned}$$

where M denotes the Hardy-Littlewood maximal function. Then we have

$$\|S_{N,A}^{a,\alpha} f(N\cdot)\|_{L^p(F(I))} \leqslant C\|A\|_{\dot{\Lambda}_\beta} N^{n+\beta-\alpha} \|M(f(N\cdot)\chi_I(\cdot))\|_{L^p(F(I))},$$

which implies

$$\|S_{N,A}^{a,\alpha} f(N\cdot)\|_{L^p(F(I))} \leqslant C\|A\|_{\dot{\Lambda}_\beta} N^{\beta+n-\alpha} \|f(N\cdot)\|_{L^p(I)}.$$

From the above estimates and the definition of $E_A^{a,\alpha}$, there is

$$\int_{F(\omega)} |E_A^{a,\alpha}(x,\omega)|^p dx \leqslant C\|A\|_{\dot{\Lambda}_\beta}^p (N2^{-\mu})^{(n+\beta-\alpha)p} \int_\omega |f(Nx)|^p dx.$$

By the above inequality and the definition of E_μ, we obtain

$$\|E_\mu\|_{L^p(I)} \leqslant C\|A\|_{\dot{\Lambda}_\beta} (N2^{-\mu})^{n+\beta-\alpha} \|f(N\cdot)\|_{L^p(I)}.$$

Now, we conclude that for the case $\mu \leqslant \mu_N$, we have

$$\sum_{\mu=0}^{\mu_N} \|E_\mu\|_{L^p(I)} \leqslant CN^\beta \|A\|_{\dot{\Lambda}_\beta} \|f(N\cdot)\|_{L^p(I)}. \tag{2.7}$$

Next, we will consider the case $\mu \geqslant \mu_N$. In order to use Lemma 2.1, we would like to mention that from [8, p. 45–p. 46], we know both $\Phi(x,y) = |x-y|^a$ and $\Psi(x,y) = \frac{1}{|x-y|^\alpha}$ satisfy (2.1) on $F(I)\times I$. Then, let $\Psi(x,y) = \frac{1}{|x-y|^\alpha}$ be a smooth function supported on $F(I)\times I$, otherwise we may choose a class of smooth function defined on $F(I)\times I$ and approximating to it.

When $p \geqslant 2$, using (2.2), we have

$$\begin{aligned}
&\int_{F(\omega)}\left|\int_\omega N^{n-\alpha}\frac{e^{iN^a|x-y|^a}}{|x-y|^\alpha}(A(Nx)-A(Ny))f(Ny)dy\right|^p dx\\
&\leqslant \int_{F(I)} 2^{-\mu n}\left|A(2^{-\mu}Nx+x_\omega)-A(2^{-\mu}Nx_0+x_\omega)\right|^p\\
&\quad\times\left|\int_I N^{n-\alpha}2^{\mu\alpha-\mu n}\frac{e^{i(2^{-\mu}N)^a|x-y|^a}}{|x-y|^\alpha}f(2^{-\mu}Ny+x_\omega)dy\right|^p dx\\
&\quad+\int_{F(I)}2^{-\mu n}\left|\int_I N^{n-\alpha}2^{\mu\alpha-\mu n}\frac{e^{i(2^{-\mu}N)^a|x-y|^a}}{|x-y|^\alpha}\right.\\
&\quad\left.\times|A(2^{-\mu}Nx_0+x_\omega)-A(2^{-\mu}Ny+x_\omega)|f(2^{-\mu}Ny+x_\omega)dy\right|^p dx\\
&\leqslant C\|A\|^p_{\dot\Lambda_\beta}N^{\beta p}N^{np-p\alpha}2^{-\mu n}2^{(\mu\alpha-\mu n)p}2^{-\mu\beta p}(2^{-\mu}N)^{-\frac{an}{p}p}\int_I|f(2^{-\mu}Ny+x_\omega)|^p dy\\
&\quad+CN^{(n-\alpha)p}2^{-\mu n}2^{(n-\alpha)p}2^{(\mu\alpha-\mu n)p}(2^{-\mu}N)^{-\frac{an}{p}p}\\
&\quad\times\int_I\left|[A(2^{-\mu}Nx_0+x_\omega)-A(2^{-\mu}Ny+x_\omega)]f(2^{-\mu}Ny+x_\omega)\right|^p dx\\
&\leqslant C\|A\|^p_{\dot\Lambda_\beta}N^{(n+\beta-\alpha)p}2^{(\mu\alpha-\mu n)p}2^{-\mu\beta p}(2^{-\mu}N)^{-an}2^{-\mu n}\int_I\left|f(2^{-\mu}Ny+x_\omega)\right|^p dy\\
&\leqslant C\|A\|^p_{\dot\Lambda_\beta}N^{(n+\beta-\alpha)p-an}2^{p\mu\alpha-\mu np-\mu\beta p+\mu an}\|f(N\cdot)\|^p_{L^p(\omega)}\\
&\leqslant C\|A\|^p_{\dot\Lambda_\beta}N^{(n+\beta-\alpha)p-an}2^{-p\mu(n+\beta-\alpha-\frac{an}{p})}\|f(N\cdot)\|^p_{L^p(\omega)}.
\end{aligned}$$

For the case $1<p\leqslant 2$, by a similar argument as in the above case, we may get

$$\begin{aligned}
&\int_{F(\omega)}\left|\int_\omega N^{n-\alpha}\frac{e^{N^a|x-y|^a}}{|x-y|^\alpha}[A(Nx)-A(Ny)]f(Ny)dy\right|^p dx\\
&\leqslant C\|A\|^p_{\dot\Lambda_\beta}N^{p(n+\beta-\alpha)-\frac{\alpha np}{p'}}2^{(\mu\alpha-kn)p}2^{-\mu\beta p}(2^{-\mu})^{-\frac{\alpha n}{p'}p}\|f(N\cdot)\|^p_{L^p(\omega)}\\
&\leqslant C\|A\|^p_{\dot\Lambda_\beta}N^{p(n+\beta-\alpha-\frac{\alpha n}{p'})}2^{\mu p(\alpha-n-\beta+\frac{\alpha n}{p'})}\|f(N\cdot)\|^p_{L^p(\omega)}.
\end{aligned}$$

By the above two estimates and the condition $p_0<p<p_0'$ with $1<p_0=\frac{an}{\beta+n-\alpha}<2$, we obtain

$$\sum_{\mu\geqslant\mu_n}\left\{\sum_{\omega\in\Omega_\mu}\int_{F(\omega)}|E_A^{a,\alpha}(x,\omega)|^p dx\right\}^{1/p}\leqslant CN^\beta\|A\|_{\dot\Lambda_\beta}\|f(N\cdot)\|_{L^p(I)}. \tag{2.8}$$

Combining (2.7)–(2.8), we finish the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

Before giving the proof of Theorem 1.2, we give some lemmas that will be very useful throughrout this section.

LEMMA 3.1. ([2]) *Let b be a function on $\mathbb{R}^n$ with $m(m\geqslant 2)$-th order derivatives in $L^q_{loc}(\mathbb{R}^n)$ for some $q>n$, then there exists a positive constant C independent of b, such that*

$$|R_m(b;x,y)|\leqslant C_{m,n}|x-y|^m\sum_{|\gamma|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}|D^\gamma b(z)|^q dz\right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having diameter $5\sqrt{n}|x-y|$.

LEMMA 3.2. ([11]) *Let $0<\beta<1$ and $1\leqslant q<\infty$, then*

$$\begin{aligned}\|f\|_{\dot{\Lambda}_\beta}&\approx\sup_Q\frac{1}{|Q|^{1+\beta/n}}\int_Q|f(x)-m_Q(f)|dx\\&\approx\sup_Q\frac{1}{|Q|^{\beta/n}}\left\{\frac{1}{|Q|}\int_Q|f(x)-m_Q(f)|^q dx\right\}^{1/q}.\end{aligned}$$

LEMMA 3.3. ([3]) *Letting $Q^*\subset Q$ and $g\in\dot{\Lambda}_\beta(0<\beta<1)$, we have*

$$|m_{Q^*}(g)-m_Q(g)|\leqslant C|Q|^{\beta/n}\|g\|_{\dot{\Lambda}_\beta}.$$

Proof of Theorem 1.2. By the same argument as in the above section, it suffices to prove

$$\int_{[0,N]^n}\left|\int_{[0,N]^n}\frac{e^{i|x-y|^a}}{|x-y|^{\alpha+m-1}}R_m(A;x,y)f(y)dy\right|^p dx\leqslant C\sum_{|\gamma|=m-1}\|D^\gamma A\|^p_{\dot{\Lambda}_\beta}\int_{[0,N]^n}|f(x)|^p dx, \tag{3.1}$$

which is equivalent to

$$\begin{aligned}&\int_I\left|\int_I N^{n-\alpha-m+1}\frac{e^{i|N(x-y)|^a}}{|x-y|^{\alpha+m-1}}R_m(A;Nx,Ny)f(Ny)dy\right|^p dx\\&\leqslant CN^{\beta p+(m-1)p}\sum_{|\gamma|=m-1}\|D^\gamma A\|^p_{\dot{\Lambda}_\beta}\int_I|f(Ny)|^p dy\end{aligned}\tag{3.2}$$

for any fixed $N\geqslant 1$.

Now, we define

$$E^{a,\alpha}_A(x,D)=N^{n-\alpha-m+1}\int_D\frac{e^{i|N(x-y)|^a}}{|x-y|^{\alpha+m-1}}R_m(A;Nx,Ny)f(Ny)dy$$

and

$$S_{N,A}^{a,\alpha}f(Nx) = N^{n-\alpha-m+1}\int_I \frac{e^{i|N(x-y)|^a}}{|x-y|^{\alpha+m-1}}R_m(A;Nx,Ny)f(Ny)dy.$$

Furthermore, we denote

$$E_\mu(x) = E_A^{a,\alpha}(x,\omega_{\mu-1}^* \setminus \omega_\mu^* \cap I),\ \ \mu \geqslant 0.$$

Then we have

$$S_{N,A}^{a,\alpha}f(Nx) \leqslant \sum_{\mu=0}^{\mu_N} E_\mu(x) + \sum_{\mu=\mu_N+1}^{\infty} E_\mu(x).$$

When $\mu \leqslant \mu_N$, denote $\tilde{A}(Nz) = [A(Nz) - \sum_{|\gamma|=m-1}\frac{z^\gamma}{\gamma!}m_{J_x}(D^\gamma A(N\cdot))]\phi(z)$, where J_x is a cube centered at x with its side length equals 4 and $\phi \in C_c^\infty(\mathbb{R}^n)$. Furthermore, we assume that when $|x|<10$, $\phi \equiv 1$ and when $|x|>20$, $\phi \equiv 0$. Thus, we get

$$\begin{aligned} S_{N,A}^{a,\alpha}f(x) &\leqslant CN^{n-\alpha-m+1}\int_I |R_{m-1}(\tilde{A}(N\cdot);x,y)||f(Ny)|dy \\ &\quad + CN^{n-\alpha-m+1}\sum_{|\gamma|=m-1}\int_I |D^\gamma A(Nx) - m_{J_x}(D^\gamma A(N\cdot))||f(Ny)|dy. \end{aligned}$$

From Lemmas 3.1–3.3, we know that for any $x \in F(I)$ and $y \in I$, there is

$$\begin{aligned} &|R_{m-1}(\tilde{A}(N\cdot);x,y)| \\ &\leqslant C|x-y|^{m-1}\sum_{|\gamma|=m-1}\left\{\frac{1}{|Q(x,y)|}\int_{Q(x,y)}|D^\gamma A(Nz) - m_{Q(x,y)}(D^\gamma A(N\cdot))|^q dz\right\}^{1/q} \\ &\quad + C|x-y|^{m-1}\sum_{|\gamma|=m-1}|m_{Q(x,y)}(D^\gamma A(N\cdot)) - m_{J_x}(D^\gamma A(N\cdot))| \\ &\leqslant C\sum_{|\gamma|=m-1}\|D^\gamma A(N\cdot)\|_{\dot{\Lambda}_\beta} = CN^\beta N^{m-1}\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta}. \end{aligned}$$

As $p>1$, we may choose a positive real number r satisfying $1<r<p$. Moreover, we denote $M_r(f)(x) = \sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q |f(y)|^r dy\right)^{1/r}$. Then by the Hölder inequality and Lemmas 3.2–3.3, we have

$$\begin{aligned} &\sum_{|\gamma|=m-1}\int_I |D^\gamma A(Ny) - m_{J_x}(D^\gamma A(N\cdot))||f(Ny)|dy \\ &\leqslant C\sum_{|\gamma|=m-1}\left(\int_I |D^\gamma A(Ny) - m_{J_x}(D^\gamma A(N\cdot))|^{r'}dy\right)^{1/r'}\left(\int_I |f(Ny)|^r dy\right)^{1/r} \\ &\leqslant CN^\beta N^{m-1}\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta}M_r(f(N\cdot)\chi_I)(x), \end{aligned}$$

for any $x \in F(I)$ and $y \in I$.

Combing the above two estimates, we may get

$$S_{N,A}^{a,\alpha} f(Nx) \leqslant C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot\Lambda_\beta} N^{n-\alpha+\beta} \left(M(f(N\cdot)\chi_I)(x) + M_r(f(N\cdot)\chi_I)(x)\right).$$

Then, by the boundedness of $M_r(f)(x)$ on $L^{\frac{p}{r}}\,(p>r)$ space, we obtain

$$\|S_{N,A}^{a,\alpha} f(N\cdot)\|_{L^p(F(I))} \leqslant C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot\Lambda_\beta} N^{n-\alpha+\beta} \|f(N\cdot)\|_{L^p(I)}.$$

By a similar argument as in Section 2, we conclude that for the case $\mu \leqslant \mu_N$, there is

$$\sum_{\mu=0}^{\mu_N} \|E_\mu\|_{L^p(I)} \leqslant C \sum_{|\gamma|=m-1} N^{n-\alpha+\beta} \|D^\gamma A\|_{\dot\Lambda_\beta} \|f(N\cdot)\|_{L^p(I)}. \tag{3.3}$$

Next, we will consider the case when $\mu \geqslant \mu_N$. First, we have the following estimates.

$$\begin{aligned}
&\int_{F(\omega)} \left|E_A^{a,\alpha}(x,\omega)\right|^p dx \\
&= \int_{F(\omega)} \left| \int_\omega N^{n-\alpha-m+1} \frac{e^{i|x-y|^a N^a}}{|x-y|^{\alpha+m-1}} R_m(A(N\cdot);x,y) f(Ny)dy \right|^p dx \\
&= \int_{F(\omega)-\omega} \left| \int_{[0,2^{-\mu}]^n} N^{n-\alpha-m+1} \frac{e^{i|x-y|^a N^a}}{|x-y|^{\alpha+m-1}} R_m(A(N\cdot);x+x_\omega,y+y_\omega) f(Ny+y_\omega)dy \right|^p dx \\
&= 2^{-\mu n} \int_{F(I)} \Bigg| \int_I N^{n-\alpha-m+1} 2^{\mu(\alpha+m-1)} 2^{-\mu n} \frac{e^{i2^{-\mu a}|x-y|^a N^a}}{|x-y|^{\alpha+m-1}} \\
&\quad \times R_m(A(N\cdot);2^{-\mu}x+x_\omega,2^{-\mu}y+y_\omega) f(2^{-\mu}Ny+y_\omega)dy \Bigg|^p dx.
\end{aligned}$$

For any $\omega \in \Omega_\mu$ and any $x_0 \in 6I\setminus 5I$, there is

$$\begin{aligned}
&\int_{F(\omega)} |E_A^{a,\alpha}(x,\omega)|^p dx \\
&\leqslant C 2^{-\mu n} \int_{F(I)} \Bigg| \int_I N^{n-\alpha-m+1} 2^{\mu(\alpha+m-1)} 2^{-kn} \frac{e^{i|2^{-\mu}N(x-y)|^a}}{|x-y|^{\alpha+m-1}} \\
&\quad \times R_{m-1}(\tilde A(N\cdot);2^{-\mu}x+x_\omega,2^{-k}y+x_\omega) f(2^{-k}Ny+x_\omega)dy \Bigg|^p dx \\
&\quad + C \sum_{|\gamma|=m-1} 2^{-\mu n} \int_{F(I)} \left| \int_I N^{n-\alpha-m+1} 2^{\mu\alpha} 2^{-\mu n} \frac{e^{i|2^{-\mu}N(x-y)|^a}}{|x-y|^{\alpha+m-1}} D^\gamma \tilde A(Ny) f(2^{-\mu}Ny+x_\omega)dy \right|^p dx \\
&\leqslant C 2^{-\mu n} \Bigg| \int_{F(I)} N^{n-\alpha-m+1} 2^{\mu(\alpha+m-1-n)} \frac{e^{i|2^{-\mu}N(x-y)|^a}}{|x-y|^{\alpha+m-1}} [R_{m-1}(\tilde A(N\cdot);2^{-\mu}x+x_\omega,2^{-\mu}y+x_\omega) \\
&\quad - R_{m-1}(\tilde A(N\cdot);2^{-\mu}x+x_\omega,2^{-\mu}x_0+x_\omega)] f(2^{-\mu}Ny+x_\omega)dy \Bigg|^p dx
\end{aligned}$$

$$+C2^{-\mu n}\int_{F(I)}\left|R_{m-1}(\tilde{A}(N\cdot);2^{-\mu}x+x_\omega,2^{-\mu}x_0+x_\omega)\right|^p$$
$$\times\left|\int_I N^{n-\alpha-m+1}2^{\mu(\alpha+m-1-n)}\frac{e^{i|2^{-\mu}N(x-y)|^a}}{|x-y|^{\alpha+m-1}}f(2^{-\mu}Ny+x_\omega)dy\right|^p dx$$
$$+C\sum_{|\gamma|=m-1}2^{-\mu n}\int_{F(I)}\left|N^{n-\alpha-m+1}2^{\mu\alpha-\mu n}\frac{e^{i|2^{-\mu}N(x-y)|^a}}{|x-y|^{\alpha+m-1}}D^\gamma\tilde{A}(Ny)f(2^{-\mu}Ny+x_\omega)dy\right|^p dx$$
$$=C(I+II+III).$$

Next, we will estimate I, II and III respectively. Note that (see [2])

$$R_{m-1}(g;x,y)-R_{m-1}(g;,x,x_0)=\sum_{|\delta|<m-1}\frac{(x-x_0)^\delta}{\delta!}R_{m-1-|\delta|}(D^\delta g;x_0,y),$$

where δ is any n-tuple index with $|\delta|<m-1$.

Then we have

$$|R_{m-1}(A(\tilde{N}\cdot);2^{-\mu}x+x_\omega,2^{-\mu}y+x_\omega)-R_{m-1}(A(\tilde{N}\cdot);2^{-\mu}x+x_\omega,2^{-\mu}x_0+x_\omega)|$$
$$=\sum_{|\delta|<m-1}\frac{(2^{-\mu}x-2^{-mu}x_0)^\delta}{\delta!}|R_{m-1-|\delta|}(D^\delta\tilde{A}(N\cdot);2^{-\mu}x_0+x_\omega,2^{-\mu}y+x_\omega)|.$$

Moreover, we denote $Q^{-\mu}_{x_0,y}=Q(2^{-\mu}x_0+x_\omega,2^{-\mu}y+x_\omega)$. Thus, using Lemmas 3.1, 3.3 and the fact $|x_0-y|\leqslant C$, we get

$$|R_{m-1-|\delta|}(D^\delta\tilde{A}(N\cdot);2^{-\mu}x_0+x_\omega,2^{-\mu}y+x_\omega)|$$
$$\leqslant 2^{-\mu(m-1-|\delta|)}|x_0-y|^{m-1-|\delta|}\sum_{|\gamma|=m-1}\left(\frac{1}{|Q^{-\mu}_{x_0,y}|}\int_{Q^{-\mu}_{x_0,y}}|D^\gamma A(Nz)-m_\omega(D^\gamma A(N\cdot))|^q dz\right)^{1/q}$$
$$\leqslant C2^{-\mu(m-1-|\delta|)}\sum_{|\gamma|=m-1}\left\{\|D^\gamma A(N\cdot)\|_{\dot{\Lambda}_\beta}(2^{-\mu})^\beta+|m_{Q^{-\mu}_{x_0,y}}(D^\gamma A(N\cdot))-m_\omega(D^\gamma A(N\cdot))|\right\}$$
$$\leqslant C2^{-\mu(m-1-|\delta|)}(2^{-\mu}N)^\beta N^{m-1}\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta}.$$

Combing the above estimates, we obtain

$$|R_{m-1}(A(\tilde{N}\cdot);2^{-\mu}x+x_\omega,2^{-\mu}y+x_\omega)-R_{m-1}(A(\tilde{N}\cdot);2^{-\mu}x+x_\omega,2^{-\mu}x_0+x_\omega)|$$
$$\leqslant C\sum_{|\delta|<m-1}2^{-\mu|\delta|}2^{-\mu(m-1-|\delta|)}(2^{-\mu}N)^\beta N^{m-1}\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta}.$$

Thus, for $p\geqslant 2$ and an n-tuple index δ with $|\delta|<m-1$, we may choose $\Psi(x,y)$ a smooth function approximating to $\frac{(x-x_0)^\delta}{|x-y|^{\alpha+m-1}}$ on $F(I)\times I$. Then using (2.2), we have

$$I\leqslant C2^{-\mu n}(2^{-\mu}N)^{-\frac{na}{p}p}2^{\mu(\alpha+m-1-n)p}N^{(n-\alpha-m+1)p}$$
$$\times\big\|[R_{m-1}(\tilde{A}(N\cdot);2^{-\mu}x_0+x_\omega,2^{-\mu}y+x_\omega)$$
$$-R_{m-1}(\tilde{A}(N\cdot);2^{-\mu}x+x_\omega,2^{-\mu}x_0+x_\omega)]f(2^{-\mu}N\cdot+x_\omega)\big\|^p_{L^p(I)}$$

$$\leqslant C \sum_{|\gamma|=m-1} \|D^\gamma A\|^p_{\dot\Lambda_\beta} N^{-na+p(n-\alpha-m+1)} \sum_{|\delta|<m-1} 2^{-\mu(m-1-2|\delta|)}$$
$$\times 2^{-\mu n} 2^{\mu(\alpha+m-1-n)p} 2^{\mu na} N^{\beta p} 2^{-\mu\beta p} \|f(2^{-\mu}N\cdot+x_\omega)\|^p_{L^p(I)}$$
$$\leqslant C \sum_{|\gamma|=m-1} \|D^\gamma A\|^p_{\dot\Lambda_\beta} N^{-na+p(n-\alpha-m+1+\beta)} N^{(m-1)p} 2^{\mu p(\alpha-n-\beta+\frac{na}{p})} \|f(N\cdot)\|^p_{L^p(I)}.$$

For II, note the following fact

$$|R_{m-1}(\tilde A(N\cdot);2^{-\mu}x+x_\omega,2^{-\mu}x_0+x_\omega)| \leqslant C2^{-\mu(m-1)}N^\beta 2^{-\mu\beta}N^{m-1} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot\Lambda_\beta}.$$

Thus, we can choose $\Psi(x,y)$ a smooth function approximating to $\frac{1}{|x-y|^{\alpha+m-1}}$ on $F(I)\times I$. So, we have

$$II \leqslant C2^{-\mu n}2^{-\mu(m-1)p}N^{\beta p}2^{-\mu\beta p} \sum_{|\gamma|=m-1} \|D^\gamma A\|^p_{\dot\Lambda_\beta} N^{p(n-\alpha-m+1)}$$
$$\times 2^{\mu p(\alpha+m-1-n)}(2^{-\mu}N)^{-\frac{na}{p}p} \|f(2^{-\mu}N\cdot+x_\omega)\|_{L^p(I)}$$
$$\leqslant C \sum_{|\gamma|=m-1} \|D^\gamma A\|^p_{\dot\Lambda_\beta} N^{p\beta+(n-\alpha-m+1)p-na} 2^{-\mu n} \|f(2^{-\mu}N\cdot+x_\omega)\|_{L^p(I)} 2^{-\mu(pn-na+\beta p-p\alpha)}$$
$$\leqslant C \sum_{|\gamma|=m-1} \|D^\gamma A\|^p_{\dot\Lambda_\beta} N^{p\beta+(n-\alpha-m+1)p-na} N^{(m-1)p} \|f(N\cdot)\|^p_{L^p(I)} 2^{-\mu(pn-na+\beta p-p\alpha)}.$$

For III, as $\omega\in\Omega_\mu$ and by the definition of Ω_μ, there is

$$|D^\gamma \tilde A(Ny)| = |D^\gamma A(Ny) - m_\omega(D^\gamma A(N\cdot))|$$
$$\leqslant C|\omega|^{\beta/n}\|D^\gamma A(N\cdot)\|_{\dot\Lambda_\beta} \leqslant C2^{-k\beta}N^\beta N^{m-1}\|D^\gamma A\|_{\dot\Lambda_\beta}.$$

Thus, we may choose $\Psi(x,y)$ a smooth function approximating to $\frac{1}{|x-y|^{\alpha+m-1}}$ on $F(I)\times I$ and we obtain from (2.2) that

$$III \leqslant C2^{-\mu n}N^{p(n-\alpha-m+1)}2^{p(\mu\alpha-\mu n)}2^{-\mu\beta p}N^{\beta p}\|D^\gamma A\|^p_{\dot\Lambda_\beta}(2^{-\mu}N)^{-\frac{an}{p}p}\|f(2^{-\mu}N\cdot+x_\omega)\|_{L^p(I)}$$
$$\leqslant C \sum_{|\gamma|=m-1} \|D^\gamma A\|^p_{\dot\Lambda_\beta} N^{p(n-\alpha-m+1)+\beta p-na} N^{(m-1)p} 2^{\mu p(\alpha-n-\beta+\frac{an}{p})} \|f(N\cdot)\|^p_{L^p(I)}.$$

Similarly, for the case $1<p\leqslant 2$, there is

$$I+II+III \leqslant C \sum_{|\gamma|=m-1} \|D^\gamma A\|^p_{\dot\Lambda_\beta} N^{p(n+\beta-\alpha-m+1-\frac{an}{p'})} N^{(m-1)p}$$
$$\times 2^{\mu p(\alpha-n-\beta+\frac{an}{p'})} \|f(N\cdot)\|^p_{L^p(I)}.$$

By the above estimates and the condition $p_0<p<p_0'$ with $1<p_0=\frac{an}{\beta+n-\alpha}<2$, we obtain

$$\sum_{\mu\geqslant\mu_n} \left\{ \sum_{\omega\in\Omega_\mu} \int_{F(\omega)} |E^{a,\alpha}_A(x,\omega)|^p dx \right\}^{1/p} \leqslant CN^\beta N^{m-1} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot\Lambda_\beta} \|f(N\cdot)\|_{L^p(I)}. \tag{3.4}$$

Combining (3.3)–(3.4), we finish the proof of Theorem 1.2. □

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