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INEQUALITIES RELATED TO THE ARITHMETIC, GEOMETRIC AND HARMONIC MEANS

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Abstract. Recent refinements of mean inequalities can be thought of certain ratios. In this article, we present this point of view and prove the relationships between the different ratios induced by the different refinements.

1. Introduction

Throughout the paper, the following notations will be used to denote the μ-weighted arithmetic mean (AM), geometric mean (GM) and harmonic mean (HM) for scalars and operators

$$\begin{array}{rr} a\nabla_{\mu}b=(1-\mu)a+\mu b, & A\nabla_{\mu}B=(1-\mu)A+\mu B, \\ a\#_{\mu}b=a^{1-\mu}b^{\mu}, & A\#_{\mu}B=A^{1/2}(A^{-1/2}BA^{-1/2})^{\mu}A^{1/2}, \\ a!_{\mu}b=((1-\mu)a^{-1}+\mu b^{-1})^{-1}, & A!_{\mu}B=((1-\mu)A^{-1}+\mu B^{-1})^{-1}. \end{array}$$

for $\mu\in[0,1]$, $a,b>0$, and A,B, invertible positive operators on a Hilbert space. When $\mu=\frac{1}{2}$, we omit μ in the above definitions. For example, $a\nabla b$ means $a\nabla_{\frac{1}{2}}b$.

Applying the well known Young's inequality $a^{\nu}b^{1-\nu}\leqslant \nu a+(1-\nu)b$, $0\leqslant\nu\leqslant 1$, one can easily obtain the following AM-GM-HM inequalities

$$a!_{\mu}b\leqslant a\#_{\mu}b\leqslant a\nabla_{\mu}b,$$
$$A!_{\mu}B\leqslant A\#_{\mu}B\leqslant A\nabla_{\mu}B,$$

where for two self-adjoint operators A and B, the notation $A\leqslant B$ $(A<B)$ means that $B-A$ is a positive (invertible positive) operator.

Investigating the relation between these different means has taken the attention of several authors due to its applications in operator theory. In this article we discuss the different relations among these quantities, in view of the inequalities appearing in the

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literature. The following inequalities, proved in [3] and its references, motivates our study. For $a,b>0$, $0\leqslant\mu\leqslant 1$ we have

$$K(h,2)^r a\#_\mu b \leqslant a\nabla_\mu b \leqslant K(h,2)^R a\#_\mu b, \tag{1.1}$$

$$a!_\mu b + 2r(a\nabla b - a!b) \leqslant a\nabla_\mu b \leqslant a!_\mu b + 2R(a\nabla b - a!b), \tag{1.2}$$

$$K(h,2)^r a!_\mu b \leqslant a\#_\mu b \leqslant K(h,2)^R a!_\mu b \tag{1.3}$$

$$K(h,2)^{2r} a!_\mu b \leqslant a\nabla_\mu b \leqslant K(h,2) a!_\mu b, \tag{1.4}$$

where $r=\min\{\mu,1-\mu\}$, $R=\max\{\mu,1-\mu\}$, $h=\frac{b}{a}$ and $K(t,2)$ is the Kantorovich constant defined by $K(t,2)=\frac{(t+1)^2}{4t}$ for $t>0$. Earlier, it was proved in [1, 2] that

$$a\#_\mu b + 2r(a\nabla b - a\#b) \leqslant a\nabla_\mu b \leqslant a\#_\mu b + 2R(a\nabla b - a\#b). \tag{1.5}$$

We remark that the first inequalities in (1.3) and (1.4) are not shown in [3], but they can be easily proved as follows:

- Since $K(h,2)^r a\#_{1-\mu} b \leqslant a\nabla_{1-\mu} b$ by (1.1) and $(a\nabla_{1-\mu} b)(a!_\mu b) = (a\#_\mu b)(a\#_{1-\mu} b)$, we have
$$K(h,2)^r a!_\mu b = K(h,2)^r \frac{a\#_{1-\mu} b}{a\nabla_{1-\mu} b}\cdot a\#_\mu b \leqslant a\#_\mu b$$
proving the first inequality of (1.3).

- By the first inequalities of (1.1) and (1.3), we have $a\nabla_\mu b \geqslant K(h,2)^r a\#_\mu b \geqslant K(h,2)^{2r} a!_\mu b$.

Inequalities (1.1), (1.2), (1.3), (1.4) and (1.5) can be written as

$$\begin{aligned}
2r &\leqslant \frac{\ln a\nabla_\mu b - \ln a\#_\mu b}{\ln a\nabla b - \ln a\#b} \leqslant 2R,\\
2r &\leqslant \frac{a\nabla_\mu b - a!_\mu b}{a\nabla b - a!b} \leqslant 2R,\\
2r &\leqslant \frac{\ln a\#_\mu b - \ln a!_\mu b}{\ln a\#b - \ln a!b} \leqslant 2R,\\
2r &\leqslant \frac{\ln a\nabla_\mu b - \ln a!_\mu b}{\ln a\nabla b - \ln a!b} \leqslant 1,\\
2r &\leqslant \frac{a\nabla_\mu b - a\#_\mu b}{a\nabla b - a\#b} \leqslant 2R,
\end{aligned} \tag{1.6}$$

respectively. Our main goal in this paper is to present the different relations among the above ratios. For this purpose, we define the following functions for given $a,b>0$ and

$\mu \in [0,1]$,

$$\begin{aligned}
LAG_{\mu}(a,b) &= (\ln a\nabla_{\mu}b - \ln a\#_{\mu}b)/(\ln a\nabla b - \ln a\#b),\\
AH_{\mu}(a,b) &= (a\nabla_{\mu}b - a!_{\mu}b)/(a\nabla b - a!b),\\
LGH_{\mu}(a,b) &= (\ln a\#_{\mu}b - \ln a!_{\mu}b)/(\ln a\#b - \ln a!b),\\
LAH_{\mu}(a,b) &= (\ln a\nabla_{\mu}b - \ln a!_{\mu}b)/(\ln a\nabla b - \ln a!b),\\
AG_{\mu}(a,b) &= (a\nabla_{\mu}b - a\#_{\mu}b)/(a\nabla b - a\#b),\\
GH_{\mu}(a,b) &= (a\#_{\mu}b - a!_{\mu}b)/(a\#b - a!b)
\end{aligned}$$

with the convention that all functions are equal to 1 when $a = b$.

In the sequel, we use the notations $r = \min\{\mu, 1-\mu\}$ and $R = \max\{\mu, 1-\mu\}$ for $0 \leqslant \mu \leqslant 1$. Moreover, the expression $\alpha \leqslant \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} \leqslant \gamma$ will be used to mean that both inequalities $\alpha \leqslant \beta_1 \leqslant \gamma$ and $\alpha \leqslant \beta_2 \leqslant \gamma$ hold.

2. Main results

2.1. The discussion of LAG_{μ} and AG_{μ}

In this part of the paper, we present several relations between the geometric mean and arithmetic mean, that lead to the main relation between LAG_{μ} and AG_{μ}, presented in Theorem 2.3.

LEMMA 2.1. *Let $a \geqslant b > 0$. Then*

$$\begin{cases} a\nabla_{\mu}b \geqslant a\nabla b, \ 0 \leqslant \mu \leqslant \frac{1}{2}\\ a\nabla_{\mu}b \leqslant a\nabla b, \ \frac{1}{2} \leqslant \mu \leqslant 1 \end{cases} \tag{2.1}$$

and

$$\begin{cases} a\#_{\mu}b \geqslant a\#b, \ 0 \leqslant \mu \leqslant \frac{1}{2}\\ a\#_{\mu}b \leqslant a\#b, \ \frac{1}{2} \leqslant \mu \leqslant 1 \end{cases}. \tag{2.2}$$

The following is a refinement of the reverse Young's inequality that will help to prove our main result in this part.

PROPOSITION 2.2. *Let $a \geqslant b > 0$ and $0 \leqslant \mu \leqslant \frac{1}{2}$. Then*

$$\frac{a\nabla_{\mu}b}{a\#_{\mu}b} \leqslant \frac{a\nabla b}{a\#b}.$$

Proof. For $0 \leqslant \mu \leqslant 1$ and $c \geqslant 1$ define $f(\mu) = (1-\mu)c^{\mu} + \mu c^{\mu-1}$. Then $f'(\mu) = c^{\mu-1}g(\mu)$ where $g(\mu) = (1-\mu)c\ln c - c + 1 + \mu\ln c$. Now $g'(\mu) = (1-c)\ln c \leqslant 0$ because $c \geqslant 1$. This implies that g is decreasing on $[0, \frac{1}{2}]$ and

$$g(\mu) \geqslant g\left(\frac{1}{2}\right) = 1 - c + \frac{(c+1)\ln c}{2} := H(c), \quad c \geqslant 1.$$

Now

$$H'(c) = -1 + \frac{c+1}{2c} + \frac{1}{2}\ln c \ \text{ and } \ H''(c) = \frac{c-1}{2c^2} \geqslant 0 \ \text{ because } \ c \geqslant 1.$$

Since $H''(c) \geqslant 0$ it follows that $H'(c) \geqslant H'(1) = 0$ hence $H(c) \geqslant H(1) = 0$.

Consequently $g(\mu) \geqslant 0$, when $0 \leqslant \mu \leqslant \frac{1}{2}$, implying that $f(\mu) \leqslant f(\frac{1}{2})$. That is, when $0 \leqslant \mu \leqslant \frac{1}{2}$, we have $(1-\mu)c^{\mu} + \mu c^{\mu-1} \leqslant \frac{\sqrt{c}+1/\sqrt{c}}{2}$, for $c \geqslant 1$ and $0 \leqslant \mu \leqslant \frac{1}{2}$. Now since $a \geqslant b$, we may replace c by $\frac{a}{b}$ to get $\frac{a\nabla_{\mu}b}{a\#_{\mu}b} \leqslant \frac{a\nabla b}{a\#b}$ for $0 \leqslant \mu \leqslant \frac{1}{2}$. □

Inequality $\frac{a\nabla_{\mu}b}{a\#_{\mu}b} \leqslant \frac{a\nabla b}{a\#b}$ for $0 \leqslant \mu \leqslant \frac{1}{2}$ is a refinement of $\frac{a\nabla_{\mu}b}{a\#_{\mu}b} \leqslant K(\frac{a}{b},2)^R$ because for these values of μ we have $\frac{a\nabla b}{a\#b} \leqslant K(\frac{a}{b},2)^R$, where $R = \max\{\mu, 1-\mu\} = 1-\mu$.

When $0 \leqslant \mu \leqslant \frac{1}{2}$, we have seen in Proposition 2.2 that $\frac{a\nabla_{\mu}b}{a\#_{\mu}b} \leqslant \frac{a\nabla b}{a\#b}$. This or the reverse inequality is not valid for $\frac{1}{2} \leqslant \mu \leqslant 1$, in general. The following theorem is the main result in this part of the paper. In the following proof, the definition of the functions f and g at $x = 1$ is understood to be the limit as $x \to 1$.

THEOREM 2.3. *Let $a, b > 0$ and $0 \leqslant \mu \leqslant 1$. Then*

$$\min\left\{\frac{a\#b}{a\#_{\mu}b}, \frac{a\nabla b}{a\nabla_{\mu}b}\right\} AG_{\mu}(a,b) \leqslant LAG_{\mu}(a,b) \leqslant \max\left\{\frac{a\#b}{a\#_{\mu}b}, \frac{a\nabla b}{a\nabla_{\mu}b}\right\} AG_{\mu}(a,b).$$

Proof. For $0 \leqslant \mu \leqslant 1$, we treat two cases. If $\frac{a\nabla b}{a\#b} \leqslant \frac{a\nabla_{\mu}b}{a\#_{\mu}b}$, then the fact that $g(x) = \frac{\ln x}{x-1}$ is decreasing on $(0,\infty)$ implies $g\left(\frac{a\nabla_{\mu}b}{a\#_{\mu}b}\right) \leqslant g\left(\frac{a\nabla b}{a\#b}\right)$, which is equivalent to saying $LAG_{\mu}(a,b) \leqslant \frac{a\#b}{a\#_{\mu}b} AG_{\mu}(a,b)$. On the other hand, if $\frac{a\nabla b}{a\#b} \geqslant \frac{a\nabla_{\mu}b}{a\#_{\mu}b}$ then using the fact that the function $f(x) = \frac{x\ln x}{x-1}$ is increasing on $(0,\infty)$ implies $LAG_{\mu}(a,b) \leqslant \frac{a\nabla b}{a\nabla_{\mu}b} AG_{\mu}(a,b)$. This proves the second desired inequality.

Now for the first inequality, if $\frac{a\nabla b}{a\#b} \leqslant \frac{a\nabla_{\mu}b}{a\#_{\mu}b}$ use the fact that $f(x) = \frac{x\ln x}{x-1}$ is increasing and if $\frac{a\nabla b}{a\#b} \geqslant \frac{a\nabla_{\mu}b}{a\#_{\mu}b}$ apply the fact that $g(x) = \frac{\ln x}{x-1}$ is decreasing. □

In view of Proposition 2.2, when $a > b$ we have $\frac{a\#b}{a\#_{\mu}b} \leqslant \frac{a\nabla b}{a\nabla_{\mu}b}$, for $0 \leqslant \mu \leqslant \frac{1}{2}$ hence the inequality of theorem 2.3 maybe restated as

$$\frac{a\#b}{a\#_{\mu}b} AG_{\mu}(a,b) \leqslant LAG_{\mu}(a,b) \leqslant \frac{a\nabla b}{a\nabla_{\mu}b} AG_{\mu}(a,b),\ 0 \leqslant \mu \leqslant \frac{1}{2}.$$

The following gives an upper bound of $LAG_{\mu}(a,b)$ in terms of a and b, and independent of μ.

PROPOSITION 2.4. *Let $a \geqslant b > 0$ and $0 \leqslant \mu \leqslant 1$ then*

$$LAG_{\mu}(a,b) \leqslant \frac{a\nabla b}{a\#b}.$$

Proof. Without loss of generality, let $b=1$ and $a \geqslant 1$ and define

$$f(\mu)=\ln(a\nabla_\mu 1)-\ln(a\#_\mu 1).$$

Then it can be easily seen that f attains its maximum on $[0,1]$ at $\mu_0=\frac{1-a+a\ln a}{(a-1)\ln a}$. Thus for $\mu\in[0,1]$ we have

$$f(\mu)\leqslant f\left(\frac{1-a+a\ln a}{(a-1)\ln a}\right)=\frac{\ln a}{a-1}+\ln(a-1)-\ln\ln a-1.$$

We assert that

$$\frac{\ln a}{a-1}+\ln(a-1)-\ln\ln a-1\leqslant\frac{a\nabla 1}{a\# 1}\left(\ln a\nabla 1-\ln a\# 1\right). \tag{2.3}$$

This is equivalent to proving that

$$g(a)=\frac{\ln a}{a-1}-\ln\left(\frac{\ln a}{a-1}\right)-1-\frac{a+1}{2\sqrt{a}}\ln\left(\frac{a+1}{2\sqrt{a}}\right)\leqslant 0.$$

Observe that

$$g'(a)=\left(\frac{\ln a}{(a-1)^2}-\frac{1}{a(a-1)}\right)\left(\frac{a-1}{\ln a}-1\right)-\left(1+\ln\frac{a+1}{2\sqrt{a}}\right)\frac{a-1}{4a\sqrt{a}}.$$

Notice that

$$\begin{aligned}
&g'(a)\leqslant 0\\
\iff&\left(\frac{a\ln a}{a-1}-1\right)\left(\frac{a-1}{\ln a}-1\right)\leqslant\left(1+\ln\frac{a+1}{2\sqrt{a}}\right)\frac{(a-1)^2}{4\sqrt{a}}\\
\iff&a+1-\left(\frac{a\ln a}{a-1}+\frac{a-1}{\ln a}\right)\leqslant\left(1+\ln\frac{a+1}{2\sqrt{a}}\right)\frac{(a-1)^2}{4\sqrt{a}}.
\end{aligned}$$

Now one can easily show that when $a\geqslant 1$ we have

$$\frac{a\ln a}{a-1}+\frac{a-1}{\ln a}\geqslant 2\sqrt{a}.$$

Therefore $g'(a)\leqslant 0$ if

$$a+1-2\sqrt{a}\leqslant\left(1+\ln\frac{a+1}{2\sqrt{a}}\right)\frac{(a-1)^2}{4\sqrt{a}}$$

which is equivalent to

$$\frac{4\sqrt{a}}{(\sqrt{a}+1)^2}\leqslant 1+\ln\frac{a+1}{2\sqrt{a}},$$

which holds trivially because the left side is at most 1 and the right side is at least 1, when $a\geqslant 1$. This proves that $g'(a)\leqslant 0$ when $a\geqslant 1$ and hence, $g(a)\leqslant g(1)=0$.

This proves (2.3), implying $f(\mu)\leqslant\frac{a\nabla 1}{a\# 1}\left(\ln a\nabla 1-\ln a\# 1\right)$, or equivalently $LAG_\mu(a,b)\leqslant\frac{a\nabla b}{a\# b}$. □

2.2. The general discussion

In this part of the paper, we present the main result of our work, where several relations among the different ratios have been obtained in the following theorem.

THEOREM 2.5. *Let* $a,b>0$ *and* $\mu\in[0,1]$. *Then*

1. *If* $(1-2\mu)(a-b)\geqslant 0$, *we have*

$$2r\leqslant LAG_\mu(a,b)\leqslant\begin{bmatrix}LAH_\mu(a,b)\leqslant LGH_\mu(a,b)\\ AG_\mu(a,b)\end{bmatrix}\leqslant AH_\mu(a,b)\leqslant\begin{bmatrix}GH_\mu(a,b)\\ 2R\end{bmatrix}. \quad (2.4)$$

2. *If* $(1-2\mu)(a-b)\leqslant 0$, *we have*

$$2R\geqslant LAG_\mu(a,b)\geqslant\begin{bmatrix}LAH_\mu(a,b)\geqslant LGH_\mu(a,b)\\ AG_\mu(a,b)\end{bmatrix}\geqslant AH_\mu(a,b)\geqslant\begin{bmatrix}GH_\mu(a,b)\\ 2r\end{bmatrix}. \quad (2.5)$$

The above inequalities can be equivalently stated in terms of r and R as follows:

1. If $a\geqslant b$, then

$$2r\leqslant LAG_r(a,b)\leqslant\begin{bmatrix}LAH_r(a,b)\leqslant LGH_r(a,b)\\ AG_r(a,b)\end{bmatrix}\leqslant AH_r(a,b)\leqslant\begin{bmatrix}GH_r(a,b)\\ 2R\end{bmatrix},$$
$$2R\geqslant LAG_R(a,b)\geqslant\begin{bmatrix}LAH_R(a,b)\geqslant LGH_R(a,b)\\ AG_R(a,b)\end{bmatrix}\geqslant AH_R(a,b)\geqslant\begin{bmatrix}GH_R(a,b)\\ 2r\end{bmatrix}.$$

2. If $a\leqslant b$, then

$$2r\leqslant LAG_R(a,b)\leqslant\begin{bmatrix}LAH_R(a,b)\leqslant LGH_R(a,b)\\ AG_R(a,b)\end{bmatrix}\leqslant AH_R(a,b)\leqslant\begin{bmatrix}GH_R(a,b)\\ 2R\end{bmatrix},$$
$$2R\geqslant LAG_r(a,b)\geqslant\begin{bmatrix}LAH_r(a,b)\geqslant LGH_r(a,b)\\ AG_r(a,b)\end{bmatrix}\geqslant AH_r(a,b)\geqslant\begin{bmatrix}GH_r(a,b)\\ 2r\end{bmatrix}.$$

Considering symmetric properties, it is enough to show (2.4) for $\mu\leqslant\frac{1}{2}$ and (2.5) for $\mu\geqslant\frac{1}{2}$, assuming $a\geqslant b$. Moreover, since they are obvious for the case $a=b$, we assume $a\neq b$ to ensure that all quantities are well defined. We also note that the inequalities $2r\leqslant LAG_\mu(a,b)$ and $AH_\mu(a,b)\leqslant 2R$ in (2.4) and $2R\geqslant LAG_\mu(a,b)$ and $AH_\mu(a,b)\geqslant 2r$ in (2.5) have been already shown in (1.6). The following lemma will be needed to prove our main result.

LEMMA 2.6. *Let* $a,b>0$ *and* $\mu\in[0,1]$.

1.

$$(a\nabla_\mu b)(a!_{1-\mu}b)=(a\#_\mu b)(a\#_{1-\mu}b). \quad (2.6)$$

2. *For* $\delta=\frac{a\nabla b-a\# b}{a\nabla b-a!b}$, *we have*

$$(a\nabla_\mu b)\nabla_\delta(a!_\mu b)\leqslant(a!_\mu b)!_\delta(a\nabla_\mu b). \quad (2.7)$$

3. For any $\nu \in [0,1]$,

$$(a\nabla_\mu b)\nabla_\nu(a!_\mu b) \leqslant a\#_\mu b,\ \forall \mu \leqslant \frac{1}{2} \Leftrightarrow (a!_\mu b)!_\nu(a\nabla_\mu b) \geqslant a\#_\mu b,\ \forall \mu \geqslant \frac{1}{2}. \tag{2.8}$$

Proof. The first statement is clear. For $c \geqslant d > 0$ and $\mu \in [0,1]$, it is easy to show that

$$c\nabla_\mu d \leqslant d!_\mu c \iff \mu \geqslant \frac{\sqrt{c}}{\sqrt{c}+\sqrt{d}}.$$

Thus (2.7) will follow by proving

$$\delta \geqslant \frac{\sqrt{a\nabla_\mu b}}{\sqrt{a\nabla_\mu b}+\sqrt{a!_\mu b}}.$$

Since δ can be written as $\frac{a\nabla b}{a\nabla b + a\# b}$, the above inequality is equivalent to

$$\left(\frac{a\nabla b}{a\# b}\right)^2 \geqslant \frac{a\nabla_\mu b}{a!_\mu b}.$$

Since $\left(\frac{a\nabla b}{a\# b}\right)^2 = \frac{a\nabla b}{a!b}$, the above follows directly by inequality $LAH_\mu(a,b) \leqslant 1$ in (1.6). The following argument proves (2.8):

$$\begin{aligned}
&(a\nabla_\mu b)\nabla_\nu(a!_\mu b) \leqslant a\#_\mu b \quad \text{for } \mu \leqslant \frac{1}{2}\\
\iff &\left(\frac{a\nabla_\mu b}{a\#_\mu b}\right)\nabla_\nu\left(\frac{a!_\mu b}{a\#_\mu b}\right) \leqslant 1 \quad \text{for } \mu \leqslant \frac{1}{2}\\
\iff &\left(\frac{a\#_{1-\mu} b}{a!_{1-\mu} b}\right)\nabla_\nu\left(\frac{a\#_{1-\mu} b}{a\nabla_{1-\mu} b}\right) \leqslant 1 \quad \text{for } \mu \leqslant \frac{1}{2}\\
\iff &\left(\frac{a\#_\mu b}{a!_\mu b}\right)\nabla_\nu\left(\frac{a\#_\mu b}{a\nabla_\mu b}\right) \leqslant 1 \quad \text{for } \mu \geqslant \frac{1}{2}\\
\iff &a\#_\mu b \leqslant (a!_\mu b)!_\nu(a\nabla_\mu b) \leqslant 1 \quad \text{for } \mu \geqslant \frac{1}{2}. \quad \square
\end{aligned}$$

Now we start proving the main theorem by proving the different inequalities the theorem is implied.

PROPOSITION 2.7. *Let* $a \geqslant b > 0$ *and* $\mu \in [0,1]$. *Then, we have*

$$\begin{aligned}
LAG_\mu(a,b) \leqslant LAH_\mu(a,b) \leqslant LGH_\mu(a,b) \quad &\textit{for } \mu \leqslant \frac{1}{2},\\
LAG_\mu(a,b) \geqslant LAH_\mu(a,b) \geqslant LGH_\mu(a,b) \quad &\textit{for } \mu \geqslant \frac{1}{2}.
\end{aligned}$$

Proof. By (2.6) of Lemma 2.6, the two results above are equivalent. Moreover, since

$$LAG_\mu(a,b) \leqslant LAH_\mu(a,b) \iff LAH_\mu(a,b) \leqslant LGH_\mu(a,b),$$

it suffices to show $LAG_\mu(a,b) \leqslant LAH_\mu(a,b)$ for $\mu \leqslant \frac{1}{2}$. Since $\frac{a\nabla b}{a!b} = \left(\frac{a\nabla b}{a\#b}\right)^2$, we have

$$\begin{aligned} & LAG_\mu(a,b) \leqslant LAH_\mu(a,b) \\ \iff & (a\nabla_\mu b)(a!_\mu b) \leqslant (a\#_\mu b)^2 \\ \iff & f(x) \geqslant 0 \quad \text{for } x \geqslant 1, \end{aligned}$$

where $f(x) = x^{1-2\mu}(\mu x + 1 - \mu) - [(1-\mu)x + \mu]$. Letting $g(x) = x^{2\mu} f'(x)$, we have

$$\begin{aligned} g(x) &= (1-2\mu)(\mu x + 1 - \mu) + \mu x - (1-\mu)x^{2\mu}, \\ g'(x) &= 2\mu(1-\mu)(1 - x^{2\mu-1}). \end{aligned}$$

Since $\mu \leqslant \frac{1}{2}$ and $x \geqslant 1$, $g'(x) \geqslant 0$. Moreover, since $g(1) = f(1) = 0$, $f(x) \geqslant 0$ for all $x \geqslant 1$. □

PROPOSITION 2.8. *Let $a \geqslant b > 0$ and $\mu \in [0,1]$. Then*

$$\begin{aligned} LGH_\mu(a,b) &\leqslant AH_\mu(a,b) \text{ for } \mu \leqslant \frac{1}{2}, \\ LGH_\mu(a,b) &\geqslant AH_\mu(a,b) \text{ for } \mu \geqslant \frac{1}{2}. \end{aligned}$$

Proof. Assume $b = 1$. Denoting a by x, we define $f(\mu)$ by

$$\begin{aligned} f(\mu) &= (x\nabla_\mu 1 - x!_\mu 1)(\ln(x\#1) - \ln(x!1)) - \left(\ln(x\#_\mu 1) - \ln(x!_\mu 1)\right)(x\nabla 1 - x!1) \\ &= \left((1-\mu)x + \mu - \frac{x}{\mu x + 1 - \mu}\right)\ln\frac{x+1}{2\sqrt{x}} - \left((1-\mu)\ln x - \ln\frac{x}{\mu x + 1 - \mu}\right)\frac{(x-1)^2}{2(x+1)}. \end{aligned}$$

Then we have

$$\begin{aligned} f'(\mu) &= \left(\frac{x}{(\mu x + 1 - \mu)^2} - 1\right)(x-1)\ln\frac{x+1}{2\sqrt{x}} - \left(\frac{x-1}{\mu x + 1 - \mu} - \ln x\right)\frac{(x-1)^2}{2(x+1)}, \\ f''(\mu) &= \frac{(x-1)^5}{2(x+1)(\mu x + 1 - \mu)^3}[\mu - u(x)], \end{aligned}$$

where

$$u(x) = \frac{4x(x+1)}{(x-1)^3}\ln\frac{x+1}{2\sqrt{x}} - \frac{1}{x-1}.$$

Let $v(x) = \frac{3(x-1)^2}{x^2+4x+1} - 4\ln\frac{x+1}{2\sqrt{x}}$. Then we can show that

$$u'(x) = \frac{x^2+4x+1}{(x-1)^4}v(x),$$

$$\begin{aligned} v'(x) &= \frac{2(x-1)}{x(x+1)(x^2+4x+1)^2}\left[9x(x+1)^2-(x^2+4x+1)^2\right] \\ &= \frac{-2(x-1)}{x(x+1)(x^2+4x+1)^2}(\sqrt{x}-1)^2(x-\sqrt{x}+1) \leqslant 0. \end{aligned}$$

Since $v(1)=0$, $u(x)$ is a decreasing function for $x \geqslant 1$. Moreover, since

$$\begin{aligned} \lim_{x\to 1} u(x) &= 8\lim_{x\to 1}\frac{1}{(x-1)^3}\left[\ln\frac{x+1}{2\sqrt{x}}-\frac{(x-1)^2}{4x(x+1)}\right] \\ &= 8\lim_{x\to 1}\frac{2x+1}{12x^2(x+1)^2} = \frac{1}{2}, \\ \lim_{x\to\infty} u(x) &= 0, \end{aligned}$$

we have $0 < u(x) \leqslant 1$ for $x \geqslant 1$.

Thus there exists $\mu_x \in [0, \frac{1}{2}]$ such that f' is decreasing on $[0, \mu_x]$ and increasing on $[\mu_x, 1]$. Now we show that $f'(0) \geqslant 0$, $f'(\frac{1}{2}) \leqslant 0$, and $f'(1) \geqslant 0$. Since

$$\begin{aligned} f'(0) &= (x-1)^2\ln\frac{x+1}{2\sqrt{x}}-(x-1-\ln x)\frac{(x-1)^2}{2(x+1)}, \\ f'(\frac{1}{2}) &= \frac{-(x-1)^3}{(x+1)^2}\ln\frac{x+1}{2\sqrt{x}}+\left(\ln x-\frac{2(x-1)}{x+1}\right)\frac{(x-1)^2}{2(x+1)}, \\ f'(1) &= \frac{-(x-1)^2}{x}\ln\frac{x+1}{2\sqrt{x}}+\left(\ln x-\frac{x-1}{x}\right)\frac{(x-1)^2}{2(x+1)}, \end{aligned}$$

we have

$$\begin{aligned} f'(0) \geqslant 0 &\iff \alpha(x) \geqslant 0, \\ f'\left(\frac{1}{2}\right) \leqslant 0 &\iff \beta(x) \geqslant 0, \\ f'(1) \geqslant 0 &\iff \gamma(x) \geqslant 0, \end{aligned}$$

where

$$\begin{aligned} \alpha(x) &= 2(x+1)\ln\frac{x+1}{2\sqrt{x}}-x+1+\ln x, \\ \beta(x) &= 2(x-1)\ln\frac{x+1}{2\sqrt{x}}-(x+1)\ln x+2(x-1), \\ \gamma(x) &= x\ln x-x+1-2(x+1)\ln\frac{x+1}{2\sqrt{x}}. \end{aligned}$$

Since

$$\begin{aligned} \alpha'(x) &= 2\ln\frac{x+1}{2\sqrt{x}} \geqslant 0, \\ \beta'(x) &= 2\ln\frac{x+1}{2\sqrt{x}}-\ln x+\frac{2(x-1)}{x+1}, \end{aligned}$$

$$\beta''(x) = \frac{2(x-1)}{x(x+1)^2} \geqslant 0,$$
$$\gamma'(x) = \ln x - 2\ln\frac{x+1}{2\sqrt{x}} + \frac{1}{x} - 1$$
$$\gamma''(x) = \frac{x-1}{x^2(x+1)} \geqslant 0$$

and $\alpha(1) = \beta(1) = \beta'(1) = \gamma'(1) = \gamma(1) = 0$, $\alpha(x), \beta(x), \gamma(x) \geqslant 0$ for all $x \geqslant 1$.

Finally, since $f(0) = f(\frac{1}{2}) = f(1) = 0$, we conclude that $f(\mu) \geqslant 0$ for $\mu \leqslant \frac{1}{2}$ and $f(\mu) \leqslant 0$ for $\mu \geqslant \frac{1}{2}$. □

For the next proposition, see Theorem 2.3.

PROPOSITION 2.9. *Let $a \geqslant b > 0$ and $\mu \in [0,1]$. Then*

$$LAG_\mu(a,b) \leqslant AG_\mu(a,b) \text{ for } \mu \leqslant \frac{1}{2},$$
$$LAG_\mu(a,b) \geqslant AG_\mu(a,b) \text{ for } \mu \geqslant \frac{1}{2}.$$

Observe that Theorem 2.3 gives a refinement and a reverse of the above proposition. This can be seen for $0 \leqslant \mu \leqslant \frac{1}{2}$ because, for these μ's, we have $\frac{a\nabla b}{a\nabla_\mu b} \leqslant 1$ and $\frac{a\#b}{a\#_\mu b} \leqslant 1$ by Lemma 2.1. A similar discussion holds for $\frac{1}{2} \leqslant \mu \leqslant 1$.

PROPOSITION 2.10. *Let $a \geqslant b > 0$ and $\mu \in [0,1]$. Then, we have*

$$AG_\mu(a,b) \leqslant AH_\mu(a,b) \leqslant GH_\mu(a,b) \quad \text{for } \mu \leqslant \frac{1}{2},$$
$$AG_\mu(a,b) \geqslant AH_\mu(a,b) \geqslant GH_\mu(a,b) \quad \text{for } \mu \geqslant \frac{1}{2}.$$

Proof. Since both $AG_\mu(a,b) \leqslant AH_\mu(a,b)$ and $AH_\mu(a,b) \leqslant GH_\mu(a,b)$ are equivalent to

$$(a\#b - a!b)a\nabla_\mu b + (a\nabla b - a\#b)a!_\mu b \leqslant (a\nabla b - a!b)a\#_\mu b,$$

it suffices to show

$$AG_\mu(a,b) \leqslant AH_\mu(a,b) \quad \text{for } \mu \leqslant \frac{1}{2},$$
$$AG_\mu(a,b) \geqslant AH_\mu(a,b) \quad \text{for } \mu \geqslant \frac{1}{2}.$$

Let $\delta = \frac{a\nabla b - a\#b}{a\nabla b - a!b} \in [0,1]$. Then we have

$$AG_\mu(a,b) \leqslant AH_\mu(a,b) \quad \text{for } \mu \leqslant \frac{1}{2}$$
$$\Longleftrightarrow \ (a\nabla_\mu b)\nabla_\delta(a!_\mu b) \leqslant a\#_\mu b \quad \text{for } \mu \leqslant \frac{1}{2}$$

$$\iff a\#_\mu b \leqslant (a!_\mu b)!_\delta(a\nabla_\mu b) \quad \text{for } \mu \geqslant \frac{1}{2},$$

where the last equivalence follows from (2.8) in Lemma 2.6. Similarly, we have

$$\begin{aligned} & AG_\mu(a,b) \geqslant AH_\mu(a,b) \quad \text{for } \mu \geqslant \frac{1}{2} \\ \iff & (a\nabla_\mu b)\nabla_\delta(a!_\mu b) \geqslant a\#_\mu b \quad \text{for } \mu \geqslant \frac{1}{2}. \end{aligned}$$

Suppose that $AG_\mu(a,b) \geqslant AH_\mu(a,b)$ holds for $\mu \geqslant \frac{1}{2}$, that is, $a\#_\mu b \leqslant (a\nabla_\mu b)\nabla_\delta(a!_\mu b)$ for $\mu \geqslant \frac{1}{2}$. Then $a\#_\mu b \leqslant (a!_\mu b)!_\delta(a\nabla_\mu b)$ for $\mu \geqslant \frac{1}{2}$ by (2.7) of Lemma 2.6, which implies that $AG_\mu(a,b) \leqslant AH_\mu(a,b)$ for $\mu \leqslant \frac{1}{2}$. Therefore it suffices to prove $AG_\mu(a,b) \geqslant AH_\mu(a,b)$ for $\mu \geqslant \frac{1}{2}$.

Letting $a = x^2$ and $b = 1$, we have

$$\begin{aligned} & AG_\mu(a,b) \geqslant AH_\mu(a,b) \\ \iff & 2((1-\mu)x^2+\mu) + \frac{x(x^2+1)}{\mu x^2+1-\mu} \geqslant (x+1)^2 x^{1-2\mu} \\ \iff & f(x) \geqslant 0, \end{aligned}$$

where

$$f(x) = (\mu x^2+1-\mu)\left[2((1-\mu)x^2+\mu) - (x+1)^2 x^{1-2\mu}\right] + x(x^2+1).$$

We will show that $f(x) \geqslant 0$ for $x \geqslant 1$ and $\mu \geqslant \frac{1}{2}$. Letting $g(x) = f'(x)/(x+1)$ and $h(x) = x^{2+2\mu} g''(x)/(2\mu(1-\mu))$, a straightforward computation shows the following:

$$\begin{aligned} g(x) &= 8\mu(1-\mu)(x-1)x + 3x + 1 \\ &\quad -x^{-2\mu}\left[\mu(5-2\mu)x^3 + \mu(3-2\mu)x^2 + (1-\mu)(3-2\mu)x + (1-\mu)(1-2\mu)\right], \\ g'(x) &= 8\mu(1-\mu)(2x-1) + 3 + x^{-1-2\mu}\left[\mu(5-2\mu)(2\mu-3)x^3\right. \\ &\quad \left.-2\mu(1-\mu)(3-2\mu)x^2 + (2\mu-1)(1-\mu)((3-2\mu)x-2\mu)\right], \\ h(x) &= 8x^{2+2\mu} + (5-2\mu)(2\mu-3)x^3 + (2\mu-1)\left[(3-2\mu)x^2 - (3-2\mu)x + 1 + 2\mu\right], \\ h'(x) &= 16(1+\mu)x^{1+2\mu} + (2\mu-3)\left[3(5-2\mu)x^2 - 2(2\mu-1)x + (2\mu-1)\right], \\ h''(x) &= 16(1+\mu)(1+2\mu)x^{2\mu} + 2(2\mu-3)\left[3(5-2\mu)x - 2\mu + 1\right], \\ h'''(x) &= 32\mu(1+\mu)(1+2\mu)x^{2\mu-1} + 6(2\mu-3)(5-2\mu). \end{aligned}$$

Since

$$\begin{aligned} & h'''(x) \geqslant h'''(1) = 2(32\mu^3+36\mu^2+64\mu-45) \geqslant 2(32/8+36/4+64/2-45) = 0, \\ & h''(1) = 80(2\mu-1) \geqslant 0, \\ & h'(1) = 8(4-\mu)(2\mu-1) \geqslant 0, \\ & h(1) = 8(2\mu-1) \geqslant 0, \\ & g'(1) = g(1) = f(1) = 0, \end{aligned}$$

we conclude $f(x) \geqslant 0$. □

COROLLARY 2.11. *Let $a,b>0$ and $\mu\in[0,1]$. Then,*

1. $\left(\frac{a\nabla b}{a!b}\right)^{2r}\leqslant\frac{a\nabla_\mu b}{a!_\mu b}\leqslant\frac{a\nabla b}{a!b}$.

2. *If $(1-2\mu)(a-b)\geqslant 0$, we have* $\left(\frac{a\#_R b}{a!_R b}\right)^2\leqslant\frac{a\nabla_\mu b}{a!_\mu b}\leqslant\left(\frac{a\#_r b}{a!_r b}\right)^2$.

3. *If $(1-2\mu)(a-b)\leqslant 0$, we have* $\left(\frac{a\#_r b}{a!_r b}\right)^2\leqslant\frac{a\nabla_\mu b}{a!_\mu b}\leqslant\left(\frac{a\#_R b}{a!_R b}\right)^2$.

Proof. The first statement is simply another expression of $2r\leqslant LAH_\mu(a,b)\leqslant 1$. Since $LAG_\mu(a,b)=LGH_{1-\mu}(a,b)$ for $\mu\in[0,1]$ by (2.6) of Lemma 2.6, the relations among $LAG_\mu(a,b)$, $LAH_\mu(a,b)$ and $LGH_\mu(a,b)$ in Proposition 2.7 imply (2) and (3). □

REMARK 2.12. By (2.6) of Lemma 2.6, this corollary can be written as follows:

1. $\left(\frac{G}{H}\right)^{4r}\leqslant\frac{a\nabla_\mu b}{a!_\mu b}\leqslant\left(\frac{G}{H}\right)^2$ for any $a,b>0$ and $\mu\in[0,1]$;

2. if $(1-2\mu)(a-b)\geqslant 0$, we have $\left(\frac{a\nabla_r b}{a\#_r b}\right)^2\leqslant\frac{a\nabla_\mu b}{a!_\mu b}\leqslant\left(\frac{a\nabla_R b}{a\#_R b}\right)^2$;

3. if $(1-2\mu)(a-b)\leqslant 0$, we have $\left(\frac{a\nabla_R b}{a\#_R b}\right)^2\leqslant\frac{a\nabla_\mu b}{a!_\mu b}\leqslant\left(\frac{a\nabla_r b}{a\#_r b}\right)^2$.

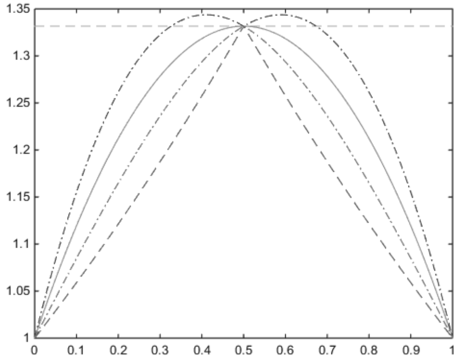

Figure 1: $\frac{a\nabla_\mu b}{a!_\mu b}$ *(Solid greenline),* $\left(\frac{a\nabla b}{a!b}\right)^{2r}$ *(Dashed blue line),* $\frac{a\nabla b}{a!b}$ *(Dashed yellow line),* $\left(\frac{a\#_R b}{a!_R b}\right)^2$ *(Dash-dot red line), and* $\left(\frac{a\#_r b}{a!_r b}\right)^2$ *(Dash-dot purple line)*

Figure 1 shows the graphs of $\frac{a\nabla_\mu b}{a!_\mu b}$, $\left(\frac{a\nabla b}{a!b}\right)^{2r}$, $\frac{a\nabla b}{a!b}$, $\left(\frac{a\#_R b}{a!_R b}\right)^2$, and $\left(\frac{a\#_r b}{a!_r b}\right)^2$ for $0\leqslant\mu\leqslant 1$, where $a=8.9092$ and $b=2.9771$.

2.3. Some related inequalities

In this section we present inequalities related to the general theme of this paper.

In the following corollary, we present a refinement and a reverse of the inequality $LGH_\mu(a,b) \geqslant GH_\mu(a,b)$ for $\frac{1}{2} \leqslant \mu \leqslant 1$.

COROLLARY 2.13. *Let* $a \geqslant b > 0$ *and* $\frac{1}{2} \leqslant \mu \leqslant 1$. *Then*

$$\frac{a!b}{a!_\mu b} GH_\mu(a,b) \leqslant LGH_\mu(a,b) \leqslant \frac{a\#b}{a\#_\mu b} GH_\mu(a,b).$$

Proof. This follows from Theorem 2.3 by observing that $\frac{a\nabla_\mu b}{a\#_\mu b} = \frac{a\#_{1-\mu} b}{a!_{1-\mu} b}$ which implies $LAG_\mu(a,b) = LGH_{1-\mu}(a,b)$, $\frac{a\nabla b}{a\nabla_\mu b} AG_\mu(a,b) = \frac{a\#b}{a\#_{1-\mu} b} GH_{1-\mu}(a,b)$ and $\frac{a\#b}{a\#_\mu b} AG_\mu(a,b) = \frac{a!b}{a!_{1-\mu} b} GH_{1-\mu}(a,b)$. □

Recall that the logarithmic mean of two positive numbers a and b is $L(a,b) = \frac{a-b}{\ln a - \ln b}$. Observe that since $a\#b \leqslant a\nabla b, a\#_\mu b \leqslant a\nabla_\mu b$ and L is a mean, we have $L(a\#b, a\#_\mu b) \leqslant L(a\nabla b, a\nabla_\mu b)$, because L is increasing in both coordinates. The following inequality is a refinement and a reverse of this relation.

COROLLARY 2.14. *Let* $a \geqslant b > 0$ *and* $0 \leqslant \mu \leqslant \frac{1}{2}$. *Then*

$$\frac{a\#b}{a\nabla b} L(a\nabla b, a\nabla_\mu b) \leqslant L(a\#b, a\#_\mu b) \leqslant \frac{a\#_\mu b}{a\nabla_\mu b} L(a\nabla b, a\nabla_\mu b).$$

Proof. For $0 \leqslant \mu \leqslant \frac{1}{2}$ we have $\frac{a\#b}{a\#_\mu b} \leqslant \frac{a\nabla b}{a\nabla_\mu b}$ by Proposition 2.2. Since $f(x) = \frac{x\ln x}{x-1}$ is increasing on $(0,\infty)$ we have $f\left(\frac{a\#b}{a\#_\mu b}\right) \leqslant f\left(\frac{a\nabla b}{a\nabla_\mu b}\right)$. Simplifying this expression gives the first inequality. On the other hand, since $g(x) = \frac{\ln x}{x-1}$ is decreasing on $(0,\infty)$, we have $g\left(\frac{a\nabla b}{a\nabla_\mu b}\right) \leqslant g\left(\frac{a\#b}{a\#_\mu b}\right)$. Simplifying this expression leads to the second inequality. □

Now noting that $\frac{a\nabla_\mu b}{a!_\mu b} \leqslant \frac{a\nabla b}{a!b}$ for $0 \leqslant \mu \leqslant 1$ and using the monotonicity of the functions f and g of the above corollary, we deduce the following inequality.

COROLLARY 2.15. *For* $a \geqslant b > 0$ *and* $0 \leqslant \mu \leqslant 1$ *we have*

$$\frac{a!b}{a!_\mu b} AH_\mu(a,b) \leqslant LAH_\mu(a,b) \leqslant \frac{a\nabla b}{a\nabla_\mu} AH_\mu(a,b), 0 \leqslant \mu \leqslant 1.$$

Note that when $0 \leqslant \mu \leqslant \frac{1}{2}$ we have $\frac{a\nabla b}{a\nabla_\mu} \leqslant 1$. Therefore the above inequality gives a refinement of the inequality $LAH_\mu(a,b) \leqslant AH_\mu(a,b)$ in Theorem 2.5 for $0 \leqslant \mu \leqslant \frac{1}{2}$. For these μ's, the above inequality gives a reverse $\frac{a!b}{a!_\mu b} AH_\mu(a,b) \leqslant LAH_\mu(a,b)$. On the other hand, when $\frac{1}{2} \leqslant \mu \leqslant 1$, the above inequality gives a reverse of $LAH_\mu(a,b) \geqslant AH_\mu(a,b)$ by introducing the factor $\frac{a\nabla b}{a\nabla_\mu} \geqslant 1$ and a refinement by introducing the factor $\frac{a!b}{a!_\mu b} \geqslant 1$.

On the other hand, the fact that $\frac{a!b}{a!_\mu b} \leqslant \frac{a\nabla b}{a\nabla_\mu b}$ for $0 \leqslant \mu \leqslant 1$ and monotonicity of the above f and g implies the following.

COROLLARY 2.16. *Let $a \geqslant b > 0$ and $0 \leqslant \mu \leqslant 1$. Then*

$$\frac{a!b}{a\nabla b} L(a\nabla b, a\nabla_\mu b) \leqslant L(a!b, a!_\mu b) \leqslant \frac{a!_\mu b}{a\nabla_\mu b} L(a\nabla b, a\nabla_\mu b).$$

Notice that this is a refinement and a reverse of the inequality $L(a!b, a!_\mu b) \leqslant L(a\nabla b, a\nabla_\mu b)$ following from the monotonicity of the mean function L.

2.4. Application to operators

In this section, we present some operator versions of the inequalities we have proved for numbers. The following is the operator versions of Corollary 2.11.

THEOREM 2.17. *Let A, B be invertible positive operators and $\mu \in [0,1]$. Then,*

1. $A\nabla_\mu B \leqslant (A\#B)(A!B)^{-1}(A!_\mu B)(A!B)^{-1}(A\#B)$.
2. *If $(1-2\mu)(A-B) \geqslant 0$, then*

$$(A\#_R B)(A!_R B)^{-1}(A!_\mu B)(A!_R B)^{-1}(A\#_R B) \leqslant A\nabla_\mu B,$$
$$A\nabla_\mu B \leqslant (A\#_r B)(A!_r B)^{-1}(A!_\mu B)(A!_r B)^{-1}(A\#_r B).$$

3. *If $(1-2\mu)(A-B) \leqslant 0$, then*

$$(A\#_r B)(A!_r B)^{-1}(A!_\mu B)(A!_r B)^{-1}(A\#_r B) \leqslant A\nabla_\mu B,$$
$$A\nabla_\mu B \leqslant (A\#_R B)(A!_R B)^{-1}(A!_\mu B)(A!_R B)^{-1}(A\#_R B).$$

Proof. We prove

$$A\nabla_\mu B \leqslant (A\#_r B)(A!_r B)^{-1}(A!_\mu B)(A!_r B)^{-1}(A\#_r B)$$

for $A \geqslant B$ and $\mu \leqslant \frac{1}{2}$. The other inequalities also hold by the same argument. Letting $a = x$ and $b = 1$, the inequality $\frac{a\nabla_\mu b}{a!_\mu b} \leqslant \left(\frac{a\#_r b}{a!_r b}\right)^2$ in (2) of Corollary 2.11 can be written by

$$(1-\mu)x + \mu \leqslant x^{1-r}\left((1-r)x^{-1}+r\right)\left((1-\mu)x^{-1}+\mu\right)^{-1}\left((1-r)x^{-1}+r\right)x^{1-r}.$$

Thus for any $X \geqslant I$, we have

$$(1-\mu)X + \mu \leqslant X^{1-r}\left((1-r)X^{-1}+r\right)\left((1-\mu)X^{-1}+\mu\right)^{-1}\left((1-r)X^{-1}+r\right)X^{1-r}$$

by the operator monotonicity of continuous functions. Replacing X by $B^{-1/2}AB^{-1/2}$, we have

$$A\nabla_\mu B \leqslant (B\#_{1-r}A)(A!_rB)^{-1}(A!_\mu B)(A!_rB)^{-1}(B\#_{1-r}A).$$

Since $B\#_{1-r}A = A\#_rB$, we have the desire inequality. □

From the relationship between AG_μ and AH_μ in Proposition 2.10, we have the following result: if $(1-2\mu)(a-b) \geqslant 0$, then

$$a\#_\mu b \leqslant \frac{a\#b}{a\nabla b + a\#b} a\nabla_\mu b + \frac{a\nabla b}{a\nabla b + a\#b} a!_\mu b \tag{2.9}$$

and if $(1-2\mu)(a-b) \leqslant 0$, then

$$a\#_\mu b \geqslant \frac{a\#b}{a\nabla b + a\#b} a\nabla_\mu b + \frac{a\nabla b}{a\nabla b + a\#b} a!_\mu b.$$

The following is their operator version.

THEOREM 2.18. *Let A,B be invertible positive operators and $\mu \in [0,1]$. Then,*

1. *if $(1-2\mu)(A-B) \geqslant 0$, then we have*

$$\begin{aligned} A\#_\mu B \leqslant\ & 2(A\#_{\frac{3}{4}}B)(A\#B+B)^{-1}(A\nabla_\mu B)(A\#B+B)^{-1}(A\#_{\frac{3}{4}}B) \\ & + ((A+B)\#B)(A\#B+B)^{-1}(A!_\mu B)(A\#B+B)^{-1}((A+B)\#B). \end{aligned}$$

2. *if $(1-2\mu)(A-B) \leqslant 0$, then*

$$\begin{aligned} A\#_\mu B \geqslant\ & 2(A\#_{\frac{3}{4}}B)(A\#B+B)^{-1}(A\nabla_\mu B)(A\#B+B)^{-1}(A\#_{\frac{3}{4}}B) \\ & + ((A+B)\#B)(A\#B+B)^{-1}(A!_\mu B)(A\#B+B)^{-1}((A+B)\#B). \end{aligned}$$

Proof. We prove the first inequality for the case that $\mu \leqslant \frac{1}{2}$ and $A \geqslant B$. The other cases follow by the same argument. Letting $a = x$ and $b = 1$ in (2.9), we have

$$\begin{aligned} x^{1-\mu} \leqslant\ & 2x^{1/4}(\sqrt{x}+1)^{-1}((1-\mu)x+\mu)(\sqrt{x}+1)^{-1}x^{1/4} \\ & + \sqrt{x+1}(\sqrt{x}+1)^{-1}\left((1-\mu)x^{-1}+\mu\right)^{-1}(\sqrt{x}+1)^{-1}\sqrt{x+1}. \end{aligned}$$

Thus replacing x by $B^{-1/2}AB^{-1/2}$, we have

$$\begin{aligned} B\#_{1-\mu}A \leqslant\ & 2(B\#_{\frac{1}{4}}A)(B\#A+B)^{-1}(A\nabla_\mu B)(B\#A+B)^{-1}(B\#_{\frac{1}{4}}A) \\ & + (B\#(A+B))(B\#A+B)^{-1}(A!_\mu B)(B\#A+B)^{-1}(B\#(A+B)). \end{aligned}$$

The desired inequality follows by the relationship $A\#_\mu B = B\#_{1-\mu}A$. □

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