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SOME REFINEMENTS OF OPERATOR INEQUALITIES FOR POSITIVE LINEAR MAPS

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Abstract. In this paper, we refine some operator inequalities as follows: Let A, B be positive operators on a Hilbert space with $0 < m \leqslant A \leqslant m' < M' \leqslant B \leqslant M$. Then for every positive unital linear map Φ and $p \geqslant 1$,

$$\Phi^p(A\nabla_t B)\Phi^p((A\sharp_t B)^{-1}) + \Phi^p((A\sharp_t B)^{-1})\Phi^p(A\nabla_t B) \leqslant \frac{(M+m)^{2p}}{2M^p m^p K^{\mu p}(h')},$$

and $p \geqslant 2$,

$$\Phi^{2p}(A\nabla B) \leqslant \left(\frac{K^2(h)(M^2+m^2)^2}{4^{\frac{2}{p}}K^{2\mu}(h')M^2m^2}\right)^p \Phi^{2p}(H_t(A,B))$$

for all $t \in [0,1]$, where $\mu = \min\{t, 1-t\}$, $K(h) = \frac{(h+1)^2}{4h}$, $K(h') = \frac{(h'+1)^2}{4h'}$, $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$.

1. Introduction

Throughout this paper, let m, m', M, M' be scalars and I be the identity operator. Other capital letters are used to denote the general elements of the C^*-algebra $B(\mathscr{H})$ of all bounded linear operators acting on a Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. The quantity $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is so called the Kantorovich constant. The operator norm is defined by $\|\cdot\|$. We write $A \geqslant 0$ to mean that the operator A is positive. If $A - B \geqslant 0$ $(A - B \leqslant 0)$, then we say that $A \geqslant B$ $(A \leqslant B)$. A^* stands for the adjoint of A. We denote the absolute value operator of A by $|A|$, that is, $|A| = (A^*A)^{\frac{1}{2}}$.

For each $t \in [0,1]$, the weighted arithmetic means ∇_t and weighted geometric mean for invertible positive operators A and B are defined as follows: $A\nabla_t B = (1-t)A + tB$ and $A\sharp_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$. When $t = \frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity respectively, see Kubo and Ando [2]. Heinz mean is defined as $H_t(A,B) = \frac{1}{2}(A\sharp_t B + B\sharp_t A)$.

In [1], Ando, Li and Mathias proposed a definition for the geometric mean of three or more positive semi-definite matrices and showed that it has many requared properties on the geometric mean. In [9], Yamazaki pointed out that the definition of the geometric mean by Ando, Li and Mathias can be extended to Hilbert space operators. For positive

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invertible operators A and B on a Hilbert space H, the geometric mean $A\sharp B$ of A and B is defined by

$$A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

As an extension of $A\sharp B$, the geometric mean $G(A_1,\ldots,A_n)$ of any $n-tuple$ of positive invertible operators $A_1,\ldots,A_n$ on a Hilbert space H is defined by induction as follows:

(i) $G(A_1,A_2) = A\sharp B$.

(ii) Assume that the geometric mean of any $(n-1)-$tuple of operators is defined. Let $G((A_j)_{j\neq i}) = G(A_1,\ldots,A_{i-1},A_{i+1}\ldots,A_n)$ and let sequences $\{A_i^{(r)}\}_{r=1}^{\infty}$ be $A_i^{(1)} = A_i$ and $A_i^{(r+1)} = G((A_j^{(r)})_{j\neq i})$. Then there exists $\lim\limits_{r\to\infty} A_i^{(r)}$ uniformly and it does not depend on i. Hence the geometric mean of $n-$operators is defined by $\lim\limits_{r\to\infty} A_i^{(r)} = G(A_1,\ldots,A_n)$ for $i=1,\ldots,n$. There is no explicit formula for $G(A_1,\ldots,A_n)$ in terms of $A_1,\ldots,A_n$ when $n\geqslant 3$. However, the only basic property that we need is

$$G(A_1,\ldots,A_n) \leqslant \frac{A_1+\ldots+A_n}{n}.$$

A linear map Φ is positive if $\Phi(A)\geqslant 0$ whenever $A\geqslant 0$. It's said to be unital if $\Phi(I)=I$.

It is well known that for two positive operator A, B,

$$A\geqslant B \not\Rightarrow A^p \geqslant B^p$$

for $p>1$.

Let $0<m\leqslant A\leqslant M$ and Φ be positive unital linear map. Lin [7, Theorem 2.10] proved the following operator inequalities:

$$|\Phi(A^{-1})\Phi(A)+\Phi(A)\Phi(A^{-1})| \leqslant \tfrac{(M+m)^2}{2Mm} \tag{1.1}$$

and

$$\Phi(A^{-1})\Phi(A)+\Phi(A)\Phi(A^{-1}) \leqslant \tfrac{(M+m)^2}{2Mm}. \tag{1.2}$$

Fu [4, Theorem 4] generalized (1.1) and (1.2) when $p\geqslant 1$:

$$|\Phi^p(A^{-1})\Phi^p(A)+\Phi^p(A)\Phi^p(A^{-1})| \leqslant \tfrac{(M+m)^{2p}}{2M^pm^p} \tag{1.3}$$

and

$$\Phi^p(A^{-1})\Phi^p(A)+\Phi^p(A)\Phi^p(A^{-1}) \leqslant \tfrac{(M+m)^{2p}}{2M^pm^p}. \tag{1.4}$$

Let $0<m\leqslant A,B\leqslant M$ and Φ is positive unital linear map. Lin [6] also proved the following operator inequalities:

$$\Phi^2(A\nabla B) \leqslant K^2(h)\Phi^2(A\sharp B) \tag{1.5}$$

and

$$\Phi^2(A\nabla B) \leqslant K^2(h)(\Phi(A)\sharp\Phi(B))^2. \tag{1.6}$$

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is the Kantorovich constant.

Zhang [11, Theorem 2.6] generalized (1.5) and (1.6) when $p \geqslant 2$:

$$\Phi^{2p}(A\triangledown B) \leqslant \frac{(K(h)(M^2+m^2))^{2p}}{16M^{2p}m^{2p}}\Phi^{2p}(A\sharp B) \tag{1.7}$$

and

$$\Phi^{2p}(A\triangledown B) \leqslant \frac{(K(h)(M^2+m^2))^{2p}}{16M^{2p}m^{2p}}(\Phi(A)\sharp\Phi(B))^{2p}. \tag{1.8}$$

Let $A_1, A_2, \dots, A_n$ be positive operators on a Hilbert space with $0 < m \leqslant A_i \leqslant M$ $(i = 1, \dots, n)$. Fujii et al. [5] showed a reverse arithmetic-geometric mean inequality of several operators

$$\frac{A_1+\ldots+A_n}{n} \leqslant (\frac{(M+m)^2}{4Mm})G(A_1, \dots, A_n). \tag{1.9}$$

Lin [6, Theorem3.2] showed that the reverse AM-GM inequality (1.9) can be squared:

$$(\frac{A_1+\ldots+A_n}{n})^2 \leqslant (\frac{(M+m)^2}{4Mm})^2 G^2(A_1, \dots, A_n). \tag{1.10}$$

Fu[4, Theorem5] showed the generalization of (1.10), that is when $p \geqslant 1$

$$(\frac{A_1+\ldots+A_n}{n})^{2p} \leqslant (\frac{(M+m)^{2p}}{4M^p m^p})^2 G^{2p}(A_1, \dots, A_n). \tag{1.11}$$

In this paper, we will focus on present some operator inequalities which are refinements of the above.

2. Main results

We need some Lemmas to prove the main theorems of this paper:

LEMMA 2.1. [3] *Let $A, B > 0$. Then the following norm inequality holds:*

$$\|AB\| \leqslant \frac{1}{4}\|A+B\|^2. \tag{2.1}$$

LEMMA 2.2. [2] *Let A and B be positive operators. Then for $1 \leqslant r < \infty$,*

$$\|A^r + B^r\| \leqslant \|(A+B)^r\|. \tag{2.2}$$

LEMMA 2.3. [7] *For any bounded operator X,*

$$|X| \leqslant tI \Leftrightarrow \|X\| \leqslant t \Leftrightarrow \begin{bmatrix} tI & X \\ X^* & tI \end{bmatrix} \geqslant 0.$$

LEMMA 2.4. [12] *Suppose that two operators A, B and positive real numbers m, m', M, M' satisfy either of the following conditions:*

(i) $0 < m \leqslant A \leqslant m' < M' \leqslant B \leqslant M$;

(ii) $0 < m \leqslant B \leqslant m' < M' \leqslant A \leqslant M$.

Then

$$A\nabla_t B \geqslant K^{\mu}(h')A\sharp_t B,$$

for all $t \in [0,1]$, *where* $\mu = \min\{t, 1-t\}$ *and* $h' = \frac{M'}{m'}$.

LEMMA 2.5. *Let* $0 < m \leqslant A \leqslant m' < M' \leqslant B \leqslant M$. *Then*

$$A\nabla_t B + MmK^{\mu}(h')(A\sharp_t B)^{-1} \leqslant M + m \tag{2.3}$$

for all $t \in [0,1]$, *where* $\mu = \min\{t, 1-t\}$, $K(h') = \frac{(h'+1)^2}{4h'}$ *and* $h' = \frac{M'}{m'}$.

Proof. It is easy to see that

$$(1-t)(M-A)(m-A)A^{-1} \leqslant 0$$

and

$$t(M-B)(m-B)B^{-1} \leqslant 0,$$

then

$$(1-t)MmA^{-1} + (1-t)A \leqslant (1-t)(M+m)$$

and

$$tMmB^{-1} + tB \leqslant t(M+m).$$

Summing up the two above inequalities, we get

$$A\nabla_t B + Mm(A^{-1}\nabla_t B^{-1}) \leqslant M + m.$$

By $(A\sharp_t B)^{-1} = A^{-1}\sharp_t B^{-1}$ and Lemma 2.4, we have

$$\begin{aligned}
& A\nabla_t B + MmK^{\mu}(h')(A\sharp_t B)^{-1} \\
& = A\nabla_t B + MmK^{\mu}(h')(A^{-1}\sharp_t B^{-1}) \\
& \leqslant A\nabla_t B + Mm(A^{-1}\nabla_t B^{-1}) \\
& \leqslant M + m.
\end{aligned}$$

This completes the proof. $\square$

THEOREM 2.6. *Let* $0 < m \leqslant A \leqslant m' < M' \leqslant B \leqslant M$ *and* $p \geqslant 1$. *Then for every positive unital linear map* Φ,

$$|\Phi^p(A\nabla_t B)\Phi^p((A\sharp_t B)^{-1}) + \Phi^p((A\sharp_t B)^{-1})\Phi^p(A\nabla_t B)| \leqslant \frac{(M+m)^{2p}}{2M^p m^p K^{\mu p}(h')} \tag{2.4}$$

and

$$\Phi^p(A\nabla_t B)\Phi^p((A\sharp_t B)^{-1})+\Phi^p((A\sharp_t B)^{-1})\Phi^p(A\nabla_t B)\leqslant \frac{(M+m)^{2p}}{2M^pm^pK^{\mu p}(h')} \tag{2.5}$$

for all $t\in[0,1]$, *where* $\mu=\min\{t,1-t\}$, $K(h)=\frac{(h+1)^2}{4h}$, $K(h')=\frac{(h'+1)^2}{4h'}$, $h=\frac{M}{m}$ *and* $h'=\frac{M'}{m'}$.

Proof. By (2.1), (2.2) and (2.3), we can compute that

$$\begin{aligned}
&\|\Phi^p(A\nabla_t B)M^pm^pK^{\mu p}(h')\Phi^p((A\sharp_t B)^{-1})\|\\
&\leqslant \tfrac{1}{4}\|\Phi^p(A\nabla_t B)+M^pm^pK^{\mu p}(h')\Phi^p((A\sharp_t B)^{-1})\|^2\\
&\leqslant \tfrac{1}{4}\|\Phi(A\nabla_t B)+MmK^{\mu}(h')\Phi((A\sharp_t B)^{-1})\|^{2p}\\
&= \tfrac{1}{4}\|\Phi(A\nabla_t B+MmK^{\mu}(h')(A\sharp_t B)^{-1})\|^{2p}\\
&\leqslant \tfrac{1}{4}(M+m)^{2p}.
\end{aligned}$$

So

$$\|\Phi^p(A\nabla_t B)\Phi^p((A\sharp_t B)^{-1})\|\leqslant \frac{(M+m)^{2p}}{4M^pm^pK^{\mu p}(h')}. \tag{2.6}$$

By Lemma 2.3 and (2.6) we obtain

$$\begin{bmatrix} \frac{(M+m)^{2p}}{4M^pm^pK^{\mu p}(h')}I & \Phi^p(A\nabla_t B)\Phi^p((A\sharp_t B)^{-1}) \\ \Phi^p((A\sharp_t B)^{-1})\Phi^p(A\nabla_t B) & \frac{(M+m)^{2p}}{4M^pm^pK^{\mu p}(h')}I \end{bmatrix}\geqslant 0$$

and

$$\begin{bmatrix} \frac{(M+m)^{2p}}{4M^pm^pK^{\mu p}(h')}I & \Phi^p((A\sharp_t B)^{-1})\Phi^p(A\nabla_t B) \\ \Phi^p(A\nabla_t B)\Phi^p((A\sharp_t B)^{-1}) & \frac{(M+m)^{2p}}{4M^pm^pK^{\mu p}(h')}I \end{bmatrix}\geqslant 0.$$

Summing up these two operator matrices, we have

$$\begin{bmatrix} \frac{(M+m)^{2p}}{2M^pm^pK^{\mu p}(h')}I & Y \\ Y^* & \frac{(M+m)^{2p}}{2M^pm^pK^{\mu p}(h')}I \end{bmatrix}\geqslant 0,$$

where denote that $Y=\Phi^p(A\nabla_t B)\Phi^p((A\sharp_t B)^{-1})+\Phi^p((A\sharp_t B)^{-1})\Phi^p(A\nabla_t B)$. It is easy to see that $Y^*=Y$. By Lemma 2.3, we can achieve (2.4) and (2.5). $\square$

REMARK 2.7. Let $t=0$ in Theorem 2.6, then $\mu=0$, we can get the inequalities (1.3) and (1.4) by (2.4) and (2.5), respectively.

THEOREM 2.8. *Let* $0<m\leqslant A\leqslant m'<M'\leqslant B\leqslant M$. *Then for every positive unital linear map* Φ *and* $p\geqslant 2$,

$$\Phi^p(A\nabla B)\leqslant \frac{2^{2p-4}K^p(h)}{K^{\mu p}(h')}\Phi^p(H_t(A,B)) \tag{2.7}$$

and

$$\Phi^p(A\nabla B) \leqslant \frac{2^{2p-4}K^p(h)}{K^{\mu p}(h')}H_t^p(\Phi(A),\Phi(B)) \tag{2.8}$$

for all $t\in[0,1]$, *where* $\mu=\min\{t,1-t\}$, $K(h)=\frac{(h+1)^2}{4h}$, $K(h')=\frac{(h'+1)^2}{4h'}$, $h=\frac{M}{m}$ *and* $h'=\frac{M'}{m'}$.

Proof. The inequality (2.7) is equivalent to

$$\|\Phi^{\frac{p}{2}}(A\nabla B)K^{\frac{\mu p}{2}}(h')\Phi^{-\frac{p}{2}}(H_t(A,B))\| \leqslant 2^{p-2}K^{\frac{p}{2}}(h). \tag{2.9}$$

By Lemma 2.1 and Lemma 2.2, (2.9) is true if

$$\Phi(A\nabla B)+MmK^{\mu}(h')\Phi^{-1}(H_t(A,B)) \leqslant M+m. \tag{2.10}$$

The well-known Choi inequality (see [4, p. 41]) says that

$$\Phi^{-1}(T)\leqslant\Phi(T^{-1}) \quad \text{for any} \quad T>0.$$

So (2.10) would follow by

$$\Phi(A\nabla B)+MmK^{\mu}(h')\Phi(H_t^{-1}(A,B)) \leqslant M+m. \tag{2.11}$$

By Lemma 2.5, we have

$$A\nabla_t B+MmK^{\mu}(h')(A\sharp_t B)^{-1}\leqslant M+m$$

and

$$B\nabla_t A+MmK^{\mu}(h')(B\sharp_t A)^{-1}\leqslant M+m.$$

Summing up these two inequalities, we get

$$A\nabla B+MmK^{\mu}(h')\frac{(A\sharp_t B)^{-1}+(B\sharp_t A)^{-1}}{2}\leqslant M+m.$$

That is,

$$A\nabla B+MmK^{\mu}(h')H_t(A^{-1},B^{-1})\leqslant M+m.$$

So,

$$\begin{aligned}
&\Phi(A\nabla B)+MmK^{\mu}(h')\Phi(H_t^{-1}(A,B))\\
&=\Phi(A\nabla B)+MmK^{\mu}(h')\Phi((\tfrac{A\sharp_t B+B\sharp_t A}{2})^{-1})\\
&=\Phi(A\nabla B)+MmK^{\mu}(h')\Phi((A\sharp_t B)^{-1}!(B\sharp_t A)^{-1})\\
&\leqslant\Phi(A\nabla B)+MmK^{\mu}(h')\Phi((A\sharp_t B)^{-1}\nabla(B\sharp_t A)^{-1})\\
&=\Phi[(A\nabla B)+MmK^{\mu}(h')H_t(A^{-1},B^{-1})]\\
&\leqslant M+m.
\end{aligned}$$

This proves (2.11). The proof of (2.8) is similar, so we omit the details. $\square$

THEOREM 2.9. *Let $0 < m \leqslant A \leqslant m' < M' \leqslant B \leqslant M$. Then for every positive unital linear map Φ and $p \geqslant 2$,*

$$\Phi^p(A\nabla_t B) \leqslant \frac{2^{2p-4}K^p(h)}{K^{\mu p}(h')}\Phi^p(A\sharp_t B) \tag{2.12}$$

and

$$\Phi^p(A\nabla_t B) \leqslant \frac{2^{2p-4}K^p(h)}{K^{\mu p}(h')}(\Phi(A)\sharp_t\Phi(B))^p \tag{2.13}$$

for all $t \in [0,1]$, where $\mu = \min\{t, 1-t\}$, $K(h) = \frac{(h+1)^2}{4h}$, $K(h') = \frac{(h'+1)^2}{4h'}$, $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$.

Proof. Use a similar argument as above, we can prove (2.12) and (2.13). □

REMARK 2.10. Letting $t = \frac{1}{2}$, $\mu = \frac{1}{2}$ and put $p = 2$, since $\frac{K^2(h)}{K^{2\mu}(h')} < K^2(h)$, so under a stronger condition as Theorem 2.9, we see (2.12) and (2.13) are refinements of (1.5) and (1.6), respectively.

THEOREM 2.11. *Let $0 < m \leqslant A \leqslant m' < M' \leqslant B \leqslant M$. Then for every positive unital linear map Φ and $p \geqslant 2$,*

$$\Phi^{2p}(A\nabla B) \leqslant \frac{1}{16}\left(\frac{K^2(h)(M^2+m^2)^2}{K^{2\mu}(h')M^2m^2}\right)^p \Phi^{2p}(H_t(A,B)) \tag{2.14}$$

and

$$\Phi^{2p}(A\nabla B) \leqslant \frac{1}{16}\left(\frac{K^2(h)(M^2+m^2)^2}{K^{2\mu}(h')M^2m^2}\right)^p H_t^{2p}(\Phi(A),\Phi(B)) \tag{2.15}$$

for all $t \in [0,1]$, where $\mu = \min\{t, 1-t\}$, $K(h) = \frac{(h+1)^2}{4h}$, $K(h') = \frac{(h'+1)^2}{4h'}$, $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$.

Proof. By the operator reverse monotonicity of inequality (2.7) and put $p = 2$, we have

$$\Phi^{-2}(H_t(A,B)) \leqslant \frac{K^2(h)}{K^{2\mu}(h')}\Phi^{-2}(A\nabla B) = L^2\Phi^{-2}(A\nabla B), \tag{2.16}$$

where $L = \frac{K(h)}{K^{\mu}(h')}$.

By $0 < m \leqslant A \leqslant M$, then for every positive unital linear map Φ, we have $m^2 \leqslant \Phi^2(A) \leqslant M^2$.

So

$$(M^2 - \Phi^2(A))(m^2 - \Phi^2(A))\Phi^{-2}(A) \leqslant 0$$

That is,

$$M^2m^2\Phi^{-2}(A) + \Phi^2(A) \leqslant M^2 + m^2, \tag{2.17}$$

By (2.1), (2.2), (2.16) and (2.17), we can compute that

$$\begin{aligned}
&\|\Phi^p(A\nabla B)M^p m^p \Phi^{-p}(H_t(A,B))\| \\
&\leqslant \tfrac{1}{4}\|L^{\frac{p}{2}}\Phi^p(A\nabla B)+(\tfrac{M^2m^2}{L})^{\frac{p}{2}}\Phi^{-p}(H_t(A,B))\|^2 \\
&\leqslant \tfrac{1}{4}\|L\Phi^2(A\nabla B)+\tfrac{M^2m^2}{L}\Phi^{-2}(H_t(A,B))\|^p \\
&\leqslant \tfrac{1}{4}\|L\Phi^2(A\nabla B)+LM^2m^2\Phi^{-2}(A\nabla B)\|^p \\
&\leqslant \tfrac{1}{4}(L(M^2+m^2))^p.
\end{aligned}$$

That is

$$\|\Phi^p(A\nabla B)\Phi^{-p}(H_t(A,B))\| \leqslant \frac{1}{4}\left(\frac{L(M^2+m^2)}{Mm}\right)^p = \frac{1}{4}\left(\frac{K^2(h)(M^2+m^2)^2}{K^{2\mu}(h')M^2m^2}\right)^{\frac{p}{2}}.$$

Thus, (2.14) holds. By inequality (2.8), the proof of (2.15) is similar, we omit the details.

This completes the proof. $\square$

THEOREM 2.12. *Let $0 < m \leqslant A \leqslant m' < M' \leqslant B \leqslant M$. Then for every positive unital linear map Φ and $p \geqslant 2$,*

$$\Phi^{2p}(A\nabla_t B) \leqslant \tfrac{1}{16}\left(\tfrac{K^2(h)(M^2+m^2)^2}{K^{2\mu}(h')M^2m^2}\right)^p \Phi^{2p}(A\sharp_t B) \tag{2.18}$$

and

$$\Phi^{2p}(A\nabla_t B) \leqslant \tfrac{1}{16}\left(\tfrac{K^2(h)(M^2+m^2)^2}{K^{2\mu}(h')M^2m^2}\right)^p (\Phi(A)\sharp_t\Phi(B))^{2p} \tag{2.19}$$

for all $t \in [0,1]$, where $\mu = \min\{t, 1-t\}$, $K(h) = \frac{(h+1)^2}{4h}$, $K(h') = \frac{(h'+1)^2}{4h'}$, $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$.

Proof. The proof are similar as Theorem 2.11, we omit the details. $\square$

REMARK 2.13. Put $t = \frac{1}{2}$, $\mu = \frac{1}{2}$, the inequalities (2.18) and (2.19) are

$$\Phi^{2p}(A\nabla B) \leqslant \tfrac{1}{16}(\tfrac{K^2(h)(M^2+m^2)^2}{K(h')M^2m^2})^p \Phi^{2p}(A\sharp B) \tag{2.20}$$

and

$$\Phi^{2p}(A\nabla B) \leqslant \tfrac{1}{16}(\tfrac{K^2(h)(M^2+m^2)^2}{K(h')M^2m^2})^p (\Phi(A)\sharp\Phi(B))^{2p}, \tag{2.21}$$

respectively.

Since $K(h') > 1$, and hence if strengthen the condition as in Theorem 2.12, inequalities (2.20) and (2.21) are stronger than (1.7) and (1.8), respectively.

THEOREM 2.14. *Let* $0 < m \leqslant A_i \leqslant M \ \ (i = 1, \ldots, n)$ *and* $p \geqslant 2$. *Then*

$$(\tfrac{A_1+\ldots+A_n}{n})^{2p} \leqslant (\tfrac{(K(h)(M^2+m^2))^p}{4M^p m^p})^2 G^{2p}(A_1, \ldots, A_n), \tag{2.22}$$

Proof. By [10, p. 40], we know that the inequality (2.22) is equivalent to

$$\|(\tfrac{A_1+\ldots+A_n}{n})^p M^p m^p G^{-p}(A_1, \ldots, A_n)\| \leqslant \tfrac{(K(h)(M^2+m^2))^p}{4}. \tag{2.23}$$

By the operator reverse monotonicity of the inequality (1.10), we have

$$G^{-2}(A_1, \ldots, A_n) \leqslant (\tfrac{(M+m)^2}{4Mm})^2 (\tfrac{A_1+\ldots+A_n}{n})^{-2} = K^2(h)(\tfrac{A_1+\ldots+A_n}{n})^{-2}. \tag{2.24}$$

By (2.1), (2.2) and (2.24), we can compute that

$$\begin{aligned}
&\|(\tfrac{A_1+\ldots+A_n}{n})^p M^p m^p G^{-p}(A_1, \ldots, A_n)\| \\
&\leqslant \tfrac{1}{4}\|K^{\frac{p}{2}}(h)(\tfrac{A_1+\ldots+A_n}{n})^p + (\tfrac{M^2m^2}{K(h)})^{\frac{p}{2}} G^{-p}(A_1, \ldots, A_n)\|^2 \\
&\leqslant \tfrac{1}{4}\|K(h)(\tfrac{A_1+\ldots+A_n}{n})^2 + \tfrac{M^2m^2}{K(h)} G^{-2}(A_1, \ldots, A_n)\|^p \\
&\leqslant \tfrac{1}{4}\|K(h)(\tfrac{A_1+\ldots+A_n}{n})^2 + M^2m^2K(h)(\tfrac{A_1+\ldots+A_n}{n})^{-2}\|^p \\
&\leqslant \tfrac{1}{4}(K(h)(M^2+m^2))^p.
\end{aligned}$$

Note that the last inequality follows from: For $0 < m \leqslant A \leqslant M$, $M^2m^2A^{-2} + A^2 \leqslant M^2 + m^2$.

That is

$$\left\|\left(\frac{A_1 + \ldots + A_n}{n}\right)^p G^{-p}(A_1, \ldots, A_n)\right\| \leqslant \frac{(K(h)(M^2+m^2))^p}{4M^p m^p}.$$

Thus, (2.22) holds. □

REMARK 2.15. If $0 < m \leqslant M \leqslant (2+\sqrt{3})m$, then

$$\frac{\dfrac{(K(h)(M^2+m^2))^p}{4M^p m^p}}{\dfrac{(M+m)^{2p}}{4M^p m^p}} = \left(\frac{M^2+m^2}{4Mm}\right)^p \leqslant 1.$$

Thus, when $p \geqslant 2$, (2.22) may be stronger than (1.11) for some M, m.

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