Journal of Mathematical Inequalities
Volume 11, Number 1 (2017), 49–58 doi:10.7153/jmi-11-05

SOME ESTIMATES OF HARDY OPERATORS AND THEIR COMMUTATORS ON MORREY–HERZ SPACES

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(*Communicated by J. Pečarić*)

Abstract. In this paper, we study the boundedness of p-adic Hardy operators $\mathscr{H}^p$ and $\mathscr{H}^{p,*}$ on the p-adic Morrey-Herz space. Furthermore, we establish the Lipschitz estimates for commutators of p-adic Hardy operators $\mathscr{H}_b^p$ and $\mathscr{H}_b^{p,*}$ on the p-adic function spaces, such as the p-adic Lebesgue space, the p-adic Herz space and the p-adic Morrey-Herz space. Moreover, we also obtain the *CMO* estimates. All our results are also true for the fractional p-adic Hardy operator.

1. Introduction

In recent years, p-adic analysis has got a lot of attention by its applications in many aspects of mathematical physics, such as quantum mechanics, the probability theory and the dynamical systems (see [15, 16] and the references therein). On the other hand, it plays an important role in pseudo-differential equations, wavelet theory and harmonic analysis, etc (see [2–4, 11–16, 19–23]).

For a prime number p, let $\mathbb{Q}_p$ be the field of p-adic numbers. This field is the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This norm is defined as follows: if $x=0$, $|0|_p=0$; if $x\neq 0$ is an arbitrary number with the unique representation $x=p^{\gamma}\frac{m}{n}$, where m, n are not multiple of p, $\gamma=\gamma(x)\in\mathbb{Z}$, then $|x|_p=p^{-\gamma}$. This norm satisfies the following properties:

(i) $|x|_p\geqslant 0$, $\forall x\in\mathbb{Q}_p$, $|x|_p=0\Leftrightarrow x=0$;

(ii) $|xy|_p=|x|_p|y|_p$, $\forall x,\ y\in\mathbb{Q}_p$;

(iii) $|x+y|_p\leqslant\max(|x|_p,|y|_p)$, $\forall x,\ y\in\mathbb{Q}_p$, and when $|x|_p\neq|y|_p$, we have $|x+y|_p=\max(|x|_p,|y|_p)$.

The space $\mathbb{Q}_p^n$ denotes a vector space over $\mathbb{Q}_p$ which consists of all points $\mathbf{x}=(x_1,\ldots,x_n)$, where $x_i\in\mathbb{Q}_p$, $i=1,\ldots,n$. The p-adic norm of $\mathbb{Q}_p^n$ is defined by $|\mathbf{x}|_p:=\max_{1\leqslant i\leqslant n}|x_i|_p$. For $\gamma\in\mathbb{Z}$, we denote the ball $B_\gamma(\mathbf{a})$ center $\mathbf{a}\in\mathbb{Q}_p^n$ and radius p^γ and its boundary $S_\gamma(\mathbf{a})$ by

$$B_\gamma(\mathbf{a})=\{\mathbf{x}\in\mathbb{Q}_p^n:\ |\mathbf{x}-\mathbf{a}|_p\leqslant p^\gamma\},\qquad S_\gamma(\mathbf{a})=\{\mathbf{x}\in\mathbb{Q}_p^n:\ |\mathbf{x}-\mathbf{a}|_p=p^\gamma\},$$

respectively.

Mathematics subject classification (2010): 42B35, 47G10.
Keywords and phrases: p-adic Hardy operator, p-adic Morrey-Herz space, Jensen's inequality.

Paper JMI-11-05

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, there exists a Haar measure $d\mathbf{x}$ on $\mathbb{Q}_p^n$. This measure is unique by normalizing $d\mathbf{x}$ such that

$$\int_{B_0(\mathbf{0})} d\mathbf{x} = |B_0(\mathbf{0})|_H = 1,$$

where $|B|_H$ denotes the Haar measure of a measure subset B of $\mathbb{Q}_p^n$. It is easy to obtain that $|B_\gamma(a)|_H = p^{\gamma n}$, $|S_\gamma(a)|_H = p^{\gamma n}(1-p^{-n})$, for any $a \in \mathbb{Q}_p^n$.

It is well known that the Hardy operator is one of the most important operators in mathematical analysis and plays an important role in many branches of mathematics, such as partial differential equations, complex analysis and harmonic analysis (for example, see [5, 8, 17, 18, 11]). Let f be a non-negative integrable function on $\mathbb{R}^+$, the one-dimensional Hardy operator is defined by

$$Hf(x) := \frac{1}{x}\int_0^x f(t)dt, \qquad x > 0.$$

A celebrated Hardy's integral inequality by Hardy [10] stated that

$$\|Hf\|_{L^q(\mathbb{R}^+)} \leqslant \frac{q}{q-1}\|f\|_{L^q(\mathbb{R}^+)}, \qquad 1 < q < \infty.$$

In [6], Faris introduced the following n-dimensional Hardy operators,

$$\mathscr{H}f(x) := \frac{1}{\Omega_n|x|^n}\int_{|t|\leqslant|x|} f(t)dt, \quad \mathscr{H}^*f(x) := \frac{1}{\Omega_n}\int_{|t|>|x|} \frac{f(t)}{|t|^n}dt, \ \ x \in \mathbb{R}^n\setminus\{0\},$$

where Ω_n is the volume of the unit ball in $\mathbb{R}^n$. Christ and Grafakos [1] obtained that the norm of Hardy operators on the Lebesgue space, i.e.

$$\|\mathscr{H}\|_{L^q(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} = \|\mathscr{H}^*\|_{L^q(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} = \frac{q}{q-1}.$$

Recently, Fu et al. [7] proved $\mathscr{H}$ is also bounded on the weighted Lebesgue space $L^q(|x|^\alpha)$ for $1 < q < \infty$ and $\alpha < n(q-1)$, and the norm is $qn/(qn-n-\alpha)$. In [9], the authors defined the p-adic Hardy operators $\mathscr{H}^p$ and $\mathscr{H}^{p,*}$ and their commutators $\mathscr{H}_b^p$ and $\mathscr{H}_b^{p,*}$. They extended the above results on $L^q(|x|_p^\alpha)$ and also obtained the boundedness of their commutators on the Herz space. In [23], Wu defined the n-dimensional fractional p-adic Hardy operators $\mathscr{H}_\beta^p$ and $\mathscr{H}_\beta^{p,*}$, and proved their boundedness on the Herz space. She also defined their commutators $\mathscr{H}_{\beta,b}^p$ and $\mathscr{H}_{\beta,b}^{p,*}$, and proved the CMO estimates on the Herz space. In [22], the authors studied the boundedness of p-adic Hardy operators and their commutators on p-adic central Morrey and BMO spaces. In this paper, we prove the boundedness of p-adic Hardy operators $\mathscr{H}^p$ and $\mathscr{H}^{p,*}$ on the p-adic Morrey-Herz space. Furthermore, we establish the Lipschitz estimates for commutators of p-adic Hardy operators $\mathscr{H}_b^p$ and $\mathscr{H}_b^{p,*}$ on some p-adic function spaces, such as the p-adic Lebesgue space, the p-adic Herz space and the p-adic Morrey-Herz space. Moreover, we also obtain the CMO estimates on the p-adic Morrey-Herz space. All our results are extended for the fractional p-adic Hardy operator.

Before recalling some definitions, we give some notation. Let $B_k = B_k(0)$, $S_k = B_k \setminus B_{k-1}$ and χ_k is the characteristic function of S_k. Throughout this article, we will use C to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. Moreover, we will denote $f \approx g$ if there exists two positive constants C_1 and C_2 such that $C_1 f(x) \leqslant g(x) \leqslant C_2 f(x)$.

DEFINITION 1. ([23]) Let $f \in L_{\mathrm{loc}}(\mathbb{Q}_p^n)$, $0 \leqslant \beta < n$. The fractional p-adic Hardy operators are defined by

$$\mathscr{H}_\beta^p f(\mathbf{x}) := \frac{1}{|\mathbf{x}|_p^{n-\beta}} \int_{|\mathbf{t}|_p \leqslant |\mathbf{x}|_p} f(\mathbf{t}) d\mathbf{t},$$

$$\mathscr{H}_\beta^{p,*} f(\mathbf{x}) := \int_{|\mathbf{t}|_p > |\mathbf{x}|_p} \frac{f(\mathbf{t})}{|\mathbf{t}|_p^{n-\beta}} d\mathbf{t}, \ \ \mathbf{x} \in \mathbb{Q}_p^n \setminus \{0\}.$$

We denote $\mathscr{H}_0^p = \mathscr{H}^p$ and $\mathscr{H}_0^{p,*} = \mathscr{H}^{p,*}$, where $\mathscr{H}^p$ and $\mathscr{H}^{p,*}$ are the p-adic Hardy operators (see [9]).

DEFINITION 2. ([23]) Let $b \in L_{\mathrm{loc}}(\mathbb{Q}_p^n)$, $0 \leqslant \beta < n$. The commutators of fractional p-adic Hardy operators are

$$\mathscr{H}_{\beta,b}^p f(\mathbf{x}) := b(\mathbf{x}) \mathscr{H}_\beta^p f(\mathbf{x}) - \mathscr{H}_\beta^p (bf)(\mathbf{x}),$$

$$\mathscr{H}_{\beta,b}^{p,*} f(\mathbf{x}) := b(\mathbf{x}) \mathscr{H}_\beta^{p,*} f(\mathbf{x}) - \mathscr{H}_\beta^{p,*} (bf)(\mathbf{x}).$$

Note that $\mathscr{H}_{0,b}^p = \mathscr{H}_b^p$ and $\mathscr{H}_{0,b}^{p,*} = \mathscr{H}_b^{p,*}$.

DEFINITION 3. ([2]) Let $\alpha \in \mathbb{R}$, $0 < l < \infty$, $0 < q < \infty$ and $\lambda \geqslant 0$. The Morrey-Herz space on p-adic fields $M\dot{K}_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)$ is defined by

$$M\dot{K}_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n) = \{ f \in L_{\mathrm{loc}}^q(\mathbb{Q}_p^n \setminus \{0\}) : \|f\|_{M\dot{K}_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)} < \infty \},$$

where

$$\|f\|_{M\dot{K}_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)} = \sup_{k_0 \in \mathbb{Z}} p^{-k_0 \lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha l} \|f\chi_k\|_{L^q(\mathbb{Q}_p^n)}^l \right)^{1/l}.$$

Obviously, $M\dot{K}_{l,q}^{\alpha,0}(\mathbb{Q}_p^n) = \dot{K}_q^{\alpha,l}(\mathbb{Q}_p^n)$ is p-adic Herz space (see [9]) and $M\dot{K}_{q,q}^{\alpha/q,0}(\mathbb{Q}_p^n) = L^q(|x|_p^\alpha)$.

DEFINITION 4. ([2]) Let γ be a positive real number. The Lipschitz space $\Lambda_\gamma(\mathbb{Q}_p^n)$ is defined to be the space of all measurable functions f on $\mathbb{Q}_p^n$ such that

$$\|f\|_{\Lambda_\gamma(\mathbb{Q}_p^n)} = \sup_{\mathbf{x},\mathbf{h} \in \mathbb{Q}_p^n, \mathbf{h} \neq 0} \frac{|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})|}{|\mathbf{h}|_p^\gamma} < \infty.$$

DEFINITION 5. ([9]) Let $1 \leqslant q < \infty$. The space $CMO^q(\mathbb{Q}_p^n)$ is defined to be the space of all measurable functions $f \in L^q_{\mathrm{loc}}(\mathbb{Q}_p^n)$ satisfying

$$\|f\|_{CMO^q(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |f(\mathbf{x}) - f_{B_\gamma}|^q d\mathbf{x} \right)^{1/q} < \infty,$$

where $f_{B_\gamma(0)} = \frac{1}{|B_\gamma|_H} \int_{B_\gamma} f(\mathbf{x}) d\mathbf{x}$. Obviously, $L^\infty(\mathbb{Q}_p^n) \subset BMO(\mathbb{Q}_p^n) \subset CMO^q(\mathbb{Q}_p^n)$.

2. Estimates for Hardy operators

THEOREM 1. *Suppose* $0 < l_1 \leqslant l_2 < \infty$, $1 < q_1, q_2 < \infty$, $0 \leqslant \beta < n$, $1/q_1 - 1/q_2 = \beta/n$ *and* $\lambda > 0$. *Then*
(i) if $\alpha < n/q_1' + \lambda$, *then*

$$\|\mathscr{H}_\beta^p f\|_{M\dot{K}^{\alpha,\lambda}_{l_2,q_2}(\mathbb{Q}_p^n)} \leqslant \frac{(1-p^{-n})^{1-\beta/n}}{1-p^{\alpha-n/q_1'-\lambda}} \frac{1}{(1-p^{-\lambda l_1})^{1/l_1}} \|f\|_{M\dot{K}^{\alpha,\lambda}_{l_1,q_1}(\mathbb{Q}_p^n)};$$

(ii) if $\alpha > -n/q_2 + \lambda$, *then*

$$\|\mathscr{H}_\beta^{p,*} f\|_{M\dot{K}^{\alpha,\lambda}_{l_2,q_2}(\mathbb{Q}_p^n)} \leqslant \frac{(1-p^{-n})^{1-\beta/n}}{p^{\alpha+n/q_2-\lambda}-1} \frac{1}{(1-p^{-\lambda l_1})^{1/l_1}} \|f\|_{M\dot{K}^{\alpha,\lambda}_{l_1,q_1}(\mathbb{Q}_p^n)}.$$

In particular, for $\beta = 0$, we get the estimates more accurately.

THEOREM 2. *Suppose* $1 < q < \infty$, $0 < l < \infty$ *and* $\lambda > 0$. *If* $\alpha < \lambda$, *then* $\mathscr{H}^p$ *is a bounded operator on* $MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)$ *and*

$$\|\mathscr{H}^p\|_{M\dot{K}^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n) \to M\dot{K}^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)} \approx \frac{1-p^{-n}}{1-p^{\alpha-n/q'-\lambda}}.$$

THEOREM 3. *Suppose* $1 < q < \infty$, $0 < l < \infty$ *and* $\lambda > 0$. *If* $-\frac{n}{q} + \lambda < \alpha < \lambda$, *then* $\mathscr{H}^{p,*}$ *is a bounded operator on* $MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)$ *and*

$$\|\mathscr{H}^{p,*}\|_{M\dot{K}^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n) \to M\dot{K}^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)} \approx \frac{1-p^{-n}}{p^{\alpha+n/q-\lambda}-1}.$$

Proof of Theorem 1. (i) First, we have

$$\begin{aligned} \|(\mathscr{H}_\beta^p f)\chi_k\|^{q_2}_{L^{q_2}(\mathbb{Q}_p^n)} &= \int_{S_k} \left| \frac{1}{|\mathbf{x}|_p^{n-\beta}} \int_{|\mathbf{t}|_p \leqslant |\mathbf{x}|_p} f(\mathbf{t}) d\mathbf{t} \right|^{q_2} d\mathbf{x} \\ &\leqslant p^{kq_2(\beta-n)} \int_{S_k} \Big(\sum_{j=-\infty}^{k} \int_{S_j} |f(\mathbf{t})| d\mathbf{t} \Big)^{q_2} d\mathbf{x}. \end{aligned}$$

Use Hölder's inequality, we get

$$\int_{S_j} |f(\mathbf{t})| d\mathbf{t} \leqslant (p^{jn}(1-p^{-n}))^{1/q_1'} \|f\chi_j\|_{L^{q_1}(\mathbb{Q}_p^n)}. \tag{1}$$

Note that $1/q_1 - 1/q_2 = \beta/n$, therefore

$$\|(\mathscr{H}_\beta^p f)\chi_k\|_{L^{q_2}(\mathbb{Q}_p^n)} \leqslant (1-p^{-n})^{1-\beta/n} \sum_{j=-\infty}^{k} p^{(j-k)n/q_1'} \|f\chi_j\|_{L^{q_1}(\mathbb{Q}_p^n)}.$$

By the definition of Morrey-Herz space on p-adic fields, we get

$$\begin{aligned}
&\|(\mathscr{H}_\beta^p f)\chi_k\|_{MK^{\alpha,\lambda}_{l_2,q_2}(\mathbb{Q}_p^n)} \\
&= (1-p^{-n})^{1-\beta/n} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha l_2} \left(\sum_{j=-\infty}^{k} p^{(j-k)n/q_1'} \|f\chi_j\|_{L^{q_1}(\mathbb{Q}_p^n)} \right)^{l_2} \right)^{1/l_2}.
\end{aligned}$$

By Jensen's inequality, so

$$\begin{aligned}
&\left(\sum_{k=-\infty}^{k_0} p^{k\alpha l_2} \left(\sum_{j=-\infty}^{k} p^{(j-k)n/q_1'} \|f\chi_j\|_{L^{q_1}(\mathbb{Q}_p^n)} \right)^{l_2} \right)^{1/l_2} \\
&\leqslant \left(\sum_{k=-\infty}^{k_0} p^{k\alpha l_1} \left(\sum_{j=-\infty}^{k} p^{(j-k)n/q_1'} \|f\chi_j\|_{L^{q_1}(\mathbb{Q}_p^n)} \right)^{l_1} \right)^{1/l_1}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|f\chi_j\|_{L^{q_1}(\mathbb{Q}_p^n)} &= p^{-j\alpha} \left(p^{j\alpha l_1} \|f\chi_j\|^{l_1}_{L^{q_1}(\mathbb{Q}_p^n)} \right)^{1/l_1} \\
&\leqslant p^{-j\alpha} \left(\sum_{i=-\infty}^{j} p^{i\alpha l_1} \|f\chi_i\|^{l_1}_{L^{q_1}(\mathbb{Q}_p^n)} \right)^{1/l_1} \\
&\leqslant p^{j(\lambda-\alpha)} \|f\|_{MK^{\alpha,\lambda}_{l_1,q_1}(\mathbb{Q}_p^n)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|(\mathscr{H}_\beta^p f)\chi_k\|_{MK^{\alpha,\lambda}_{l_2,q_2}(\mathbb{Q}_p^n)} \\
&\leqslant (1-p^{-n})^{1-\beta/n} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\lambda l_1} \left(\sum_{j=-\infty}^{k} p^{(k-j)(\alpha-n/q_1'-\lambda)} \right)^{l_1} \right)^{1/l_1} \|f\|_{MK^{\alpha,\lambda}_{l_1,q_1}(\mathbb{Q}_p^n)}.
\end{aligned}$$

When $\alpha < n/q_1' + \lambda$, the series

$$\sum_{j=-\infty}^{k} p^{(k-j)(\alpha-n/q_1'-\lambda)} = \frac{1}{1-p^{\alpha-n/q_1'-\lambda}}$$

and for $\lambda > 0$,

$$\left(\sum_{k=-\infty}^{k_0} p^{k\lambda l_1} \right)^{1/l_1} = \frac{p^{k_0\lambda}}{(1-p^{-\lambda l_1})^{1/l_1}}.$$

So we get

$$\|\mathscr{H}_\beta^p f\|_{MK^{\alpha,\lambda}_{l_2,q_2}(\mathbb{Q}_p^n)} \leqslant \frac{(1-p^{-n})^{1-\beta/n}}{1-p^{\alpha-n/q_1'+\lambda}} \frac{1}{(1-p^{-\lambda l_1})^{1/l_1}} \|f\|_{MK^{\alpha,\lambda}_{l_1,q_1}(\mathbb{Q}_p^n)}.$$

(ii) First, by (1) and $1/q_1 - 1/q_2 = \beta/n$, we have

$$\begin{aligned}
\|(\mathscr{H}_\beta^{p,*} f)\chi_k\|^{q_2}_{L^{q_2}(\mathbb{Q}_p^n)} &= \int_{S_k} \left| \sum_{j=k+1}^{\infty} \int_{S_j} \frac{f(\mathbf{t})}{|\mathbf{t}|_p^{n-\beta}} d\mathbf{t} \right|^{q_2} d\mathbf{x} \\
&= p^{kn}(1-p^{-n}) \left(\sum_{j=k+1}^{\infty} p^{j(\beta-n)} \int_{S_j} |f(\mathbf{t})| d\mathbf{t} \right)^{q_2} \\
&\leqslant (1-p^{-n})^{q_2(1-\beta/n)} \left(\sum_{j=k+1}^{\infty} p^{(k-j)n/q_2} \|f\chi_j\|_{L^{q_1}(\mathbb{Q}_p^n)} \right)^{q_2}.
\end{aligned}$$

Therefore, by a similar argument as the proof of (i), and note that $\alpha > -n/q_2 + \lambda$, we have

$$\|\mathscr{H}_\beta^{p,*} f\|_{MK^{\alpha,\lambda}_{l_2,q_2}(\mathbb{Q}_p^n)} \leqslant \frac{(1-p^{-n})^{1-\beta/n}}{p^{\alpha+n/q_2-\lambda}-1} \frac{1}{(1-p^{-\lambda l})^{1/l_1}} \|f\|_{MK^{\alpha,\lambda}_{l_1,q_1}(\mathbb{Q}_p^n)}. \qquad \square$$

Proof of Theorem 2. First, by Theorem 1 (i), we have $\mathscr{H}^p$ is bounded on $MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)$ and

$$\|\mathscr{H}^p\|_{MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)\to MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)} \leqslant \frac{1-p^{-n}}{1-p^{\alpha-n/q'-\lambda}} \frac{1}{(1-p^{-\lambda l})^{1/l}} \tag{2}$$

On the other hand, we choose the function

$$f_0(\mathbf{t}) = |\mathbf{t}|_p^{-\alpha-\frac{n}{q}+\lambda}.$$

For $\alpha < \lambda$, by computation, we can obtain $f_0 \in MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)$ (see also [2]) and

$$\begin{aligned}
\mathscr{H}^p f_0(\mathbf{x}) &= \frac{1}{|x|_p^n} \sum_{j=-\infty}^{\log_p |\mathbf{x}|_p} \int_{S_j} |\mathbf{t}|_p^{-\alpha-\frac{n}{q}+\lambda} d\mathbf{t} \\
&= \frac{1-p^{-n}}{|\mathbf{x}|_p^n} \sum_{j=-\infty}^{\log_p |\mathbf{x}|_p} p^{j(-\alpha+\frac{n}{q'}+\lambda)} \\
&= \frac{1-p^{-n}}{1-p^{\alpha-n/q'-\lambda}} |\mathbf{x}|_p^{-\alpha-\frac{n}{q}+\lambda}
\end{aligned}$$

for $\alpha < \lambda < \frac{n}{q'} + \lambda$. Therefore

$$\mathscr{H}^p f_0(\mathbf{x}) = \frac{1-p^{-n}}{1-p^{\alpha-n/q'-\lambda}} f_0(\mathbf{x}).$$

Suppose $\mathscr{H}^p$ is bounded on $MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}^n_p)$, it follows that

$$\frac{1-p^{-n}}{1-p^{\alpha-n/q'-\lambda}} \leqslant \|\mathscr{H}^p\|_{MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}^n_p)\to MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}^n_p)} < \infty. \tag{3}$$

By (2) and (3), we complete the proof. □

Proof of Theorem 3. The proof is similar to that of Theorem 2. Here note that we choose the same function $f_0(\mathbf{t}) = |\mathbf{t}|_p^{-\alpha-\frac{n}{q}+\lambda}$. For $\lambda > \alpha$, we have $f_0 \in MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}^n_p)$ and

$$\begin{aligned}
\mathscr{H}^{p,*} f_0(\mathbf{x}) &= \sum_{j=\log_p |x|_p+1}^{\infty} \int_{S_j} |\mathbf{t}|_p^{-\alpha-\frac{n}{q}+\lambda-n} d\mathbf{t} \\
&= (1-p^{-n}) \sum_{j=\log_p |\mathbf{x}|_p+1}^{\infty} p^{j(-\alpha-n/q+\lambda)} \\
&= \frac{1-p^{-n}}{p^{\alpha+n/q-\lambda}} f_0(\mathbf{x}).
\end{aligned}$$

for $\alpha > -\frac{n}{q}+\lambda$. □

3. Estimates for commutators of Hardy operators

THEOREM 4. *Suppose* $1 < q_1, q_2 < \infty$, $0 \leqslant \beta < n$, $1/q_1 - 1/q_2 = (\beta+\gamma)/n$ *and* $\gamma > 0$. *If* $b \in \Lambda_\gamma(\mathbb{Q}^n_p)$, *then*

$$\|\mathscr{H}^p_{\beta,b} f\|_{L^{q_2}(\mathbb{Q}^n_p)} \leqslant C\|b\|_{\Lambda_\gamma(\mathbb{Q}^n_p)} \|f\|_{L^{q_1}(\mathbb{Q}^n_p)};$$

and

$$\|\mathscr{H}^{p,*}_{\beta,b} f\|_{L^{q_2}(\mathbb{Q}^n_p)} \leqslant C\|b\|_{\Lambda_\gamma(\mathbb{Q}^n_p)} \|f\|_{L^{q_1}(\mathbb{Q}^n_p)}.$$

THEOREM 5. *Suppose* $0 < l_1 \leqslant l_2 < \infty$, $1 < q_1, q_2 < \infty$, $0 \leqslant \beta < n$, $1/q_1 - 1/q_2 = (\beta+\gamma)/n$ *and* $\gamma > 0$. *If* $b \in \Lambda_\gamma(\mathbb{Q}^n_p)$, *we have*

(i) if $\alpha < n/q_1'$, *then*

$$\|\mathscr{H}^p_{\beta,b} f\|_{\dot{K}^{\alpha,l_2}_{q_2}(\mathbb{Q}^n_p)} \leqslant C\|b\|_{\Lambda_\gamma(\mathbb{Q}^n_p)} \|f\|_{\dot{K}^{\alpha,l_1}_{q_1}(\mathbb{Q}^n_p)};$$

(ii) if $\alpha > -n/q_2$, *then*

$$\|\mathscr{H}^{p,*}_{\beta,b} f\|_{\dot{K}^{\alpha,l_2}_{q_2}(\mathbb{Q}^n_p)} \leqslant C\|b\|_{\Lambda_\gamma(\mathbb{Q}^n_p)} \|f\|_{\dot{K}^{\alpha,l_1}_{q_1}(\mathbb{Q}^n_p)}.$$

THEOREM 6. *Suppose* $0 < l_1 \leqslant l_2 < \infty$, $1 < q_1, q_2 < \infty$, $0 \leqslant \beta < n$, $1/q_1 - 1/q_2 = (\beta+\gamma)/n$ *and* $\gamma, \lambda > 0$. *If* $b \in \Lambda_\gamma(\mathbb{Q}^n_p)$, *we have* *(i) if $\alpha < n/q_1' + \lambda$, then*

$$\|\mathscr{H}^p_{\beta,b} f\|_{M\dot{K}^{\alpha,\lambda}_{l_2,q_2}(\mathbb{Q}^n_p)} \leqslant \frac{(1-p^{-n})^{1-(\beta+\gamma)/n}}{1-p^{\alpha-n/q_1'-\lambda}} \frac{1}{(1-p^{-\lambda l_1})^{1/l_1}} \|b\|_{\Lambda_\gamma(\mathbb{Q}^n_p)} \|f\|_{M\dot{K}^{\alpha,\lambda}_{l_1,q_1}(\mathbb{Q}^n_p)};$$

(ii) if $\alpha > -n/q_2 + \lambda$, then

$$\|\mathscr{H}^{p,*}_{\beta,b} f\|_{M\dot{K}^{\alpha,\lambda}_{l_2,q_2}(\mathbb{Q}^n_p)} \leqslant \frac{(1-p^{-n})^{1-(\beta+\gamma)/n}}{p^{\alpha+n/q_2-\lambda}-1} \frac{1}{(1-p^{-\lambda l_1})^{1/l_1}} \|b\|_{\Lambda_\gamma(\mathbb{Q}^n_p)} \|f\|_{M\dot{K}^{\alpha,\lambda}_{l_1,q_1}(\mathbb{Q}^n_p)}.$$

THEOREM 7. *Suppose $0 < l_1 \leqslant l_2 < \infty$, $1 < q_1, q_2 < \infty$, $0 \leqslant \beta < n$, $1/q_1 - 1/q_2 = (\beta+\gamma)/n$ and $\gamma, \lambda > 0$. If $b \in CMO^{\max\{q_1', q_2\}}(\mathbb{Q}^n_p)$, we have*
(i) if $\alpha < n/q_1' + \lambda$, then

$$\|\mathscr{H}^p_{\beta,b} f\|_{M\dot{K}^{\alpha,\lambda}_{l_2,q_2}(\mathbb{Q}^n_p)} \leqslant C\|b\|_{CMO^{\max\{q_1',q_2\}}(\mathbb{Q}^n_p)} \|f\|_{M\dot{K}^{\alpha,\lambda}_{l_1,q_1}(\mathbb{Q}^n_p)};$$

(ii) if $\alpha > -n/q_2 + \lambda$, then

$$\|\mathscr{H}^{p,*}_{\beta,b} f\|_{M\dot{K}^{\alpha,\lambda}_{l_2,q_2}(\mathbb{Q}^n_p)} \leqslant C\|b\|_{CMO^{\max\{q_1',q_2\}}(\mathbb{Q}^n_p)} \|f\|_{M\dot{K}^{\alpha,\lambda}_{l_1,q_1}(\mathbb{Q}^n_p)}.$$

REMARK 1. If $\beta = 0$ in all of the above theorems, then we obtain the corresponding estimates for the p-adic Hardy operators.

Proof of Theorem 4. Because of $\dot{K}^{0,q}_q(\mathbb{Q}^n_p) = L^q(\mathbb{Q}^n_p)$, Theorem 4 is a special case of Theorem 5. $\square$

Proof of Theorem 5. The proofs of (i) and (ii) are similar, so we only prove (i). First, we have

$$\begin{aligned}\|(\mathscr{H}^p_{\beta,b} f)\chi_k\|^{q_2}_{L^{q_2}(\mathbb{Q}^n_p)} &\leqslant \int_{S_k} \left(\frac{1}{|\mathbf{x}|_p^{n-\beta}} \int_{|\mathbf{t}|_p \leqslant |\mathbf{x}|_p} |f(\mathbf{t})||b(\mathbf{x})-b(\mathbf{t})| d\mathbf{t} \right)^{q_2} d\mathbf{x} \\ &= p^{kq_2(\beta-n)} \int_{S_k} \left(\sum_{i=-\infty}^{k} \int_{S_i} |f(\mathbf{t})||b(\mathbf{x})-b(\mathbf{t})| d\mathbf{t} \right)^{q_2} d\mathbf{x}.\end{aligned}$$

By $b \in \Lambda_\gamma(\mathbb{Q}^n_p)$,

$$|b(\mathbf{x}) - b(\mathbf{t})| \leqslant |\mathbf{x}-\mathbf{t}|_p^\gamma \|b\|_{\Lambda_\gamma(\mathbb{Q}^n_p)}.$$

Since $|\mathbf{t}|_p \leqslant |\mathbf{x}|_p$,

$$|\mathbf{x}-\mathbf{t}|_p^\gamma \leqslant \max\{|\mathbf{x}|_p, |\mathbf{t}|_p\}^\gamma = |\mathbf{x}|_p^\gamma.$$

Therefore, by (1) and $1/q_1 - 1/q_2 = (\beta+\gamma)/n$, we have

$$\|(\mathscr{H}^p_{\beta,b} f)\chi_k\|_{L^{q_2}(\mathbb{Q}^n_p)} \leqslant (1-p^{-n})^{1-(\beta+\gamma)/n} \|b\|_{\Lambda_\gamma(\mathbb{Q}^n_p)} \sum_{i=-\infty}^{k} p^{(i-k)n/q_1'} \|f\chi_i\|_{L^{q_1}(\mathbb{Q}^n_p)} \quad (4).$$

In view of the definition of Herz space and Jensen's inequality, we get

$$\begin{aligned}&\|\mathscr{H}^p_{\beta,b}f\|_{\dot{K}^{\alpha,l_2}_{q_2}(\mathbb{Q}_p^n)}\\ &\leqslant (1-p^{-n})^{1-(\beta+\gamma)/n}\|b\|_{\Lambda_\gamma(\mathbb{Q}_p^n)}\left(\sum_{k\in\mathbb{Z}}p^{k\alpha l_1}\left(\sum_{i=-\infty}^{k}p^{(i-k)n/q_1'}\|f\chi_i\|_{L^{q_1}(\mathbb{Q}_p^n)}\right)^{l_1}\right)^{1/l_1}.\end{aligned}$$

For $\alpha < n/q_1'$, we apply the same method as in [9] (see p. 148) to obtain the following estimate:

$$\sum_{k\in\mathbb{Z}}p^{k\alpha l_1}\left(\sum_{i=-\infty}^{k}p^{\frac{(i-k)n}{q_1'}}\|f\chi_i\|_{L^{q_1}(\mathbb{Q}_p^n)}\right)^{l_1}\leqslant C\|f\|^{l_1}_{\dot{K}^{\alpha,l_1}_{q_1}(\mathbb{Q}_p^n)},$$

where

$$C=\begin{cases}\frac{1}{1-p^{(\alpha-n/q_1')l_1}}, & 0<1/l_1\leqslant 1.\\ \frac{1}{1-p^{(\alpha-n/q_1')l_1/2}}\left(\frac{1}{1-p^{(\alpha-n/q_1')l_1'/2}}\right)^{l_1/l_1'}, & 1/l_1>1.\end{cases}$$

Therefore, we have

$$\sum_{k\in\mathbb{Z}}p^{k\alpha l_1}\left(\sum_{i=-\infty}^{k}p^{\frac{(i-k)n}{q_1'}}\|f\chi_i\|_{L^{q_1}(\mathbb{Q}_p^n)}\right)^{l_1}=C\|f\|^{l_1}_{\dot{K}^{\alpha,l_1}_{q_1}(\mathbb{Q}_p^n)}. \qquad \square$$

Proof of Theorem 6. By (4), we can prove it by the same procedure of the proof of Theorem 1. $\square$

Proof of Theorem 7. The proof is similar to Theorem 3, the difference is we need the following estimate and it is proved in [23],

$$\|(\mathscr{H}^p_{\beta,b}f)\chi_k\|_{L^{q_2}(\mathbb{Q}_p^n)}\leqslant C\|b\|_{CMO^{\max\{q_1',q_2\}}(\mathbb{Q}_p^n)}\sum_{i=-\infty}^{k}(k-i)p^{(i-k)n/q_2'}\|f\chi_i\|_{L^{q_1}(\mathbb{Q}_p^n)}. \qquad \square$$

Acknowledgements. This research was supported by the National Natural Science Foundation of China grants 11526067 and 51505295, Chinese Universities Scientific Fund WK0010000028, and the National Science Foundation of Zhejiang Province of China grant LY16A010013.

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(Received February 10, 2015)

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