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SOME BIVARIATE DURRMEYER OPERATORS BASED ON q–INTEGERS

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Abstract. In the present paper we introduce a q-analogue of the bivariate Durrmeyer operators. A convergence theorem for these operators is established and the rate of convergence in terms of modulus of continuity is determined. Also, a Voronovskaja type theorem has been investigated for these operators.

1. Introduction

Durrmeyer introduced in 1967 an integral modification of the well known Bernstein operator in order to approximate Lebesque integrable functions on the interval $[0,1]$. These operators, called now Durrmeyer operators, were defined in [11] by the formula

$$D_m(f;x) = (m+1)\sum_{k=0}^{m} p_{m,k}(x)\int_0^1 f(t)p_{m,k}(t)dt \tag{1.1}$$

for each $f \in L_1[0,1]$, $x \in [0,1]$ where the Bernstein basis function is defined by

$$p_{m,k}(x) = \binom{m}{k} x^k(1-x)^{m-k}.$$

In 1981 Derriennic [8] is the first who studied the operators (1.1) in details.

Lupas [14] in 1987, and independently Phillips [16] in 1997, introduced q-analogues of Bernstein polynomials. After that, several researches have studied these polynomials and established many interesting properties. Important results in this direction were obtained in [1], [2], [3], [4], [5], [12], [13], [15], [16], [17], [19], [20].

First, let us to recall some basic definitions of q-calculus. Let $q>0$ be given. For each non-negative integer k, the q-integer $[k]_q$ and the q-factorial $[k]_q!$ are respectively defined by

$$[k]_q := \begin{cases} (1-q^k)/(1-q), & q \neq 1, \\ k, & q = 1, \end{cases}$$

and

$$[k]_q! := \begin{cases} [k]_q[k-1]_q \dots [1]_q, & k \geqslant 1, \\ 1, & k = 0. \end{cases}$$

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For the integers $m \geqslant k \geqslant 0$, the q-binomial coefficients are defined by

$$\begin{bmatrix} m \\ k \end{bmatrix}_q := \frac{[m]_q!}{[k]_q![m-k]_q!}.$$

Denote

$$(a+b)_q^m := \prod_{j=0}^{m-1}(a+q^j b) = (a+b)(a+qb)\dots(a+q^{m-1}b),$$

$$p_{m,k}(q;x) := \begin{bmatrix} m \\ k \end{bmatrix}_q x^k(1-x)_q^{m-k}.$$

Using the above notations, the q-Bernstein polynomials introduced by Phillips [16] can be expressed as

$$B_{m,q}(f;x) := \sum_{k=0}^{m} p_{m,k}(q;x) f\left(\frac{[k]_q}{[m]_q}\right)$$

for each positive integer m and $f \in C[0,1]$.

The q-analogue of integration, introduced by Thomae [21] is defined by

$$\int_0^a f(t)d_qt := (1-q)\sum_{k=1}^{\infty}(aq^k)q^k,\ 0<q<1.$$

Using the above definitions, Gupta [12] introduced the q-Durrmeyer operators defined as

$$D_{m,q}(f;x) := [m+1]_q \sum_{k=0}^{m} q^{-k} p_{m,k}(q;x) \int_0^1 f(t)p_{m,k}(q;qt)d_qt. \tag{1.2}$$

In the case $q=1$ the operators (1.2) reduce to the classical Durrmeyer operators (1.1).

Other q-variants of the Durrmeyer operators were studied by Derriennic [10], Muraru and Acu [15].

Let $C([0,1]^2)$ be the space of bivariate real valued functions continuous on $[0,1]^2$. Suppose m,n are positive integers and q_1,q_2 are real parameters satisfying the conditions $0<q_1\leqslant 1$, $0<q_2\leqslant 1$. Barbosu [6] introduced the following bivariate q-Bernstein operators

$$B_{m,n,q_1,q_2}(f;x,y) := \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(q_1;x)p_{n,j}(q_2;y) f\left(\frac{[k]_{q_1}}{[m]_{q_1}}, \frac{[j]_{q_2}}{[n]_{q_2}}\right) \tag{1.3}$$

defined for each positive integers m,n and $f \in C([0,1]^2)$.

When $q_1=q_2=1$, the Bernstein operators (1.3) reduce to the classical bivariate Bernstein operators.

The present note deals with the study of the bivariate q-Durrmeyer operators, which are a q-integral modification of operators (1.3). First, we estimate the moments for these q-operators and then, applying the Korovkin-type theorem for bivariate linear positive operators we obtain a convergence property of the sequence of operators. Using the Shisha-Mond theorem for the bivariate case, we give an estimation of the rate of convergence for the sequence of bivariate q-Durrmeyer. In the last section we give a Voronovskaja type theorem.

2. Definition of the bivariate q-Durrmeyer operators and estimation of their moments

Let $f \in C([0,1]^2)$. Using the method of parametric extensions [7] and the operator (1.2), it follows that the bivariate q-Durrmeyer operators are defined for each positive integers m,n by the formula

$$D_{m,n,q_1,q_2}(f;x,y) := [m+1]_{q_1}[n+1]_{q_2} \sum_{k=0}^{m} \sum_{j=0}^{n} q_1^{-k} q_2^{-j} p_{m,k}(q_1;x) q_{n,j}(q_2;y)$$
$$\times \int_0^1 \int_0^1 f(s,t) p_{m,k}(q_1;q_1 s) p_{n,j}(q_2;q_2 t) d_{q_1} s \, d_{q_2} t. \qquad (2.1)$$

In order to prove the main results we will establish the following statements for the q-Durrmeyer operators (1.2).

LEMMA 2.1. *Let $s \in \mathbb{N}$ and $c_i(s) > 0$, $i = \overline{0,s}$ such that*

$$[k+1]_q [k+2]_q \cdots [k+s]_q = \sum_{i=0}^{s} c_i(s) [k]_q^i.$$

For the q-Durrmeyer operators (1.2) the following statement is true

$$D_{m,q}(e_s;x) = \frac{[m+1]_q!}{[m+s+1]_q!} \sum_{i=0}^{s} c_i(s) [m]_q^i B_{m,q}(e_i;x), \qquad (2.2)$$

where $e_i(x) = x^i$.

Proof. Let $s = 0, 1, \ldots$. We have

$$\int_0^1 e_s(t) p_{m,k}(q;qt) d_q t = \begin{bmatrix} m \\ k \end{bmatrix}_q q^k \int_0^1 t^{k+s} (1-qt)_q^{m-k} d_q t$$
$$= \begin{bmatrix} m \\ k \end{bmatrix}_q q^k \beta_q(k+s+1, m-k+1) = q^k \frac{[m]_q! [k+s]_q!}{[k]_q! [m+s+1]_q!},$$

where

$$\beta_q(a,b) := \int_0^1 t^{a-1} (1-qt)_q^{b-1} d_q t, \quad a,b > 0$$

is the q-Beta function.

Using the above result we obtain

$$D_{m,q}(e_s;x) = [m+1]_q \sum_{k=0}^{m} q^{-k} p_{m,k}(q;x) \int_0^1 e_s(t) p_{m,k}(q;qt) d_q t$$
$$= \frac{[m+1]_q!}{[m+s+1]_q!} \sum_{k=0}^{m} [k+1]_q [k+2]_q \cdots [k+s]_q p_{m,k}(q;x)$$
$$= \frac{[m+1]_q!}{[m+s+1]_q!} \sum_{k=0}^{m} \sum_{i=0}^{s} c_i(s) [k]_q^i p_{m,k}(q;x)$$
$$= \frac{[m+1]_q!}{[m+s+1]_q!} \sum_{i=0}^{s} c_i(s) [m]_q^i B_{m,q}(e_i;x). \quad \square$$

LEMMA 2.2. *The q-Durrmeyer operators* (1.2) *verify*

i) $D_{m,q}(e_0;x)=1$,

ii) $D_{m,q}(e_1;x)=\dfrac{1+qx[m]_q}{[m+2]_q}$,

iii) $D_{m,q}(e_2;x)=\dfrac{q^3x^2[m]_q([m]_q-1)+(1+q)^2qx[m]_q+1+q}{[m+3]_q[m+2]_q}$,

iv) $D_{m,q}(e_3;x)$

$$=\frac{q^6([m]_q-[2]_q)[m]_q([m]_q-1)x^3+q^3[3]_q^2[m]_q([m]_q-1)x^2+q[2]_q[3]_q^2[m]_qx+[2]_q[3]_q}{[m+2]_q[m+3]_q[m+4]_q},$$

v) $D_{m,q}(e_4;x)$

$$=\frac{q^{10}[m]_q([m]_q-1)([m]_q-[2]_q)([m]_q-[3]_q)x^4+q^6(q^2+1)^2[2]_q^2[m]_q([m]_q-[2]_q)([m]_q-1)x^3}{[m+2]_q[m+3]_q[m+4]_q[m+5]_q}$$

$$+\frac{q^3(q^2+1)^2[2]_q[3]_q^2[m]_q([m]_q-1)x^2+q(q^2+1)^2[2]_q^3[3]_q[m]_qx+[2]_q[3]_q[4]_q}{[m+2]_q[m+3]_q[m+4]_q[m+5]_q}.$$

Proof. This result is obtained using the relation (2.2) and the values of the q-Bernstein operators for the test functions, namely:

$$B_{m,q}(e_0;x)=1,\ B_{m,q}(e_1;x)=x,\ B_{m,q}(e_2;x)=x^2+\frac{x(1-x)}{[m]_q},$$

$$B_{m,q}(e_3;x)=\left(-\frac{2+q}{[m]_q}+1+\frac{1+q}{[m]_q^2}\right)x^3+\frac{2+q}{[m]_q}\left(1-\frac{1}{[m]_q}\right)x^2+\frac{x}{[m]_q^2},$$

$$B_{m,q}(e_4;x)=\frac{([m]_q-1)([m]_q-1-q)([m]_q-1-q-q^2)}{[m]_q^3}x^4$$

$$+\frac{([m]_q-1)([m]_q-1-q)(q^2+2q+3)}{[m]_q^3}x^3+\frac{([m]_q-1)(q^2+3q+3)}{[m]_q^3}x^2+\frac{x}{[m]_q^3}.\qquad\square$$

Using Lemma 2.2, we shall prove

THEOREM 2.1. *The bivariate q-Durrmeyer operators* (2.1) *satisfy the following equations*

$(i)\quad D_{m,n,q_1,q_2}(1;x,y)=1$,

$(ii)\quad D_{m,n,q_1,q_2}(s;x,y)=\dfrac{1+q_1x[m]_{q_1}}{[m+2]_{q_1}}$,

$(iii)\quad D_{m,n,q_1,q_2}(t;x,y)=\dfrac{1+q_2y[n]_{q_2}}{[n+2]_{q_2}}$,

$(iv)\quad D_{m,n,q_1,q_2}(st;x,y)=\dfrac{1+q_1x[m]_{q_1}}{[m+2]_{q_1}}\cdot\dfrac{1+q_2y[n]_{q_2}}{[n+2]_{q_2}}$,

$(v)\quad D_{m,n,q_1,q_2}(s^2;x,y)=\dfrac{q_1^3x^2[m]_{q_1}([m]_{q_1}-1)+(1+q_1)^2q_1x[m]_{q_1}+1+q_1}{[m+3]_{q_1}[m+2]_{q_1}}$,

$(vi)\quad D_{m,n,q_1,q_2}(t^2;x,y)=\dfrac{q_2^3y^2[n]_{q_2}([n]_{q_2}-1)+(1+q_2)^2q_2x[n]_{q_2}+1+q_2}{[n+3]_{q_2}[n+2]_{q_2}}$.

Proof. Taking the definition (2.1) into account one obtains the following identities

$$\begin{aligned} D_{m,n,q_1,q_2}(1;x,y) &= D_{m,q_1}(1;x)D_{n,q_2}(1;y),\\ D_{m,n,q_1,q_2}(s;x,y) &= D_{m,q_1}(s;x)D_{n,q_2}(1;y),\\ D_{m,n,q_1,q_2}(t;x,y) &= D_{m,q_1}(1;x)D_{n,q_2}(t;y),\\ D_{m,n,q_1,q_2}(st;x,y) &= D_{m,q_1}(s;x)D_{n,q_2}(t;y),\\ D_{m,n,q_1,q_2}(s^2;x,y) &= D_{m,q_1}(s^2;x)D_{n,q_2}(1;y),\\ D_{m,n,q_1,q_2}(t^2;x,y) &= D_{m,q_1}(1;x)D_{n,q_2}(t^2;y). \end{aligned}$$

Next, one applies Lemma 2.2. □

COROLLARY 2.1. *The following identities hold true*

$$D_{m,n,q_1,q_2}((s-x)^2;x,y) = \frac{x^2\{[m]_{q_1}([m]_{q_1}-1)q_1^3-2[m]_{q_1}[m+3]_{q_1}q_1+[m+2]_{q_1}[m+3]_{q_1}\}}{[m+2]_{q_1}[m+3]_{q_1}} + \frac{x\{[m]_{q_1}q_1(1+q_1)^2-2[m+3]_{q_1}\}+1+q_1}{[m+2]_{q_1}[m+3]_{q_1}},$$

$$D_{m,n,q_1,q_2}((t-y)^2;x,y) = \frac{y^2\{[n]_{q_2}([n]_{q_2}-1)q_2^3-2[n]_{q_2}[n+3]_{q_2}q_2+[n+2]_{q_2}[n+3]_{q_2}\}}{[n+2]_{q_2}[n+3]_{q_2}} + \frac{y\{[n]_{q_2}q_2(1+q_2)^2-2[n+3]_{q_2}\}+1+q_2}{[n+2]_{q_2}[n+3]_{q_2}}.$$

Proof. The linearity of D_{m,n,q_1,q_2} leads to

$$D_{m,n,q_1,q_2}((s-x)^2;x,y) = D_{m,n,q_1,q_2}(s^2;x,y) - 2xD_{m,n,q_1,q_2}(s;x,y) + x^2D_{m,n,q_1,q_2}(1;x,y).$$

Next, one applies Theorem 2.1. The second identity follows in a similar way. □

LEMMA 2.3. *Let $m,n\in\mathbb{N}$ and $q_1,q_2\in(0,1)$. Then*

i) $D_{m,n,q_1,q_2}\left((s-x)^2;x,y\right) \leqslant \dfrac{2}{[m+2]_{q_1}}\delta^2_{m,q_1}(x)$,

ii) $D_{m,n,q_1,q_2}\left((t-y)^2;x,y\right) \leqslant \dfrac{2}{[n+2]_{q_2}}\delta^2_{n,q_2}(y)$,

where $\delta^2_{m,q_1}(x) = \varphi^2(x) + \dfrac{3}{2[m+2]_{q_1}}$, $\delta^2_{n,q_2}(y) = \varphi^2(y) + \dfrac{3}{2[n+2]_{q_2}}$, $\varphi^2(x) = x(1-x)$, $x\in[0,1]$.

Proof. Using Corollary 2.1, we obtain

$$D_{m,n,q_1,q_2}\left((s-x)^2;x,y\right) = \frac{x^2}{[m+2]_{q_1}[m+3]_{q_1}}\left\{[m]_{q_1}q_1(q_1+1)^2-2[m+3]_{q_1}\right.$$

$$+ [m]_{q_1}([m]_{q_1}-1)q_1^3 - 2[m]_{q_1}[m+3]_{q_1}q_1 + [m+2]_{q_1}[m+3]_{q_1}\}$$
$$+ \frac{\varphi^2(x)}{[m+2]_{q_1}[m+3]_{q_1}}\left\{[m]_{q_1}q_1(q_1+1)^2 - 2[m+3]_{q_1}\right\} + \frac{q_1+1}{[m+2]_{q_1}[m+3]_{q_1}}$$
$$= \frac{x^2}{[m+2]_{q_1}[m+3]_{q_1}}A(m,q_1) + \frac{\varphi^2(x)}{[m+2]_{q_1}[m+3]_{q_1}}B(m,q_1) + \frac{q_1+1}{[m+2]_{q_1}[m+3]_{q_1}},$$

where

$$\begin{aligned} A(m,q_1) &:= [m]_{q_1}q_1(q_1+1)^2 - 2[m+3]_{q_1} + [m]_{q_1}([m]_{q_1}-1)q_1^3 - 2[m]_{q_1}[m+3]_{q_1}q_1 \\ &\quad + [m+2]_{q_1}[m+3]_{q_1}, \\ B(m,q_1) &:= [m]_{q_1}q_1(q_1+1)^2 - 2[m+3]_{q_1}. \end{aligned}$$

By direct computations, using the following relation of q-integers

$$[m+i]_{q_1} = [i]_{q_1} + q_1^i[m]_{q_1},$$

it follows

$$\begin{aligned} A(m,q_1) &= [m]_{q_1}q_1(q_1+1)^2 - 2(1+q_1+q_1^2+q_1^3[m]_{q_1}) + [m]_{q_1}([m]_{q_1}-1)q_1^3 \\ &\quad - 2[m]_{q_1}q_1(1+q_1+q_1^2+q_1^3[m]_{q_1}) + (1+q_1+q_1^2[m]_{q_1})(1+q_1+q_1^2+q_1^3[m]_{q_1}) \\ &= q_1^3(q_1-1)^2[m]_{q_1}^2 + q_1(q_1-1)(2q_1^2+1)[m]_{q_1} + (q_1-1)(q_1^2+q_1+1). \end{aligned}$$

Since $(q_1-1)(q_1^2+q_1+1) < 0$ we have

$$\begin{aligned} A(m,q_1) &\leqslant q_1^3(q_1-1)^2[m]_{q_1}^2 + q_1(q_1-1)(2q_1^2+1)[m]_{q_1} \\ &= \left(\frac{1-q_1^m}{1-q_1}\right)^2 q_1^3(q_1-1)^2 + q_1(q_1-1)(2q_1^2+1)\frac{1-q_1^m}{1-q_1} \\ &= (1-q_1^m)^2q_1^3 - q_1(2q_1^2+1)(1-q_1^m) \leqslant (1-q_1^m)^2q_1^3 \leqslant 1, \\ B(m,q_1) &= [m]_{q_1}q_1(q_1+1)^2 - 2(1+q_1+q_1^2+q_1^3[m]_{q_1}) \\ &= q_1(-q_1^2+2q_1+1)[m]_{q_1} - 2(q_1^2+q_1+1) \\ &\leqslant q_1(-q_1^2+2q_1+1)[m]_{q_1} \leqslant 2[m]_{q_1}. \end{aligned}$$

Therefore,

$$\begin{aligned} D_{m,n,q_1,q_2}\left((s-x)^2;x,y\right) &\leqslant \frac{x^2}{[m+2]_{q_1}[m+3]_{q_1}} + \frac{2[m]_{q_1}}{[m+2]_{q_1}[m+3]_{q_1}}\varphi^2(x) + \frac{2}{[m+2]_{q_1}[m+3]_{q_1}} \\ &\leqslant \frac{2}{[m+2]_{q_1}}\varphi^2(x) + \frac{3}{[m+2]_{q_1}^2} = \frac{2}{[m+2]_{q_1}}\left(\varphi^2(x) + \frac{3}{2[m+2]_{q_1}}\right). \end{aligned}$$

In a similar way can be proven the relation ii). □

3. Approximation properties of the bivariate q-Durrmeyer operators

To study the convergence of the sequence $\{D_{m,n,q_{1,m},q_{2,n}}(f;x,y)\}$ we shall use the following Korovkin type theorem, established by Volkov [22].

THEOREM 3.1. ([7], [22]) *Let $I,J \subseteq \mathbb{R}$ be compact intervals of the real axis and let $\{L_{m,n}f\}$ be a sequence of linear positive operators applying the space $C(I\times J)$ into itself. Suppose that the following relations*

(i) $L_{m,n}(1;x,y) = 1 + a_{m,n}(x,y)$,

(ii) $L_{m,n}(s;x,y) = x + b_{m,n}(x,y)$,

(iii) $L_{m,n}(t;x,y) = y + c_{m,n}(x,y)$,

(iv) $L_{m,n}(s^2+t^2;x,y) = x^2+y^2+d_{m,n}(x,y)$,

hold, for each $(x,y)\in I\times J$.

If the sequences $\{a_{m,n}(x,y)\}$, $\{b_{m,n}(x,y)\}$, $\{c_{m,n}(x,y)\}$, $\{d_{m,n}(x,y)\}$ converge to zero uniformly on $I\times J$, then the sequence $\{L_{m,n}f\}$ converges to f, uniformly on $I\times J$, for each $f\in C(I\times J)$.

Sufficient conditions for the uniform convergence of the sequence

$$\{D_{m,n,q_{1,m},q_{2,n}}(f;x,y)\}$$

are contained in the following

THEOREM 3.2. *Let $q_{1,m},q_{2,n}\in(0,1)$. If $\lim\limits_{m\to\infty} q_{1,m} = 1$ and $\lim\limits_{n\to\infty} q_{2,n} = 1$, the sequence $\{D_{m,n,q_{1,m},q_{2,n}}(f;x,y)\}$ converges to $f(x,y)$, uniformly on $[0,1]^2$ for each $f\in C([0,1]^2)$.*

Proof. When $q_{1,m}\to 1$ and $q_{2,n}\to 1$, then $[m]_{q_{1,m}}\to\infty$ and $[n]_{q_{2,n}}\to\infty$. For $\ell = 1,2,3$ we have $\lim\limits_{m\to\infty}\dfrac{[m+\ell]_{q_{1,m}}}{[m]_{q_{1,m}}} = 1$ and $\lim\limits_{n\to\infty}\dfrac{[n+l]_{q_{2,n}}}{[n]_{q_{2,n}}} = 1$. Applying Theorem 2.1, we get that $\lim\limits_{m,n\to\infty} D_{m,n,q_{1,m},q_{2,n}}(e_{ij};x,y) = e_{ij}(x,y)$ uniformly on $[0,1]^2$, where $e_{ij}(x,y) = x^iy^j$, $0\leqslant i+j\leqslant 2$ are the test functions. By virtue of Theorem 3.1, it follows that $\lim\limits_{m,n\to\infty} D_{m,n,q_{1,m},q_{2,n}}(f;x,y) = f(x,y)$, uniformly on $[0,1]^2$, for each $f\in C([0,1]^2)$. $\square$

In the following we give a numerical result which shows the rate of convergence of the operator D_{m,n,q_1,q_2} to certain function using Matlab algorithms.

EXAMPLE 3.1. Let us consider $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y) = x^2y^2+x^2y-2y^2$. The convergence of the bivariate q-Durrmeyer operator to the function f is illustrated in Figure 1 and Figure 2, respectively for $n_1 = n_2 = 50$, $q_1 = q_2 = 0.6$ and $n_1 = n_2 = 500$, $q_1 = q_2 = 0.9$, respectively. We remark that as the values of n_1 and n_2 increase, the error in the approximation of the function by the operator becomes smaller.

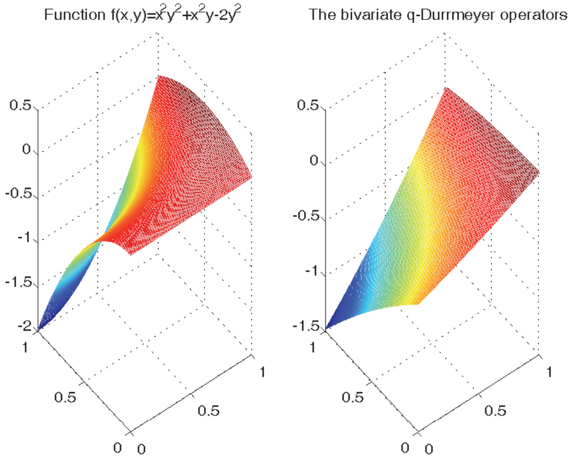

Figure 1: *Approximation process by* D_{n_1,n_2,q_1,q_2} *for* $n_1 = n_2 = 50$ *and* $q_1 = q_2 = 0.6$

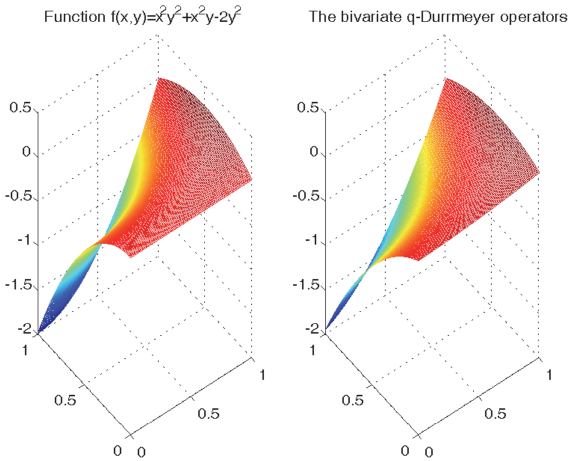

Figure 2: *Approximation process by* D_{n_1,n_2,q_1,q_2} *for* $n_1 = n_2 = 500$ *and* $q_1 = q_2 = 0.9$

An estimation of the rate of convergence can be obtained using the modulus of continuity for bivariate real valued functions. Recall that if $I, J \subseteq \mathbb{R}$ are compact intervals and $f \in \mathbb{R}^{I\times J}$ is bounded, the modulus of continuity is the function $\omega : [0,+\infty)^2 \to [0,+\infty)$, defined as

$$\omega(\delta_1,\delta_2) = \sup_{(x',y'),(x'',y'')\in I\times J} \left\{ |f(x',y') - f(x'',y'')| : |x'-x''| \leqslant \delta_1, |y'-y''| \leqslant \delta_2 \right\}.$$

Recall also the following variant of Shisha-Mond theorem [18].

THEOREM 3.3. ([7], [18]) *Let $I,J \subseteq \mathbb{R}$ be compact intervals, $B(I\times J)=\{f\in\mathbb{R}^{I\times J} | f$ bounded on $I\times J\}$ and $L: C(I\times J)\to B(I\times J)$ be a linear positive operator. For each $f\in C(I\times J)$, $(x,y)\in I\times J$ and any $\delta_1>0$, $\delta_2>0$, the following inequality*

$$|Lf(x,y)-f(x,y)| \leqslant |f(x,y)|\cdot|L(1;x,y)-1|$$
$$+\Big\{L(1;x,y)+\delta_1^{-1}\sqrt{L(1;x,y)L((s-x)^2;x,y)}+\delta_2^{-1}\sqrt{L(1;x,y)L((t-y)^2;x,y)}$$
$$+\delta_1^{-1}\delta_2^{-1}L(1;x,y)\sqrt{L((s-x)^2;x,y)L((t-y)^2;x,y)}\Big\}\,\omega(\delta_1,\delta_2)$$

holds.

THEOREM 3.4. *Suppose $q_{1,m},q_{2,n}\in(0,1)$ such that $\lim_{m\to\infty}q_{1,m}=1$, $\lim_{n\to\infty}q_{2,n}=1$. The following inequality*

$$|D_{m,n,q_{1,m},q_{2,n}}(f;x,y)-f(x,y)| \leqslant 4\omega\left(\sqrt{\frac{2}{[m+2]_{q_{1,m}}}\delta_{m,q_{1,m}}(x)},\sqrt{\frac{2}{[n+2]_{q_{2,n}}}\delta_{n,q_{2,n}}(y)}\right)$$

holds for each $f\in C([0,1]^2)$, $(x,y)\in[0,1]^2$, where $\delta_{m,q_{1,m}}(x)$ and $\delta_{n,q_{2,n}}(y)$ are defined as in Lemma 2.3.

Proof. Applying Theorem 3.3, taking Theorem 2.1 (i) and Lemma 2.3 into account, one arrives to

$$\begin{aligned}&\left|D_{m,n,q_{1,m},q_{2,n}}(f;x,y)-f(x,y)\right|\\ &\leqslant\Big\{1+\delta_1^{-1}\sqrt{D_{m,n,q_{1,m},q_{2,n}}((s-x)^2;x,y)}+\delta_2^{-1}\sqrt{D_{m,n,q_{1,m},q_{2,n}}((t-y)^2;x,y)}\\ &\quad+\delta_1^{-1}\delta_2^{-1}\sqrt{D_{m,n,q_{1,m},q_{2,n}}((s-x)^2;x,y)D_{m,n,q_{1,m},q_{2,n}}((t-y)^2;x,y)}\Big\}\,\omega(\delta_1,\delta_2)\\ &\leqslant 4\omega\left(\sqrt{\frac{2}{[m+2]_{q_{1,m}}}\delta_{m,q_{1,m}}(x)},\sqrt{\frac{2}{[n+2]_{q_{2,n}}}\delta_{n,q_{2,n}}(y)}\right).\quad\square\end{aligned}$$

For $0<\alpha_1\leqslant 1$ and $0<\alpha_2\leqslant 1$, we define the Lipschitz class $Lip_M(\alpha_1,\alpha_2)$ for the bivariate case as follows:

$$|f(s,t)-f(x,y)| \leqslant M|s-x|^{\alpha_1}|t-y|^{\alpha_2},$$

where $(t,s),(x,y)\in[0,1]\times[0,1]$ are arbitrary.

The next result gives the degree of approximation for the Durrmeyer operators.

THEOREM 3.5. *Let $f\in Lip_M(\alpha_1,\alpha_2)$ and $q_{1,m},q_{2,n}\in(0,1)$ such that $\lim_{m\to\infty}q_{1,m}=1$, $\lim_{n\to\infty}q_{2,n}=1$. Then, for all $(x,y)\in[0,1]\times[0,1]$, we have*

$$|D_{m,n,q_{1,m},q_{2,n}}(f;x,y)-f(x,y)| \leqslant M\delta_{m,q_{1,m}}^{\alpha_1/2}(x)\,\delta_{n,q_{2,n}}^{\alpha_1/2}(y),$$

where $\delta_{m,q_{1,m}}(x)$ and $\delta_{n,q_{2,n}}(y)$ are defined as in Lemma 2.3.

Proof. From $f \in Lip_M(\alpha_1, \alpha_2)$, it follows

$$\begin{aligned}\left|D_{m,n,q_{1,m},q_{2,n}}\left(|f(s,t)-f(x,y)|;x,y\right)\right| &\leqslant D_{m,n,q_{1,m},q_{2,n}}\left(|f(s,t)-f(x,y)|;x,y\right)\\ &\leqslant MD_{m,n,q_{1,m},q_{2,n}}\left(|s-x|^{\alpha_1}|t-y|^{\alpha_2};x,y\right)\\ &= MD_{m,q_{1,m}}\left(|s-x|^{\alpha_1};x)D_{n,q_{2,n}}(|t-y|^{\alpha_2};y\right).\end{aligned}$$

Using the Hölder's inequality with $\tilde{p} = \dfrac{2}{\alpha_1}$, $\tilde{q} = \dfrac{2}{2-\alpha_1}$ and $\tilde{p} = \dfrac{2}{\alpha_2}$, $\tilde{q} = \dfrac{2}{2-\alpha_2}$, respectively, we get

$$\begin{aligned}&\left|D_{m,n,q_{1,m},q_{2,n}}(f;x,y) - f(x,y)\right|\\ &\leqslant M\left\{D_{m,q_{1,m}}\left((s-x)^2;x\right)\right\}^{\frac{\alpha_1}{2}}\left\{D_{m,q_{1,m}}(1;x)\right\}^{\frac{2-\alpha_1}{2}}\\ &\quad\times\left\{D_{n,q_{2,n}}\left((t-y)^2;y\right)\right\}^{\frac{\alpha_2}{2}}\left\{D_{n,q_{2,n}}(1;y)\right\}^{\frac{2-\alpha_2}{2}}\\ &\leqslant M\delta_{m,q_{1,m}}^{\alpha_1/2}(x)\delta_{n,q_{2,n}}^{\alpha_1/2}(y). \quad \square\end{aligned}$$

Let $I = [0,1]$ and $C(I^2)$ be the space of all real valued continuous function on $I^2 = I \times I$ endowed with the norm $\|f\|_{C(I^2)} = \sup\limits_{(x,y)\in I^2} |f(x,y)|$.

THEOREM 3.6. *Let $f \in C^1(I^2)$ and $q_{1,m}, q_{2,n} \in (0,1)$ such that $\lim\limits_{m\to\infty} q_{1,m} = 1$ and $\lim\limits_{n\to\infty} q_{2,n} = 1$. Then, we have*

$$\left|D_{m,n,q_{1,m},q_{2,n}}(f;x,y) - f(x,y)\right| \leqslant \|f'_x\|_{C(I^2)}\delta_{m,q_{1,m}}(x) + \|f'_y\|_{C(I^2)}\delta_{n,q_{2,n}}(y),$$

where $\delta_{m,q_{1,m}}(x)$ and $\delta_{n,q_{2,n}}(y)$ are defined as in Lemma 2.3.

Proof. For $(s,t) \in I^2$ we have

$$f(s,t) - f(x,y) = \int_x^s f'_u(u,t)du + \int_y^t f'_v(x,v)dv.$$

Applying the Durrmeyer operator on both sides we get

$$\begin{aligned}&\left|D_{m,n,q_{1,m},q_{2,n}}(f;x,y) - f(x,y)\right|\\ &\leqslant D_{m,n,q_{1,m},q_{2,n}}\left(\left|\int_x^s f'_u(u,t)du\right|;x,y\right) + D_{m,n,q_{1,m},q_{2,n}}\left(\left|\int_y^t f'_v(x,v)dv\right|;x,y\right).\end{aligned}$$

Since

$$\left|\int_x^s f'_u(u,t)du\right| \leqslant \|f'_x\|_{C(I^2)}|s-x| \text{ and } \left|\int_y^t f'_v(x,v)dv\right| \leqslant \|f'_y\|_{C(I^2)}|t-y|,$$

it follows

$$\begin{aligned}&\left|D_{m,n,q_{1,m},q_{2,n}}(f;x,y) - f(x,y)\right|\\ &\leqslant \|f'_x\|_{C(I^2)}D_{m,q_{1,m}}(|s-x|;x) + \|f'_y\|_{C(I^2)}D_{n,q_{2,n}}(|t-y|;y).\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\left|D_{m,n,q_{1,m},q_{2,n}}(f;x,y)-f(x,y)\right|\\ &\leqslant \|f_x'\|_{C(I^2)}\left\{D_{m,q_{1,m}}\left((s-x)^2;x\right)\right\}^{1/2}\left\{D_{m,q_{1,m}}(1;x)\right\}^{1/2}\\ &\quad+\|f_y'\|_{C(I^2)}\left\{D_{n,q_{2,n}}\left((t-y)^2;x\right)\right\}^{1/2}\left\{D_{n,q_{2,n}}(1;y)\right\}^{1/2}\\ &\leqslant \|f_x'\|_{C(I^2)}\delta_{m,q_{1,m}}(x)+\|f_y'\|_{C(I^2)}\delta_{n,q_{2,n}}(y). \quad \square\end{aligned}$$

Let $f\in C(I^2)$ and $\delta>0$. In what follows, we shall use the following modulus of continuity for bivariate real valued functions

$$\omega(f;\delta)=\sup\left\{|f(t,s)-f(x,y)|:(t,s),(x,y)\in I^2 \text{ and } \sqrt{(t-x)^2+(s-y)^2}\leqslant\delta\right\}.$$

The partial moduli of continuity with respect to x and y is given by

$$\omega_1(f;\delta)=\sup\{|f(x_1,y)-f(x_2,y)|:y\in I,|x_1-x_2|<\delta\}$$

and

$$\omega_2(f;\delta)=\sup\{|f(x,y_1)-f(x,y_2)|:x\in I,|y_1-y_2|<\delta\}.$$

Let

$$C^2(I^2)=\left\{f\in C(I^2):\frac{\partial^i f}{\partial x^i},\frac{\partial^i f}{\partial y^i}\in C(I^2),\ \text{for } i=1,2\right\}$$

equipped with the norm

$$\|f\|_{C^2(I^2)}=\|f\|_{C(I^2)}+\sum_{i=1}^{2}\left(\left\|\frac{\partial^i f}{\partial x^i}\right\|_{C(I^2)}+\left\|\frac{\partial^i f}{\partial y^i}\right\|_{C(I^2)}\right).$$

The Peetre's K-functional of the function $f\in C(I^2)$ is defined by

$$K(f;\delta)=\inf_{g\in C^2(I^2)}\left\{\|f-g\|_{C(I^2)}+\delta\|g\|_{C^2(I^2)},\delta>0\right\}.$$

It is known the following inequality (see [9])

$$K(f;\delta)\leqslant M\left\{\overline{\omega}_2(f;\sqrt{\delta})+\min(1,\delta)\|f\|_{C(I^2)},\right\} \text{ for all } \delta>0,$$

where $\overline{\omega}_2(f;\sqrt{\delta})$ is the second order modulus of continuity and the constant M is independent of δ and f.

In the following result we give the order of approximation of the Durrmeyer operators to the function $f\in C^2(I^2)$ by K-functional.

THEOREM 3.7. *For the function $f\in C(I^2)$, we have the following inequality*

$$\left|D_{m,n,q_1,q_2}(f;x,y)-f(x,y)\right|\leqslant 4K\left(f;\frac{1}{4}A_{m,n,q_1,q_2}(x,y)\right)+\omega\left(f;\sqrt{\nu_{m,n,q_1,q_2}(x,y)}\right),$$

where

$$\nu_{m,n,q_1,q_2}(x,y)=\left(\frac{1+q_1x[m]_{q_1}}{[m+2]_{q_1}}-x\right)^2+\left(\frac{1+q_2y[n]_{q_2}}{[n+2]_{q_2}}-y\right)^2,$$

$$A_{m,n,q_1,q_2}(x,y)=\frac{2}{[m+2]_{q_1}}\delta^2_{m,q_1}(x)+\frac{2}{[n+2]_{q_2}}\delta^2_{n,q_1}(y)+\nu_{m,n,q_1,q_2}(x,y).$$

Proof. We define

$$\tilde{D}_{m,n,q_1,q_2}(f;x,y) = D_{m,n,q_1,q_2}(f;x,y) - f\left(\frac{1+q_1x[m]_{q_1}}{[m+2]_{q_1}}, \frac{1+q_2y[n]_{q_2}}{[n+2]_{q_2}}\right) + f(x,y).$$

From Lemma 2.2 we have

$$\tilde{D}_{m,n,q_1,q_2}(s-x;x,y) = 0, \tilde{D}_{m,n,q_1,q_2}(t-y;x,y) = 0.$$

Using the Taylor's theorem, for $g \in C^2(I^2)$, it follows

$$\begin{aligned} g(s,t) - g(x,y) = &\frac{\partial g(x,y)}{\partial x}(s-x) + \int_x^s (s-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial \alpha^2}d\alpha \\ &+ \frac{\partial g(x,y)}{\partial y}(t-y) + \int_y^t (t-\beta)\frac{\partial^2 g(x,\beta)}{\partial \beta^2}d\beta. \end{aligned} \tag{3.1}$$

Applying $\tilde{D}_{m,n,q_1,q_2}$ on both side of (3.1), we get

$$\begin{aligned} &\tilde{D}_{m,n,q_1,q_2}(g;x,y) - g(x,y) \\ &= \tilde{D}_{m,n,q_1,q_2}\left(\int_x^s (s-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial \alpha^2}d\alpha;x,y\right) + \tilde{D}_{m,n,q_1,q_2}\left(\int_y^t (t-\beta)\frac{\partial^2 g(x,\beta)}{\partial \beta^2}d\beta;x,y\right) \\ &= D_{m,n,q_1,q_2}\left(\int_x^s (s-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial \alpha^2}d\alpha;x,y\right) \\ &\quad - \int_x^{\frac{1+q_1x[m]_{q_1}}{[m+2]_{q_1}}} \left(\frac{1+q_1x[m]_{q_1}}{[m+2]_{q_1}} - \alpha\right)\frac{\partial^2 g(\alpha,y)}{\partial \alpha^2}d\alpha \\ &\quad + D_{m,n,q_1,q_2}\left(\int_y^t (t-\beta)\frac{\partial^2 g(x,\beta)}{\partial \beta^2}d\beta;x,y\right) \\ &\quad - \int_y^{\frac{1+q_2y[n]_{q_2}}{[n+2]_{q_2}}} \left(\frac{1+q_2y[n]_{q_2}}{[n+2]_{q_2}} - \beta\right)\frac{\partial^2 g(x,\beta)}{\partial \beta^2}d\beta. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left|\tilde{D}_{m,n,q_1,q_2}(g;x,y) - g(x,y)\right| \\ &\leqslant D_{m,n,q_1,q_2}\left(\left|\int_x^s |s-\alpha|\left|\frac{\partial^2 g(\alpha,y)}{\partial \alpha^2}\right|d\alpha\right|;x,y\right) \\ &\quad + \left|\int_x^{\frac{1+q_1x[m]_{q_1}}{[m+2]_{q_1}}} \left|\frac{1+q_1x[m]q_1}{[m+2]_{q_1}} - \alpha\right| \cdot \left|\frac{\partial^2 g(\alpha,y)}{\partial \alpha^2}\right|d\alpha\right| \\ &\quad + D_{m,n,q_1,q_2}\left(\left|\int_y^t |t-\beta|\left|\frac{\partial^2 g(x,\beta)}{\partial \beta^2}\right|d\beta\right|;x,y\right) \\ &\quad + \left|\int_y^{\frac{1+q_2y[n]_{q_2}}{[n+2]_{q_2}}} \left|\frac{1+q_2y[n]_{q_2}}{[n+2]_{q_2}} - \beta\right| \cdot \left|\frac{\partial^2 g(x,\beta)}{\partial \beta^2}\right|d\beta\right| \\ &\leqslant \left\{D_{m,n,q_1,q_2}\left((s-x)^2;x,y\right) + \left(\frac{1+q_1x[m]_{q_1}}{[m+2]_{q_1}} - x\right)^2\right\}\|g\|_{C^2(I^2)} \\ &\quad + \left\{D_{m,n,q_1,q_2}\left((t-y)^2;x,y\right) + \left(\frac{1+q_2y[n]_{q_2}}{[n+2]_{q_2}} - y\right)^2\right\}\|g\|_{C^2(I^2)} \end{aligned}$$

$$\leqslant \left\{ \frac{2}{[m+2]_{q_1}} \delta_{m,q_1}^2(x) + \left(\frac{1+q_1x[m]_{q_1}}{[m+2]_{q_1}} - x \right)^2 + \frac{2}{[n+2]_{q_2}} \delta_{n,q_2}^2(y) + \left(\frac{1+q_2y[n]_{q_2}}{[n+2]_{q_2}} - y \right)^2 \right\} \|g\|_{C^2(I^2)}$$

But,

$$\left|\tilde{D}_{m,n,q_1,q_2}(f;x,y)\right| \leqslant 3\|f\|_{C(I^2)}.$$

Now, we have

$$\begin{aligned}
&\left|D_{m,n,q_1,q_2}(f;x,y)-f(x,y)\right| \\
&\leqslant \left|\tilde{D}_{m,n,q_1,q_2}(f-g;x,y)\right| + \left|\tilde{D}_{m,n,q_1,q_2}(g;x,y)-g(x,y)\right| \\
&\quad + |g(x,y)-f(x,y)| + \left| f\left(\frac{1+q_1x[m]_{q_1}}{[m+2]_{q_1}}, \frac{1+q_2y[n]_{q_2}}{[n+2]_{q_2}} \right) - f(x,y) \right| \\
&\leqslant 4\|f-g\|_{C(I^2)} + A_{m,n,q_1,q_2}(x,y)\|g\|_{C^2(I^2)} + \omega\left(f; \sqrt{\nu_{m,n,q_1,q_2}(x,y)}\right) \\
&= 4\left\{ \|f-g\|_{C(I^2)} + \frac{1}{4}A_{m,n,q_1,q_2}(x,y)\|g\|_{C^2(I^2)} \right\} + \omega\left(f; \sqrt{\nu_{m,n,q_1,q_2}(x,y)}\right).
\end{aligned}$$

Taking the infimum on the right hand side over all $g \in C^2(I^2)$ it follows

$$\left|D_{m,n,q_1,q_2}(f;x,y)-f(x,y)\right| \leqslant 4K\left(f; \frac{1}{4}A_{m,n,q_1,q_2}(x,y)\right) + \omega\left(f; \sqrt{\nu_{m,n,q_1,q_2}(x,y)}\right). \quad \square$$

REMARKS 3.1. There exists a constant M independent of δ and f such that

$$\begin{aligned}
&\left|D_{m,n,q_1,q_2}(f;x,y)-f(x,y)\right| \\
&\leqslant M\left\{ \overline{\omega}_2\left(f; \frac{1}{2}\sqrt{A_{m,n,q_1,q_2}(x,y)}\right) + \min\left(1, \frac{1}{4}A_{m,n,q_1,q_2}(x,y)\right) \right\} + \omega\left(f; \sqrt{\nu_{m,n,q_1,q_2}(x,y)}\right),
\end{aligned}$$

where $\overline{\omega}_2$ is the second order modulus of continuity.

4. A Voronovskaya theorm for the bivariate q-Durrmeyer operators

In this section we shall establish a Voronovskaya type theorem for the operators D_{m,n,q_1,q_2}. First, we need the auxiliary result contained in the following lemma.

LEMMA 4.1. *Assume that $0 < q_m < 1$, $q_m \to 1$ and $q_m^m \to a$, $a \in [0,1)$ as $m \to \infty$. Then we have*

$$\begin{aligned}
&\lim_{m\to\infty} [m]_{q_m} D_{m,q_m}(s-x;x) = 1-(a+1)x, \\
&\lim_{n\to\infty} [m]_{q_m} D_{m,q_m}((s-x)^2;x) = 2x(1-x), \\
&\lim_{n\to\infty} [m]_{q_m} D_{m,q_m}((s-x)^4;x) = 12x^2(1-x)^2.
\end{aligned}$$

Proof. To prove the lemma we use the formulas for $D_{m,q_m}(e_i;x)$, $i=0,1,2$ given in Lemma 2.2.

$$\begin{aligned}
\lim_{m\to\infty}[m]_{q_m}D_{m,q_m}(s-x;x) &= \lim_{m\to\infty}[m]_{q_m}\left(\frac{1+q_m[m]_{q_m}x}{[m+2]_{q_m}}-x\right)\\
&= \lim_{m\to\infty}[m]_{q_m}\frac{1+q_m[m]_{q_m}x-x(1+q_m+q_m^2[m]_{q_m})}{[m+2]_{q_m}}\\
&= \lim_{m\to\infty}\frac{[m]_{q_m}}{[m+2]_{q_m}}\left\{1+([m]_{q_m}q_m(1-q_m)-1-q_m)x\right\}\\
&= \lim_{m\to\infty}\frac{[m]_{q_m}}{[m+2]_{q_m}}\{1+((1-q_m^m)q_m-1-q_m)x\}=1-(a+1)x;
\end{aligned}$$

$$\begin{aligned}
&\lim_{m\to\infty}[m]_{q_m}D_{m,q_m}\left((s-x)^2;x\right)\\
&=\lim_{m\to\infty}[m]_{q_m}\left\{D_{m,q_m}(s^2;x)-2xD_{m,q_m}(s;x)+x^2\right\}\\
&=\frac{[m]_{q_m}}{[m+2]_{q_m}[m+3]_{q_m}}\left\{q_m^3x^2[m]_{q_m}([m]_{q_m}-1)+(1+q_m)^2q_mx[m]_{q_m}+1+q_m\right.\\
&\quad-2x(1+q_mx[m]_{q_m})\left(1+q_m+q_m^2+q_m^3[m]_{q_m}\right)\\
&\quad\left.+x^2\left(1+q_m+q_m^2[m]_{q_m}\right)\left(1+q_m+q_m^2+q_m^3[m]_{q_m}\right)\right\}\\
&=\lim_{m\to\infty}\frac{[m]_{q_m}}{[m+2]_{q_m}[m+3]_{q_m}}\left\{(q_m-1)^2q_m^3x^2[m]_{q_m}^2\right.\\
&\quad+q_mx\left(1+2q_m-q_m^2-q_m^2x+2xq_m^3-q_mx-2x\right)[m]_{q_m}\\
&\quad\left.+q_m-2x(1+q_m+q_m^2)+1+x^2(1+q_m)(1+q_m+q_m^2)\right\}\\
&=2x(1-x)+\lim_{m\to\infty}\frac{[m]_{q_m}^3}{[m+2]_{q_m}[m+3]_{q_m}}(q_m-1)^2q_m^3x^3\\
&=2x(1-x)+\lim_{m\to\infty}\frac{[m]_{q_m}^2}{[m+2]_{q_m}[m+3]_{q_m}}\frac{1-q_m^m}{1-q_m}(1-q_m)^2q_m^3x^2\\
&=2x(1-x)+\lim_{m\to\infty}\frac{[m]_{q_m}^2}{[m+2]_{q_m}[m+3]_{q_m}}(1-q_m^m)(1-q_m)q_m^3x^2=2x(1-x);
\end{aligned}$$

$$\begin{aligned}
&\lim_{m\to\infty}[m]_{q_m}^2D_{m,q_m}\left((s-x)^4;x\right)\\
&=\lim_{m\to\infty}[m]_{q_m}^2\left\{D_{m,q_m}(s^4;x)-4xD_{m,q_m}(s^3;x)+6x^2D_{m,q_m}(s^2;x)-4x^3D_{m,q_m}(s;x)+x^4\right\}\\
&=\lim_{m\to\infty}\frac{[m]_{q_m}^2}{[m+2]_{q_m}[m+3]_{q_m}[m+4]_{q_m}[m+5]_{q_m}}\left\{q_m^{10}x^4(q_m-1)^4[m]^4+x^3q_m^6(q_m-1)^2\right.\\
&\quad\times(4xq_m^5-3xq_m^4-q_m^4+4q_m^3-q_m^3x+6q_m^2-4q_m^2x+4q_m-6q_mx-4x+1)[m]_{q_m}^3\\
&\quad+[m]_{q_m}^2q_m^3x^2(1-10x^2q_m^5-4x+6x^2+3q_m+7q_m^2+7q_m^3+2q_m^4-6q_m^5-3q_m^6+q_m^7\\
&\quad+q_m^9+5q_m^9x^2+q_m^{10}x^2-8q_m^8x^2+7x^2q_m^2-3xq_m^8-xq_m^{10}+x^2q_m^4+8x^2q_m-9q_m^6x^2
\end{aligned}$$

$$+9xq_m^6+14xq_m^7-3xq_m^4-q_m^8-12q_mx+4xq_m^5-18q_m^2x-10q_m^3x+11q_m^3x^2)\Big\}$$
$$=12x^2(1-x)^2. \quad \square$$

The main result of this section is the following Voronovskaya type theorem:

THEOREM 4.1. *Let $f\in C^2([0,1]\times[0,1])$ and $(q_m)_m$ be a sequence in the interval $(0,1)$ such that $q_m\to 1$ and $q_m^m\to a$, $a\in[0,1)$ as $m\to\infty$. Then for every $(x,y)\in[0,1]\times[0,1]$, one has*

$$\lim_{m\to\infty}[m]_{q_m}\left\{D_{m,m,q_m,q_m}(f;x,y)-f(x,y)\right\}$$
$$=[1-(a+1)x]f'_x(x,y)+[1-(a+1)y]f'_y(x,y)+x(1-x)f''_{x^2}(x,y)+y(1-y)f''_{y^2}(x,y).$$

Proof. Let $(x_0,y_0)\in[0,1]\times[0,1]$ be a fixed point. By the Taylor formula, it follows

$$\begin{aligned}f(s,t)&=f(x_0,y_0)+f'_x(x_0,y_0)(s-x_0)+f'_y(x_0,y_0)(t-y_0)\\&\quad+\frac{1}{2}\left\{f''_{x^2}(x_0,y_0)(s-x_0)^2+2f''_{xy}(x_0,y_0)(s-x_0)(t-y_0)+f''_{y^2}(x_0,y_0)(t-y_0)^2\right\}\\&\quad+\varphi(s,t)\left((s-x_0)^2+(t-y_0)^2\right),\end{aligned}$$

where $(s,t)\in[0,1]\times[0,1]$ and $\lim\limits_{(s,t)\to(x_0,y_0)}\varphi(s,t)=0$.

From the linearity of D_{m,m,q_m,q_m}, we have

$$\begin{aligned}&D_{m,m,q_m,q_m}(f(s,t);x_0,y_0)\\&=f(x_0,y_0)+f'_x(x_0,y_0)D_{m,m,q_m,q_m}(s-x_0;x_0,y_0)\\&\quad+f'_y(x_0,y_0)D_{m,m,q_m,q_m}(t-y_0;x_0,y_0)\\&\quad+\frac{1}{2}\{f''_{x^2}(x_0,y_0)D_{m,m,q_m,q_m}((s-x_0)^2;x_0,y_0)\\&\quad+2f''_{xy}(x_0,y_0)D_{m,m,q_m,q_m}((s-x_0)(t-y_0);x_0,y_0)\\&\quad+f''_{y^2}(x_0,y_0)D_{m,m,q_m,q_m}((t-y_0)^2;x_0,y_0)\Big\}\\&\quad+D_{m,m,q_m,q_m}\left(\varphi(s,t)\left((s-x_0)^2+(t-y_0)^2\right);x_0,y_0\right)\\&=f(x_0,y_0)+f'_x(x_0,y_0)D_{m,q_m}(s-x_0;x_0)+f'_y(x_0,y_0)D_{m,q_m}(t-y_0;y_0)\\&\quad+\frac{1}{2}\Big\{f''_{x^2}(x_0,y_0)D_{m,q_m}((s-x_0)^2;x_0)+f''_{y^2}(x_0,y_0)D_{m,q_m}((t-y_0)^2;y_0)\\&\quad+2f''_{xy}(x_0,y_0)D_{m,q_m}((s-x_0);x_0)D_{m,q_m}((t-y_0);y_0)\}\\&\quad+D_{m,m,q_m,q_m}\left(\varphi(s,t)\left((s-x_0)^2+(t-y_0)^2\right);x_0,y_0\right).\end{aligned}$$

By the Hölder inequality, we have

$$\begin{aligned}&\left|D_{m,m,q_m,q_m}\left(\varphi(s,t)\left((s-x_0)^2+(t-y_0)^2\right);x_0,y_0\right)\right|\\&\leqslant\left\{D_{m,m,q_m,q_m}\left(\varphi^2(s,t);x_0,y_0\right)\right\}^{1/2}\left\{D_{m,m,q_m,q_m}\left(\left((s-x_0)^2+(t-y_0)^2\right)^2;x_0,y_0\right)\right\}^{1/2}\end{aligned}$$

$$\leqslant \sqrt{2}\left\{D_{m,m,q_m,q_m}\left(\varphi^2(s,t);x_0,y_0\right)\right\}^{1/2}$$
$$\times\left\{D_{m,m,q_m,q_m}\left((s-x_0)^4;x_0,y_0\right)+D_{m,m,q_m,q_m}\left((t-y_0)^4;x_0,y_0\right)\right\}^{1/2}.$$

By Theorem 3.2, we get

$$\lim_{n\to\infty} D_{m,m,q_m,q_m}\left(\varphi^2(s,t);x_0,y_0\right)=\varphi^2(x_0,y_0)=0,$$

and using Lemma 4.1 we have

$$\lim_{m\to\infty}[m]_{q_m}D_{m,m,q_m,q_m}\left(\varphi(s,t)\left((s-x_0)^2+(t-y_0)^2\right);x_0,y_0\right)=0.$$

Applying Lemma 4.1 theorem is proved. □

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