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NOTE ON A LI–STEVIĆ INTEGRAL–TYPE OPERATOR FROM MIXED–NORM SPACES TO nTH WEIGHTED SPACES

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Abstract. The boundedness and compactness of a Li-Stević integral-type operator from mixed-norm spaces to nth weighted spaces are characterized in this paper.

1. Introduction

Let $\mathbb{D}$ denote the unit disk in the complex plane $\mathbb{C}$, $\mathscr{H}(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$ and $\mathbb{N}$ the set of natural numbers. A positive continuous function ϕ on [0,1) is called normal if there exist two positive numbers a and b with $0<a<b$, and $\delta\in[0,1)$ such that (see [9])

$$\frac{\phi(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1), \quad \lim_{r\to 1}\frac{\phi(r)}{(1-r)^a}=0;$$

$$\frac{\phi(r)}{(1-r)^b} \text{ is increasing on } [\delta,1), \quad \lim_{r\to 1}\frac{\phi(r)}{(1-r)^b}=\infty.$$

For $p,q\in(0,\infty)$ and ϕ normal, the mixed-norm space $H(p,q,\phi)(\mathbb{D})=H(p,q,\phi)$ is the space of all functions $f\in\mathscr{H}(\mathbb{D})$ such that

$$\|f\|_{H(p,q,\phi)}=\left(\int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r}dr\right)^{\frac{1}{p}}<\infty,$$

where

$$M_q(f,r)=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^q d\theta\right)^{\frac{1}{q}}.$$

For $1\leqslant p,q<\infty$, $H(p,q,\phi)$ is a Banach space equipped with the norm $\|f\|_{H(p,q,\phi)}$, while for the other vales of p and q, $\|\cdot\|_{H(p,q,\phi)}$ is a quasinorm on $H(p,q,\phi)$, $H(p,q,\phi)$ is a Fréchet space but not a Banach space. Note that if $\phi(r)=(1-r)^{\frac{\alpha+1}{p}}$, then

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$H(p,q,\phi)$ is equivalent to the weighted Bergman space $A^p_\alpha(\mathbb{D}) = A^p_\alpha$ defined for $0 < p < \infty$ and $\alpha > -1$, as the spaces of all $f \in \mathscr{H}(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\alpha} = (\alpha+1)\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dm(z) < \infty,$$

where $dm(z) = \frac{1}{\pi} r dr d\theta$ is the normalized Lebesgue area measure on $\mathbb{D}$. We would like to say that studying various concrete operators from or to mixed-norm spaces attracted a considerable attention recently (see, e.g., [2, 4, 7, 10, 12, 13, 14, 17, 18, 20, 21].

Let $\mu(z) = \mu(|z|)$ be a normal function on $\mathbb{D}$. The nth weighted space on $\mathbb{D}$, denoted by $\mathscr{W}^{(n)}_\mu = \mathscr{W}^{(n)}_\mu(\mathbb{D})$ which was introduced by Stević in [16], consists of all $f \in \mathscr{H}(\mathbb{D})$ such that

$$b_{\mathscr{W}^{(n)}_\mu}(f) = \sup_{z\in\mathbb{D}} \mu(z)|f^{(n)}(z)| < \infty.$$

For $n = 0$ the space becomes the weighted-type space $H^\infty_\mu(\mathbb{D})$ in [7, 11, 19, 21], for $n = 1$ the Bloch-type space $\mathscr{B}_\mu(\mathbb{D})$ and for $n = 2$ the Zygmund-type space $\mathscr{Z}_\mu(\mathbb{D})$ in [1, 3, 5]. From now on, we will assume that $n \in \mathbb{N}$. Set

$$\|f\|_{\mathscr{W}^{(n)}_\mu} = \sum_{j=0}^{n-1} |f^{(j)}(0)| + b_{\mathscr{W}^{(n)}_\mu}(f).$$

With this norm the nth weighted space becomes a Banach space.

Assume that $g : \mathbb{D} \to \mathbb{C}$ is a holomorphic map, φ is an analytic self-map of $\mathbb{D}$. For $f \in \mathscr{H}(\mathbb{D})$, we define a linear operator as follows:

$$\left(C^g_\varphi f\right)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi, \quad z \in \mathbb{D}.$$

The operator C^g_φ is now usually called the Li-Stević integral-type operator and was introduced in [5, 10] (they called it the generalized composition operator, but it should be noted that another operator has practically the same name, see, e.g., [6] and references therein). When $g = \varphi'$, we see that

$$\left(C^g_\varphi f\right)(z) = \int_0^z f'(\varphi(\xi))\varphi'(\xi)d\xi = f(\varphi(z)) - f(\varphi(0)),$$

since $f(\varphi(0))$ is a point-evaluation functional, this operator is closely related to composition operator. Its n-dimensional extension was introduced in [15]. These one and n-dimensional operators have been considerably studied so far (see, e.g., [5, 10, 12, 14, 15]). A natural counterpart of the operator was introduced in [11], and later studied, for example, in [13, 20, 22]. In all above mentioned papers can be found a plenty of information on integral-type operators and their products with composition operators between spaces of analytic functions on the unit disk or the unit ball. Motivated by Stević's papers [11, 12, 16, 17, 21], here we study the boundedness and compactness of the Li-Stević integral-type operator from mixed-norm spaces to Stević's nth weighted spaces. We have to point out that the results are closely related to the ones in Stević's paper [17], but we consider here more generalized mixed norm spaces which can be found, for example in [2].

Throughout this paper, we use the letter C to denote a positive constant whose value may vary at each occurrence.

2. Auxiliary results

In this section we formulate some auxiliary results which will be used in the proofs of the main results. Lemma 1 can be found in ([11]).

LEMMA 1. *Assume $0<p,q<\infty$, ϕ is normal and $f\in H(p,q,\phi)$. Then for every $n\in\mathbb{N}$, there is a positive constant C independent of f such that*

$$|f^{(n)}(z)| \leqslant \frac{C\|f\|_{H(p,q,\phi)}}{\phi(|z|)(1-|z|^2)^{\frac{1}{q}+n}}, \quad z\in\mathbb{D}.$$

From Lemma 2.4 in [17], we get the following result.

LEMMA 2. *Assume $a>0$ and $a\neq 1$,*

$$D_n = \begin{vmatrix} a-1 & a & \cdots & a+n-2 \\ (a-1)a & a(a+1) & \cdots & (a+n-2)(a+n-1) \\ \vdots & \vdots & & \vdots \\ \prod_{j=0}^{n-1}(a-1+j) & \prod_{j=0}^{n-1}(a+j) & \cdots & \prod_{j=0}^{n-1}(a+n-2+j) \end{vmatrix}.$$

Then, $D_n=\prod_{j=0}^{n-1}(a-1+j)\prod_{j=0}^{n-1}j!$.

From Lemma 4 in [19], we can get the following result.

LEMMA 3. *Assume $n\in\mathbb{N}$, $u,f\in\mathscr{H}(\mathbb{D})$ and ϕ is an analytic self-map of $\mathbb{D}$. Then,*

$$\left(C_\varphi^g f\right)^{(n)}(z) = \sum_{k=0}^{n-1} f^{(k+1)}(\varphi(z)) \sum_{l=k}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,k}\left(\varphi'(z),\cdots,\varphi^{(l-k+1)}(z)\right),$$

where

$$B_{l,k}\left(\varphi'(z),\cdots,\varphi^{(l-k+1)}(z)\right) = \sum_{k_1,\cdots,k_l} \frac{l!}{k_1!,\cdots,k_l!} \prod_{j=1}^{l} \left(\frac{\varphi^{(j)}(z)}{j!}\right)^{k_j},$$

and the sum is over all non-negative integers $k_1,\cdots,k_l$ satisfying $k_1+k_2+\cdots+k_l=k$ and $k_1+2k_2+\cdots+lk_l=l$.

The following lemma can be proved by using standard Schwartz's arguments in [8].

LEMMA 4. *Suppose that $g\in\mathscr{H}(\mathbb{D})$, $n\in\mathbb{N}$, φ is an analytic self-map of $\mathbb{D}$. Then, $C_\varphi^g : H(p,q,\phi)\to\mathscr{W}_\mu^{(n)}$ is compact if and only if $C_\varphi^g : H(p,q,\phi)\to\mathscr{W}_\mu^{(n)}$ is bounded and for any bounded sequence $(f_i)_{i\in\mathbb{N}}$ in $H(p,q,\phi)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $i\to\infty$, we have $\|C_\varphi^g f_i\|_{\mathscr{W}_\mu^{(n)}}\to 0$ as $i\to\infty$.*

3. The boundedness and compactness of $C_\varphi^g : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}$

THEOREM 5. *Let $g \in \mathscr{H}(\mathbb{D})$, $n \in \mathbb{N}$, $0 < p,q < \infty$, φ is an analytic self-map of $\mathbb{D}$. Then $C_\varphi^g : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}$ is bounded if and only if for each $k \in \{0,1,\cdots,n-1\}$,*

$$I_k := \sup_{z\in\mathbb{D}} \frac{\mu(z)\big|\sum_{l=k}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,k}\big(\varphi'(z),\cdots,\varphi^{(l-k+1)}(z)\big)\big|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+k+1}} < \infty. \tag{1}$$

Proof. Assume that (1) holds, then for each $f \in H(p,q,\phi)$, by Lemmas 1 and 3, we have

$$\begin{aligned}
&\mu(z)\big|(C_\varphi^g f)^{(n)}(z)\big| \\
&= \mu(z)\Big|\sum_{k=0}^{n-1} f^{(k+1)}(\varphi(z)) \sum_{l=k}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,k}\big(\varphi'(z),\cdots,\varphi^{(l-k+1)}(z)\big)\Big| \\
&\leqslant \mu(z)\sum_{k=0}^{n-1} \big|f^{(k+1)}(\varphi(z))\big| \Big|\sum_{l=k}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,k}\big(\varphi'(z),\cdots,\varphi^{(l-k+1)}(z)\big)\Big| \\
&\leqslant C\|f\|_{H(p,q,\phi)} \sum_{k=0}^{n-1} \frac{\mu(z)\big|\sum_{l=k}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,k}\big(\varphi'(z),\cdots,\varphi^{(l-k+1)}(z)\big)\big|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+k+1}}.
\end{aligned} \tag{2}$$

We also have that for each $s \in \{1,\cdots,n-1\}$

$$\begin{aligned}
&\big|(C_\varphi^g f)^{(s)}(0)\big| \\
&= \Big|\sum_{k=0}^{s-1} f^{(k+1)}(\varphi(z)) \sum_{l=k}^{s-1} C_{s-1}^l g^{(s-1-l)}(0) B_{l,k}\big(\varphi'(0),\cdots,\varphi^{(l-k+1)}(0)\big)\Big| \\
&\leqslant C\|f\|_{H(p,q,\phi)} \sum_{k=0}^{s-1} \frac{\big|\sum_{l=k}^{s-1} C_{s-1}^l g^{(s-1-l)}(0) B_{l,k}\big(\varphi'(0),\cdots,\varphi^{(l-k+1)}(0)\big)\big|}{\phi(|\varphi(0)|)(1-|\varphi(0)|^2)^{\frac{1}{q}+k+1}},
\end{aligned} \tag{3}$$

and

$$\big|(C_\varphi^g f)(0)\big| = \int_0^0 f'(\varphi(\xi))g(\xi)d\xi = 0. \tag{4}$$

From (1), (2), (3) and (4), we see that $C_\varphi^g : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}$ is bounded.

Conversely, suppose that $C_\varphi^g : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}$ is bounded, i.e., there exists $C > 0$ such that $\|C_\varphi^g f\|_{\mathscr{W}_\mu^{(n)}} \leqslant C\|f\|_{H(p,q,\phi)}$ for all $f \in H(p,q,\phi)$. For a fixed $\omega \in \mathbb{D}$, set

$$h_\omega(z) = \frac{(1-|\omega|^2)^b}{\phi(|\omega|)} \sum_{j=1}^{n} \frac{c_j(1-|\omega|^2)^j}{(1-\overline{\omega}z)^{\frac{1}{q}+b+j}}, \tag{5}$$

where the constant b is from the definition of the normality of the function ϕ and $c_j (j = 1,2,\cdots,n)$ are fixed. Then we know that $h_\omega(z) \in H(p,q,\phi)$, and moreover, $\sup_{\omega\in\mathbb{D}} \|h_\omega\|_{H(p,q,\phi)} \leqslant C$ (see [2]).

Now we show that for each $s\in\{1,2,\cdots,n\}$, there are constants $c_1,c_2,\cdots,c_n$ such that

$$h_\omega^{(s)}(\omega)=\frac{\overline{\omega}^s}{\phi(|\omega|)(1-|\omega|^2)^{\frac{1}{q}+s}},\quad h_\omega^{(t)}(\omega)=0,\quad t\in\{1,2,\cdots,n\}\backslash\{s\}. \tag{6}$$

In fact, by differentiating function h_ω, for each $s\in\{1,2,\cdots,n\}$, (6) is equivalent to the following system of liner equations

$$\begin{cases}(\frac{1}{q}+b+1)c_1+(\frac{1}{q}+b+2)c_2+\cdots+(\frac{1}{q}+b+n)c_n=0,\\ (\frac{1}{q}+b+1)(\frac{1}{q}+b+2)c_1+(\frac{1}{q}+b+2)(\frac{1}{q}+b+3)c_2+\cdots\\ \qquad+(\frac{1}{q}+b+n)(\frac{1}{q}+b+n+1)c_n=0,\\ \cdots\cdots\\ \prod_{j=1}^{s}(\frac{1}{q}+b+j)c_1+\prod_{j=2}^{s+1}(\frac{1}{q}+b+j)c_2+\cdots+\prod_{j=n}^{n+s-1}(\frac{1}{q}+b+j)c_n=1,\\ \cdots\cdots\\ \prod_{j=1}^{n}(\frac{1}{q}+b+j)c_1+\prod_{j=2}^{n+1}(\frac{1}{q}+b+j)c_2+\cdots+\prod_{j=n}^{2n-1}(\frac{1}{q}+b+j)c_n=0.\end{cases} \tag{7}$$

Applying Lemma 2 with $a=\frac{1}{q}+b+2$, we see that the determinant of system (7) is different from zero, as claimed. For each $k\in\{0,1,\cdots,n-1\}$, we choose the corresponding family of function that satisfy (6) with $s=k+1$ and denote it by $h_{\omega,k}$. Then, from Lemma 3 and the boundedness of $C_\varphi^g:H(p,q,\phi)\to\mathscr{W}_\mu^{(n)}$, for $\omega\in\mathbb{D}$ such that $|\varphi(\omega)|>\frac{1}{2}$,

$$\begin{aligned}&\frac{\mu(\omega)|\varphi(\omega)|^{k+1}\big|\sum_{l=k}^{n-1}C_{n-1}^{l}g^{(n-1-l)}(\omega)B_{l,k}\big(\varphi'(\omega),\cdots,\varphi^{(l-k+1)}(\omega)\big)\big|}{\phi(|\varphi(\omega)|)(1-|\varphi(\omega)|^2)^{\frac{1}{q}+k+1}}\\ &\leqslant C\sup_{\omega\in\mathbb{D}}\big\|C_\varphi^g(h_{\varphi(\omega),k})\big\|_{\mathscr{W}_\mu^{(n)}}\leqslant C\big\|C_\varphi^g\big\|_{H(p,q,\phi)\to\mathscr{W}_\mu^{(n)}}.\end{aligned} \tag{8}$$

From (8), it follows that for each $k\in\{0,1,\cdots,n-1\}$,

$$\sup_{|\varphi(z)|>\frac{1}{2}}\frac{\mu(z)\big|\sum_{l=k}^{n-1}C_{n-1}^{l}g^{(n-1-l)}(z)B_{l,k}\big(\varphi'(z),\cdots,\varphi^{(l-k+1)}(z)\big)\big|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+k+1}}\leqslant C\big\|C_\varphi^g\big\|_{H(p,q,\phi)\to\mathscr{W}_\mu^{(n)}}. \tag{9}$$

Let

$$p_k(z)=z^{k+1},\qquad k=0,1,\cdots,n-1. \tag{10}$$

Then clearly $\|p_k\|_{H(p,q,\phi)}<\infty$. By applying Lemma 3 to $p_0(z)=z$, we get

$$\begin{aligned}\big(C_\varphi^g p_0\big)^{(n)}(z)&=p_0'(\varphi(z))\sum_{l=0}^{n-1}C_{n-1}^{l}g^{(n-1-l)}(z)B_{l,0}\big(\varphi'(z),\cdots,\varphi^{(l+1)}(z)\big)\\ &=\sum_{l=0}^{n-1}C_{n-1}^{l}g^{(n-1-l)}(z)B_{l,0}\big(\varphi'(z),\cdots,\varphi^{(l+1)}(z)\big),\end{aligned}$$

which along with the boundedness of $C_\varphi^g : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}$ implies that

$$\sup_{z\in\mathbb{D}} \mu(z)\left|\sum_{l=0}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,0}\big(\varphi'(z),\cdots,\varphi^{(l+1)}(z)\big)\right| \leqslant C\big\|C_\varphi^g\big\|_{H(p,q,\phi)\to\mathscr{W}_\mu^{(n)}}. \tag{11}$$

Assume now that we have proved the following inequalities

$$\sup_{z\in\mathbb{D}} \mu(z)\left|\sum_{l=j}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,j}\big(\varphi'(z),\cdots,\varphi^{(l-j+1)}(z)\big)\right| \leqslant C\big\|C_\varphi^g\big\|_{H(p,q,\phi)\to\mathscr{W}_\mu^{(n)}}. \tag{12}$$

for $j\in\{0,1,\cdots,k-1\}, k\leqslant n-1$.

Apply Lemma 3 to $p_k(z)=z^{k+1}, k\in\{0,1,\cdots,n-1\}$, we get

$$\begin{aligned}
&\big(C_\varphi^g p_k\big)^{(n)}(z)\\
&=\sum_{j=0}^{k-1}(k+1)\cdots(k-j+1)(\varphi(z))^{k-j}\sum_{l=j}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,j}\big(\varphi'(z),\cdots,\varphi^{(l-j+1)}(z)\big)\\
&\quad+(k+1)!\sum_{l=j}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,j}\big(\varphi'(z),\cdots,\varphi^{(l-j+1)}(z)\big),
\end{aligned}$$

from which, along with the boundedness of $C_\varphi^g : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}$, the fact that $\|\varphi\|_\infty \leqslant 1$, the triangle inequality, and using hypothesis (12) we get

$$\sup_{z\in\mathbb{D}} \mu(z)\left|\sum_{l=k}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,k}\big(\varphi'(z),\cdots,\varphi^{(l-k+1)}(z)\big)\right| \leqslant C\big\|C_\varphi^g\big\|_{H(p,q,\phi)\to\mathscr{W}_\mu^{(n)}}. \tag{13}$$

for each $k\in\{0,1,\cdots,n-1\}$. Then for each $k\in\{0,1,\cdots,n-1\}$

$$\begin{aligned}
&\sup_{|\varphi(z)|\leqslant\frac{1}{2}} \frac{\mu(z)\big|\sum_{l=k}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,k}\big(\varphi'(z),\cdots,\varphi^{(l-k+1)}(z)\big)\big|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+k+1}}\\
&\leqslant C\sup_{z\in\mathbb{D}} \mu(z)\left|\sum_{l=k}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,k}\big(\varphi'(z),\cdots,\varphi^{(l-k+1)}(z)\big)\right|\\
&\leqslant C\big\|C_\varphi^g\big\|_{H(p,q,\phi)\to\mathscr{W}_\mu^{(n)}}.
\end{aligned} \tag{14}$$

From (9) and (14), we get (1). □

THEOREM 6. *Let $g\in\mathscr{H}(\mathbb{D})$, $n\in\mathbb{N}$, $0<p,q<\infty$, and φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi^g : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}$ is compact if and only if $C_\varphi^g : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}$ is bounded and for each $k\in\{0,1,\cdots,n-1\}$,*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)\big|\sum_{l=k}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,k}\big(\varphi'(z),\cdots,\varphi^{(l-k+1)}(z)\big)\big|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+k+1}}=0. \tag{15}$$

Proof. Suppose that $C_\varphi^g : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}$ is compact, then clearly $C_\varphi^g : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}$ is bounded. Let $(z_i)_{i\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_i)| \to 1$ as $i \to \infty$. If such a sequence does not exist, then the conditions in (15) automatically hold.

Let $h_{\varphi(z_i),k}, k \in \{0,1,\cdots,n-1\}$ be as in Theorem 5. Then the sequence $(h_{\varphi(z_i),k})_{i\in\mathbb{N}}$ is bounded and $h_{\varphi(z_i),k} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$. Since $C_\varphi^g : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}$ is compact, from Lemma 4, we have that for each $k \in \{0,1,\cdots,n-1\}$,

$$\lim_{i\to\infty} \|C_\varphi^g h_{\varphi(z_i),k}\|_{\mathscr{W}_\mu^{(n)}} = 0. \tag{16}$$

From (8) we obtain

$$\begin{aligned}
&\|C_\varphi^g h_{\varphi(z_i),k}\|_{\mathscr{W}_\mu^{(n)}} \\
&\geqslant \frac{C\mu(z_i)|\varphi(z_i)|^{k+1}\left|\sum_{l=k}^{n-1} C_{n-1}^l g^{(n-1-l)}(z_i) B_{l,k}\big(\varphi'(z_i),\cdots,\varphi^{(l-k+1)}(z_i)\big)\right|}{\phi(|\varphi(z_i)|)(1-|\varphi(z_i)|^2)^{\frac{1}{q}+k+1}},
\end{aligned}$$

which along with $|\varphi(z_i)| \to 1$ as $i \to \infty$ and (16) implies that

$$\lim_{i\to\infty} \frac{\mu(z_i)\left|\sum_{l=k}^{n-1} C_{n-1}^l g^{(n-1-l)}(z_i) B_{l,k}\big(\varphi'(z_i),\cdots,\varphi^{(l-k+1)}(z_i)\big)\right|}{\phi(|\varphi(z_i)|)(1-|\varphi(z_i)|^2)^{\frac{1}{q}+k+1}} = 0,$$

for each $k \in \{0,1,\cdots,n-1\}$, from which (15) holds in this case.

On the other hand, we assume that $C_\varphi^g : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}$ is bounded and (15) holds. Let $(f_i)_{i\in\mathbb{N}}$ be a sequence in $H(p,q,\phi)$ such that $\sup_{i\in\mathbb{N}} \|f_i\|_{H(p,q,\phi)} \leqslant L$ and f_i converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$. By the assumption, for any $\varepsilon > 0$, there is a $\delta \in (0,1)$ such that for each $k \in \{0,1,\cdots,n-1\}$ and $\delta < |\varphi(z)| < 1$,

$$\frac{\mu(z)\left|\sum_{l=k}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,k}\big(\varphi'(z),\cdots,\varphi^{(l-k+1)}(z)\big)\right|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+k+1}} < \varepsilon. \tag{17}$$

We have

$$\begin{aligned}
&\|C_\varphi^g f_i\|_{\mathscr{W}_\mu^{(n)}} \\
&= \sum_{j=0}^{n-1} |(C_\varphi^g f_i)^{(j)}(0)| + \sup_{z\in\mathbb{D}} \mu(z)|(C_\varphi^g f_i)^{(n)}(z)| \\
&\leqslant \sum_{j=0}^{n-1} \left| \sum_{k=0}^{j-1} f_i^{(k+1)}(\varphi(0)) \sum_{l=k}^{j-1} C_{j-1}^l g^{(j-1-l)}(0) B_{l,k}\big(\varphi'(0),\cdots,\varphi^{(l-k+1)}(0)\big) \right| \\
&\quad + \sup_{|\varphi(z)|\leqslant\delta} \mu(z) \left| \sum_{k=0}^{n-1} f_i^{(k+1)}(\varphi(z)) \sum_{l=k}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,k}\big(\varphi'(z),\cdots,\varphi^{(l-k+1)}(0)\big) \right| \\
&\quad + \sup_{|\varphi(z)|>\delta} \mu(z) \left| \sum_{k=0}^{n-1} f_i^{(k+1)}(\varphi(z)) \sum_{l=k}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,k}\big(\varphi'(z),\cdots,\varphi^{(l-k+1)}(0)\big) \right| \\
&= J_1 + J_2 + J_3.
\end{aligned}$$

Now we estimate J_1, J_2 and J_3, by Cauchy's estimate we see that

$$f_i^{(k+1)}(\varphi(0)) \to 0 \quad \text{and} \quad \sup_{|\omega|\leqslant\delta} |f_i^{(k+1)}(\omega)| \to 0. \tag{18}$$

From (18) and (13) in Theorem 5, we can easily get that

$$J_1 = \sum_{j=0}^{n-1} \left| \sum_{k=0}^{j-1} f_i^{(k+1)}(\varphi(0)) \sum_{l=k}^{j-1} C_{j-1}^l g^{(j-1-l)}(0) B_{l,k}\big(\varphi'(0), \cdots, \varphi^{(l-k+1)}(0)\big) \right| \to 0, \tag{19}$$

and

$$\begin{aligned} J_2 = & \sup_{|\varphi(z)|\leqslant\delta} \mu(z) \\ & \times \left| \sum_{k=0}^{n-1} f_i^{(k+1)}(\varphi(z)) \sum_{l=k}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,k}\big(\varphi'(z), \cdots, \varphi^{(l-k+1)}(0)\big) \right| \to 0. \end{aligned} \tag{20}$$

By Lemma 1 and (17), we have that

$$\begin{aligned} J_3 = & \sup_{|\varphi(z)|>\delta} \mu(z) \\ & \times \left| \sum_{k=0}^{n-1} f_i^{(k+1)}(\varphi(z)) \sum_{l=k}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,k}\big(\varphi'(z), \cdots, \varphi^{(l-k+1)}(z)\big) \right| \\ \leqslant & C\|f_i\|_{H(p,q,\phi)} \\ & \times \sum_{k=0}^{n-1} \sup_{|\varphi(z)|>\delta} \frac{\mu(z)|\sum_{l=k}^{n-1} C_{n-1}^l g^{(n-1-l)}(z) B_{l,k}\big(\varphi'(z), \cdots, \varphi^{(l-k+1)}(z)\big)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+k+1}} \\ < & CnL\varepsilon. \end{aligned} \tag{21}$$

From (19), (20) and (21) we obtain $\lim_{i\to\infty} \|C_\varphi^g f_i\|_{\mathscr{W}_\mu^{(n)}} = 0$. From this and applying Lemma 4 the implication follows. □

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