FUNCTIONAL INEQUALITIES IN MATRIX BANACH SPACES

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Abstract. Using the fixed point method, we prove the Hyers-Ulam stability of the following additive functional inequality and quadratic functional inequality

\[\|f(x+y) - f(x) - f(y)\| \leq \|f\left(\frac{x+y}{2}\right) - f(x) - f\left(\frac{x+y}{2}\right)\|,\]
\[\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \|f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x) - f\left(\frac{x-y}{2}\right)\|\]

in matrix Banach spaces, respectively.

1. Introduction and preliminaries


Gilányi [4] and Rätz [19] proved that if, for a function $f : G \to E$ mapping from Abelian group $G$ divisible by 2 into an inner product space $E$, the functional inequality

\[\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\|, \quad \forall x, y \in G \tag{1.1}\]

holds, then $f$ satisfies the Jordan-Von Neumann functional equation

\[2f(x) + 2f(y) = f(xy) + f(xy^{-1}), \quad \forall x, y \in G.\]

Fechner [3] and Gilányi [5] have proved the generalized Hyers-Ulam stability of the functional inequality (1.1). Park et. al. [14] have investigated the generalized Hyers-Ulam stability of the following inequalities associated with Jordon-Von Neumann type additive functional equations:

\[\|f(x) + f(y) + f(z)\| \leq \|2f\left(\frac{x+y+z}{2}\right)\|,\]
\[\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\|,\]
\[\|f(x) + f(y) + 2f(z)\| \leq \|2f\left(\frac{x+y+z}{2}\right)\|.\]


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In 2013, Kim, Jun and Son [10] considered the following quadratic functional inequality
\[ \|f(x - y) + f(y - z) + f(x - z) - 3f(x) - 3f(y) - 3f(z)\| \leq \|f(x + y + z)\|. \]  
(1.2)
They established the general solution of the quadratic functional inequality (1.2), and then investigated the generalized Hyers-Ulam stability of this inequality in Banach spaces and in non-Archimedean Banach spaces. Recently, the stability results of several functional equations and inequalities were investigated [11, 12, 15, 16, 17] in matrix paranormed spaces and in matrix fuzzy normed spaces.

In 2015, Park [13] considered the following functional inequalities:
\[ \|f(x + y) - f(x) - f(y)\| \leq \|f\left(\frac{x + y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y)\|, \]  
(1.3)
\[ \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \|f\left(\frac{x + y}{2}\right) + f\left(\frac{x - y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y)\|. \]  
(1.4)
Using the direct method, he proved Hyers-Ulam stability of the functional inequalities (1.3) and (1.4) in Banach spaces. The main purpose of this paper is to apply the fixed point method to investigate the Hyers-Ulam stability of functional inequalities (1.3) and (1.4) in matrix Banach spaces, respectively.

Next, we will also use the following notations:

The set of all $m \times n$-matrices in $X$ will be denoted by $M_{m,n}(X)$. When $m = n$, the matrix $M_{m,n}(X)$ will be written as $M_n(X)$. The symbols $e_j \in M_{1,n}(\mathbb{C})$ will denote the row vector whose $j$th component is 1 and the other components are 0. Similarly, $E_{ij} \in M_n(\mathbb{C})$ will denote the $n \times n$ matrix whose $(i,j)$-component is 1 and the other components are 0. The $n \times n$ matrix whose $(i,j)$-component is $x$ and the other components are 0 will be denoted by $E_{ij} \otimes x \in M_n(X)$.

For $x \in M_n(X), y \in M_k(X)$,
\[ x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}. \]

Let $(X, \| \cdot \|)$ be a normed space. Note that $(X, \{\| \cdot \|_n\})$ is a matrix normed space if and only if $(M_n(X), \| \cdot \|_n)$ is a normed space for each positive integer $n$ and $\|AxB\|_k \leq \|A\| \|B\| \|x\|_n$ holds for $A \in M_{k,n}, x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}$. And that $(X, \{\| \cdot \|_n\})$ is a matrix Banach space if and only if $X$ is a Banach space and $(X, \{\| \cdot \|_n\})$ is a matrix normed space.

A matrix normed space $(X, \| \cdot \|_n)$ is called an $L^\infty$-matrix normed space if $\|x \oplus y\|_{n+k} = \max\{\|x\|_n, \|y\|_k\}$ holds for all $x \in M_n(X)$ and all $y \in M_k(X)$.

Let $E, F$ be vector spaces. For a given mapping $h : E \rightarrow F$ and a given positive integer $n$, define $h_n : M_n(E) \rightarrow M_n(F)$ by
\[ h_n([x_{ij}]) = [h(x_{ij})] \]
for all $[x_{ij}] \in M_n(E)$.

Let $S$ be a set. A function $d : S \times S \rightarrow [0, \infty]$ is called a generalized metric on $S$ if $d$ satisfies
(1) \(d(x, y) = 0\) if and only if \(x = y\);
(2) \(d(x, y) = d(y, x), \forall x, y \in S\);
(3) \(d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in S\).

**Lemma 1.1.** (cf. [2]) Let \((S, d)\) be a complete generalized metric space and \(J : S \to S\) be a strictly contractive mapping with Lipschitz constant \(L < 1\). Then for each fixed element \(x \in S\), either
\[
d(J^n x, J^{n+1} x) = \infty \quad \forall n \geq 0,
\]
or
\[
d(J^n x, J^{n+1} x) < \infty \quad \forall n \geq n_0,
\]
for some natural number \(n_0\). Moreover, if the second alternative holds then:
(i) The sequence \(\{J^n x\}\) is convergent to a fixed point \(y^*\) of \(J\);
(ii) \(y^*\) is the unique fixed point of \(J\) in the set \(S^* := \{y \in S \mid d(J^{n_0} x, y) < +\infty\}\) and
\[
d(y, y^*) \leq \frac{1}{1-L}d(y, Jy), \quad \forall x, y \in S^*.
\]

Throughout this paper, let \((X, \| \cdot \|_n)\) be a matrix normed space and \((Y, \{\| \cdot \|_n\})\) a matrix Banach space.

### 2. Hyers-Ulam stability of the functional inequality (1.3) in matrix Banach spaces

In this section, we prove the Hyers-Ulam stability of the additive functional inequality (1.3) in matrix Banach spaces by using the fixed point method. We need the following Lemmas:

**Lemma 2.1.** (cf. [11, 12, 15, 16]) Let \((X, \{\| \cdot \|_n\})\) be a matrix normed space. Then
(1) \(\| E_{kl} \otimes x \|_n = \| x \|_n \) for \(x \in X\);
(2) \(\| x_{kl} \| \leq \sum_{i,j=1}^{n} \| x_{ij} \|_n \) for \([x_{ij}] \in M_n(X)\);
(3) \(\lim_{n \to \infty} x_n = x\) if and only if \(\lim_{n \to \infty} x_{ijn} = x_{ij}\) for \(x_n = [x_{ijn}], x = [x_{ij}] \in M_k(X)\).

**Lemma 2.2.** (cf. [13]) A mapping \(f : X \to Y\) satisfies
\[
\| f(a + b) - f(a) - f(b) \| \leq \left\| f\left(\frac{a + b}{2}\right) - \frac{1}{2} f(a) - \frac{1}{2} f(b)\right\|
\]
for all \(a, b \in X\) if and only if \(f : X \to Y\) is additive.

**Theorem 2.1.** Let \(\varphi : X^2 \to [0, \infty)\) be a function such that there exists an \(\alpha < 1\) with
\[
\varphi(a, b) \leq 2\alpha \varphi\left(\frac{a}{2}, \frac{b}{2}\right)
\]
(2.1)
for all $a, b \in X$. Suppose that $f : X \to Y$ is a mapping satisfying

\[
\|f_n(x_{ij} + [y_{ij}]) - f_n([x_{ij}]) - f_n([y_{ij}])\|_n \\
\leq \left\| f_n\left(\frac{[x_{ij}]}{2} + \frac{[y_{ij}]}{2}\right) - \frac{1}{2} f_n([x_{ij}]) - \frac{1}{2} f_n([y_{ij}])\right\|_n + \sum_{i,j=1}^n \phi(x_{ij}, y_{ij}) \tag{2.2}
\]

for all $x = [x_{ij}], y = [y_{ij}] \in \mathbb{M}_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that

\[
\|f_n([x_{ij}]) - A_n([x_{ij}])\|_n \leq \sum_{i,j=1}^n \frac{1}{2(1 - \alpha)} \phi(x_{ij}, x_{ij}) \tag{2.3}
\]

for all $x = [x_{ij}] \in \mathbb{M}_n(X)$.

**Proof.** When $n = 1$, (2.2) is equivalent to

\[
\|f(a + b) - f(a) - f(b)\| \leq \left\| f\left(\frac{a + b}{2}\right) - \frac{1}{2} f(a) - \frac{1}{2} f(b)\right\| + \phi(a, b) \tag{2.4}
\]

for all $a, b \in X$. Letting $b = a$ in (2.4), we get

\[
\|f(2a) - 2f(a)\| \leq \phi(a, a) \tag{2.5}
\]

for all $a \in X$. So

\[
\|f(a) - \frac{1}{2} f(2a)\| \leq \frac{1}{2} \phi(a, a) \tag{2.6}
\]

for all $a \in X$.

Let $S_1 := \{g_1 : X \to Y\}$, and introduce a generalized metric $d_1$ on $S_1$ as follows:

\[
d_1(g_1, h_1) := \inf \left\{ \lambda \in \mathbb{R}_+ \left| \|g_1(a) - h_1(a)\| \leq \lambda \phi(a, a), \forall a \in X \right. \right\}.
\]

It is easy to prove that $(S_1, d_1)$ is a complete generalized metric space [8, 9].

Now we consider the mapping $\mathcal{J}_1 : S_1 \to S_1$ defined by

\[
\mathcal{J}_1 g_1(a) := \frac{1}{2} g_1(2a), \text{ for all } g_1 \in S_1 \text{ and } a \in X. \tag{2.7}
\]

Let $g_1, h_1 \in S_1$ and let $\lambda \in \mathbb{R}_+$ be an arbitrary constant with $d_1(g_1, h_1) \leq \lambda$. From the definition of $d_1$, we get

\[
\|g_1(a) - h_1(a)\| \leq \lambda \phi(a, a)
\]

for all $a \in X$. Therefore, using (2.1), we get

\[
\|\mathcal{J}_1 g_1(a) - \mathcal{J}_1 h_1(a)\| = \left\| \frac{1}{2} g_1(2a) - \frac{1}{2} h_1(2a)\right\| \leq \frac{\lambda}{2} \phi(2a, 2a) \leq \alpha \lambda \phi(a, a) \tag{2.8}
\]

for some $\alpha < 1$ and for all $a \in X$. Hence, it holds that $d_1(\mathcal{J}_1 g_1, \mathcal{J}_1 h_1) \leq \alpha \lambda$, that is, $d_1(\mathcal{J}_1 g_1, \mathcal{J}_1 h_1) \leq \alpha d_1(g_1, h_1)$ for all $g_1, h_1 \in S_1$.

It follows from (2.6) that $d_1(f, \mathcal{J}_1 f) \leq \frac{1}{2}$. Therefore according to Lemma 1.1, the sequence $\mathcal{J}_1^n f$ converges to a fixed point $A$ of $\mathcal{J}_1$, that is,

\[
A : X \to Y, \quad \lim_{n \to \infty} \frac{1}{2^n} f(2^n a) = A(a)
\]
for all \( a \in X \), and

\[
A(2a) = 2A(a)
\]  
(2.9)

for all \( a \in X \). Also \( A \) is the unique fixed point of \( \mathcal{J}_1 \) in the set \( S_1^* = \{ g_1 \in S_1 : d_1(f, g_1) < \infty \} \). This implies that \( A \) is a unique mapping satisfying (2.9) such that there exists a \( \lambda \in \mathbb{R}_+ \) such that

\[
\| f(a) - A(a) \| \leq \lambda \varphi(a, a)
\]

for all \( a \in X \). Also,

\[
d_1(f, A) \leq \frac{1}{1 - \alpha} d_1(f, \mathcal{J}_1 f) \leq \frac{1}{2(1 - \alpha)}.
\]

So

\[
\| f(a) - A(a) \| \leq \frac{1}{2(1 - \alpha)} \varphi(a, a)
\]  
(2.10)

for all \( a \in X \).

It follows from (2.1) and (2.4) that

\[
\lim_{l \to \infty} \frac{1}{2l} \| f(2^l(a + b)) - f(2^l a) - f(2^l b) \| \leq \lim_{l \to \infty} \left( \frac{1}{2l} \left\| f\left(\frac{2^l(a + b)}{2}\right) - \frac{1}{2} f(2^l a) - \frac{1}{2} f(2^l b) \right\| + \frac{1}{2} \varphi(2^l a, 2^l b) \right)
\]

(2.11)

for all \( a, b \in X \). By (2.11), we get

\[
\| A(a + b) - A(a) - A(b) \| \leq \| A\left(\frac{a + b}{2}\right) - \frac{1}{2} A(a) - \frac{1}{2} A(b) \|
\]

for all \( a, b \in X \). By Lemma 2.2, the mapping \( A : X \to Y \) is additive.

By Lemma 2.1 and (2.10),

\[
\| f_n([x_{ij}]) - A_n([x_{ij}]) \| \leq \sum_{i,j=1}^{n} \| f(x_{ij}) - A(x_{ij}) \| \leq \sum_{i,j=1}^{n} \frac{1}{2(1 - \alpha)} \varphi(x_{ij}, x_{ij})
\]

for all \( x = [x_{ij}] \in M_n(X) \). Thus \( A : X \to Y \) is a unique additive mapping satisfying (2.3), as desired. This completes the proof of the theorem. \( \square \)

**Corollary 2.1.** Let \( r, \theta \) be positive real numbers with \( r < 1 \). Suppose that \( f : X \to Y \) is a mapping satisfying

\[
\| f_n([x_{ij}] + [y_{ij}]) - f_n([x_{ij}]) - f_n([y_{ij}]) \| \leq \left\| f_n\left(\frac{[x_{ij}] + [y_{ij}]}{2}\right) - \frac{1}{2} f_n([x_{ij}]) - \frac{1}{2} f_n([y_{ij}]) \right\| + \sum_{i,j=1}^{n} \theta \left( \| x_{ij} \|^r + \| y_{ij} \|^r \right)
\]

(2.12)

for all \( x = [x_{ij}], y = [y_{ij}] \in M_n(X) \). Then there exists a unique additive mapping \( A : X \to Y \) such that

\[
\| f_n([x_{ij}]) - A_n([x_{ij}]) \| \leq \sum_{i,j=1}^{n} \frac{2}{2 - 2r} \theta \| x_{ij} \|^r
\]

(2.13)

for all \( x = [x_{ij}] \in M_n(X) \).
Proof. The proof follows immediately by taking \( \varphi(a,b) = \theta(\|a\|^r + \|b\|^r) \) for all \( a, b \in X \) and choosing \( \alpha = 2^{r-1} \) in Theorem 2.1. 

**THEOREM 2.2.** Let \( \varphi : X^2 \to [0, \infty) \) be a function such that there exists an \( \alpha < 1 \) with
\[
\varphi(a,b) \leq \frac{\alpha}{2} \varphi(2a,2b)
\] (2.14)
for all \( a, b \in X \). Suppose that \( f : X \to Y \) is a mapping satisfying (2.2) for all \( x = [x_{ij}], y = [y_{ij}] \in M_n(X) \). Then there exists a unique additive mapping \( A : X \to Y \) such that
\[
\|f_n([x_{ij}]) - A_n([x_{ij}])\|_n \leq \frac{\alpha}{2(1 - \alpha)} \varphi(x_{ij}, x_{ij})
\] (2.15)
for all \( x = [x_{ij}] \in M_n(X) \).

**Proof.** Let \((S_1,d_1)\) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the mapping \( \mathcal{J}_1 : S_1 \to S_1 \) defined by
\[
\mathcal{J}_1 g_1(a) := 2g_1\left(\frac{a}{2}\right), \text{ for all } g_1 \in S_1 \text{ and } a \in X.
\] (2.16)

It follows from (2.5) that
\[
\left\|f(a) - 2f\left(\frac{a}{2}\right)\right\| \leq \frac{\alpha}{2} \varphi(a,a)
\] (2.17)
for all \( a \in X \). Thus \( d_1(f, \mathcal{J}_1 f) \leq \frac{\alpha}{2} \). So
\[
d_1(f,A) \leq \frac{1}{1 - \alpha} d_1(f, \mathcal{J}_1 f) \leq \frac{\alpha}{2(1 - \alpha)}.
\]
The rest of the proof is similar to the proof of Theorem 2.1. 

**COROLLARY 2.2.** Let \( r, \theta \) be positive real numbers with \( r > 1 \). Suppose that \( f : X \to Y \) is a mapping satisfying (2.12) for all \( x = [x_{ij}], y = [y_{ij}] \in M_n(X) \). Then there exists a unique additive mapping \( A : X \to Y \) such that
\[
\|f_n([x_{ij}]) - A_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2}{2^r - 2} \theta \|x_{ij}\|^r
\] (2.18)
for all \( x = [x_{ij}] \in M_n(X) \).

**Proof.** By choosing \( \varphi(a,b) = \theta(\|a\|^r + \|b\|^r) \) for all \( a, b \in X \) and \( \alpha = 2^{1-r} \) in Theorem 2.2, we obtain the inequality (2.18). 

\[\square\]
3. Hyers-Ulam stability of the functional inequality (1.4) in matrix Banach spaces

In this section, we prove the Hyers-Ulam stability of the quadratic functional inequality (1.4) in matrix Banach spaces by using the fixed point method.

We need the following result.

**Lemma 3.1.** (cf. [13]) A mapping \( f : X \to Y \) satisfies
\[
\|f(a + b) + f(a - b) - 2f(a) - 2f(b)\| \leq \|f\left(\frac{a + b}{2}\right) + f\left(\frac{a - b}{2}\right) - \frac{1}{2}f(a) - \frac{1}{2}f(b)\|
\]
for all \( a, b \in X \) if and only if \( f : X \to Y \) is quadratic.

**Theorem 3.1.** Let \( \phi : X^2 \to [0, \infty) \) be a function with \( \phi(0, 0) = 0 \) such that there exists an \( \alpha < 1 \) with
\[
\phi(a, b) \leq 4\alpha\phi\left(\frac{a}{2}, \frac{b}{2}\right)
\]
for all \( a, b \in X \). Suppose that \( f : X \to Y \) is a mapping satisfying
\[
\|f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\|_n \\
\leq \left\|f_n\left(\frac{[x_{ij}] + [y_{ij}]}{2}\right) + f_n\left(\frac{[x_{ij}] - [y_{ij}]}{2}\right) - \frac{1}{2}f_n([x_{ij}]) - \frac{1}{2}f_n([y_{ij}])\right\|_n \\
+ \sum_{i,j=1}^n \phi(x_{ij}, y_{ij})
\]
for all \( x = [x_{ij}], y = [y_{ij}] \in M_n(X) \). Then there exists a unique quadratic mapping \( Q : X \to Y \) such that
\[
\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^n \frac{1}{4(1 - \alpha)} \phi(x_{ij}, x_{ij})
\]
for all \( x = [x_{ij}] \in M_n(X) \).

**Proof.** When \( n = 1 \), (3.2) is equivalent to
\[
\|f(a + b) + f(a - b) - 2f(a) - 2f(b)\| \\
\leq \left\|f\left(\frac{a + b}{2}\right) + f\left(\frac{a - b}{2}\right) - \frac{1}{2}f(a) - \frac{1}{2}f(b)\right\| + \phi(a, b)
\]
for all \( a, b \in X \). Letting \( a = b = 0 \) in (3.4), we get \( \|2f(0)\| \leq \|f(0)\| \). So \( f(0) = 0 \).

Letting \( b = a \) in (3.4), we get
\[
\|f(2a) - 4f(a)\| \leq \phi(a, a)
\]
for all \( a \in X \). So
\[
\|f(a) - \frac{1}{4}f(2a)\| \leq \frac{1}{4}\phi(a, a)
\]
for all \( a \in X \).
Let $S_2 := \{g_2 : X \to Y\}$, and introduce a generalized metric $d_2$ on $S_2$ as follows:

$$d_2(g_2, h_2) := \inf \left\{ \mu \in \mathbb{R}_+ \middle| \|g_2(a) - h_2(a)\| \leq \mu \phi(a, a), \forall a \in X \right\}.$$ 

It is easy to prove that $(S_2, d_2)$ is a complete generalized metric space [8, 9].

Now we consider the mapping $\mathcal{J}_2 : S_2 \to S_2$ defined by

$$\mathcal{J}_2 g_2(a) := \frac{1}{4} g_2(2a), \text{ for all } g_2 \in S_2 \text{ and } a \in X. \quad (3.7)$$

Let $g_2, h_2 \in S_2$ and let $\mu \in \mathbb{R}_+$ be an arbitrary constant with $d_2(g_2, h_2) \leq \mu$. From the definition of $d_2$, we get

$$\|g_2(a) - h_2(a)\| \leq \mu \phi(a, a)$$

for all $a \in X$. Therefore, using (3.1), we get

$$\|\mathcal{J}_2 g_2(a) - \mathcal{J}_2 h_2(a)\| = \left\| \frac{1}{4} g_2(2a) - \frac{1}{4} h_2(2a) \right\| \leq \frac{\mu}{4} \phi(2a, 2a) \leq \alpha \mu \phi(a, a) \quad (3.8)$$

for some $\alpha < 1$ and for all $a \in X$. Hence, it holds that $d_2(\mathcal{J}_2 g_2, \mathcal{J}_2 h_2) \leq \alpha \mu$, that is, $d_2(\mathcal{J}_2 g_2, \mathcal{J}_2 h_2) \leq \alpha d_2(g_2, h_2)$ for all $g_2, h_2 \in S_2$.

It follows from (3.6) that $d_2(f, \mathcal{J}_2 f) \leq \frac{1}{4}$. Therefore according to Lemma 1.1, the sequence $\mathcal{J}_2^n f$ converges to a fixed point $Q$ of $\mathcal{J}_2$, that is,

$$Q : X \to Y, \quad \lim_{n \to \infty} \frac{1}{4^n} f(2^n a) = Q(a)$$

for all $a \in X$, and

$$Q(2a) = 4Q(a) \quad (3.9)$$

for all $a \in X$. Also $Q$ is the unique fixed point of $\mathcal{J}_2$ in the set $S_2^* = \{g_2 \in S_2 : d_2(f, g_2) < \infty\}$. This implies that $Q$ is a unique mapping satisfying (3.9) such that there exists a $\mu \in \mathbb{R}_+$ such that

$$\|f(a) - Q(a)\| \leq \mu \phi(a, a)$$

for all $a \in X$. Also,

$$d_2(f, Q) \leq \frac{1}{1 - \alpha} d_2(f, \mathcal{J}_2 f) \leq \frac{1}{4(1 - \alpha)}.$$

So

$$\|f(a) - Q(a)\| \leq \frac{1}{4(1 - \alpha)} \phi(a, a) \quad (3.10)$$

for all $a \in X$.

It follows from (3.1) and (3.4) that

$$\lim_{l \to \infty} \frac{1}{4^l} \|f(2^l (a + b)) + f(2^l (a - b)) - 2f(2^l a) - 2f(2^l b)\|$$

$$\leq \lim_{l \to \infty} \left( \frac{1}{4^l} \|f \left( \frac{2^l (a + b)}{2} \right) + f \left( \frac{2^l (a - b)}{2} \right) - \frac{1}{2} f \left( 2^l a \right) - \frac{1}{2} f \left( 2^l b \right) \| + \frac{1}{4^l} \phi(2^l a, 2^l b) \right)$$

(3.11)
for all \( a, b \in X \). By (3.11), we get
\[
\|Q(a + b) + Q(a - b) - 2Q(a) - 2Q(b)\| \\
\leq \|Q\left(\frac{a + b}{2}\right) + Q\left(\frac{a - b}{2}\right) - \frac{1}{2}Q(a) - \frac{1}{2}Q(b)\|
\]
for all \( a, b \in X \). By Lemma 3.1, the mapping \( Q : X \to Y \) is quadratic.

By Lemma 2.1 and (3.10),
\[
\|f_n([x_{ij}]) - Q_n([y_{ij}])\| \leq \sum_{i,j=1}^{n} \|f(x_{ij}) - Q(x_{ij})\| \leq \sum_{i,j=1}^{n} \frac{1}{4(1 - \alpha)} \phi(x_{ij}, x_{ij})
\]
for all \( x = [x_{ij}] \in M_n(X) \). Then there exists a unique quadratic mapping \( Q : X \to Y \) such that
\[
\|f_n([x_{ij}]) - Q_n([x_{ij}])\| \leq \sum_{i,j=1}^{n} \frac{2}{4 - 2r} \theta \|x_{ij}\|^r
\]
for all \( x = [x_{ij}] \in M_n(X) \).

Proof. The proof follows immediately by taking \( \phi(a, b) = \theta(\|a\|^r + \|b\|^r) \) for all \( a, b \in X \) and choosing \( \alpha = 2^{r-2} \) in Theorem 3.1. \( \square \)

\textbf{Corollary 3.1.} Let \( r, \theta \) be positive real numbers with \( r < 2 \). Suppose that \( f : X \to Y \) is a mapping satisfying
\[
\|f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\|_n \\
\leq \|f_n\left(\frac{[x_{ij}] + [y_{ij}]}{2}\right) + f_n\left(\frac{[x_{ij}] - [y_{ij}]}{2}\right) - \frac{1}{2}f_n([x_{ij}]) - \frac{1}{2}f_n([y_{ij}])\|_n \\
+ \sum_{i,j=1}^{n} \theta(\|x_{ij}\|^r + \|y_{ij}\|^r)
\]
for all \( x = [x_{ij}], y = [y_{ij}] \in M_n(X) \). Then there exists a unique quadratic mapping \( Q : X \to Y \) such that
\[
\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2}{4 - 2r} \theta \|x_{ij}\|^r
\]
for all \( x = [x_{ij}] \in M_n(X) \).

Proof. The proof follows immediately by taking \( \phi(a, b) = \theta(\|a\|^r + \|b\|^r) \) for all \( a, b \in X \) and choosing \( \alpha = 2^{r-2} \) in Theorem 3.1. \( \square \)

\textbf{Theorem 3.2.} Let \( \phi : X^2 \to [0, \infty) \) be a function with \( \phi(0, 0) = 0 \) such that there exists an \( \alpha < 1 \) with
\[
\phi(a, b) \leq \frac{\alpha}{4} \phi(2a, 2b)
\]
for all \( a, b \in X \). Suppose that \( f : X \to Y \) is a mapping satisfying (3.2) for all \( x = [x_{ij}], y = [y_{ij}] \in M_n(X) \). Then there exists a unique quadratic mapping \( Q : X \to Y \) such that
\[
\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{\alpha}{4(1 - \alpha)} \phi(x_{ij}, x_{ij})
\]
for all \( x = [x_{ij}] \in M_n(X) \).
Proof. Let \((S_2,d_2)\) be the generalized metric space defined in the proof of Theorem 3.1.

Now we consider the mapping \(J_2 : S_2 \to S_2\) defined by
\[
J_2 g(a) := 4g\left(\frac{a}{2}\right), \quad \text{for all } g \in S_2 \text{ and } a \in X.
\]

(3.16)

It follows from (3.5) that
\[
\|f(a) - 4f\left(\frac{a}{2}\right)\| \leq \frac{\alpha}{4} \phi(a,a)
\]
for all \(a \in X\). Thus \(d_2(f, J_2f) \leq \frac{\alpha}{4}\). So
\[
d_2(f, Q) \leq \frac{1}{1 - \alpha} d_2(f, J_2f) \leq \frac{\alpha}{4(1 - \alpha)}.
\]

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.2. Let \(r, \theta\) be positive real numbers with \(r > 2\). Suppose that \(f : X \to Y\) is a mapping satisfying (3.12) for all \(x = [x_{ij}], y = [y_{ij}] \in M_n(X)\). Then there exists a unique quadratic mapping \(Q : X \to Y\) such that
\[
\|f_n([x_{ij}]) - Q_n([x_{ij}])\| \leq \sum_{i,j=1}^{n} \frac{2}{2r-4} \theta \|x_{ij}\|^r
\]
for all \(x = [x_{ij}] \in M_n(X)\).

Proof. By choosing \(\phi(a,b) = \theta(\|a\|^r + \|b\|^r)\) for all \(a, b \in X\) and \(\alpha = 2^{2-r}\) in Theorem 3.2, we obtain the inequality (3.18). □

References


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