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FUNCTIONAL INEQUALITIES IN MATRIX BANACH SPACES

ZHIHUA WANG

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Abstract. Using the fixed point method, we prove the Hyers-Ulam stability of the following additive functional inequality and quadratic functional inequality

$$\|f(x+y)-f(x)-f(y)\| \leqslant \|f(\frac{x+y}{2})-\frac{1}{2}f(x)-\frac{1}{2}f(y)\|,$$
$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\|$$
$$\leqslant \|f(\frac{x+y}{2})+f(\frac{x-y}{2})-\frac{1}{2}f(x)-\frac{1}{2}f(y)\|$$

in matrix Banach spaces, respectively.

1. Introduction and preliminaries

In 1940, Ulam [20] posed the first stability problem concerning group homomorphisms. In the next year, Hyers [7] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' result was generalized by Aoki [1] for additive mappings and by Rassias [18] for linear mappings. Găvruta [6] obtained generalized Rassias' result which allows the Cauchy difference to be controlled by a general unbounded function in the spirit of Rassias' approach.

Gilányi [4] and Rätz [19] proved that if, for a function $f: G \to E$ mapping from Abelian group G divisible by 2 into an inner product space E, the functional inequality

$$\|2f(x)+2f(y)-f(xy^{-1})\| \leqslant \|f(xy)\|, \quad \forall x,y \in G \tag{1.1}$$

holds, then f satisfies the Jordan-Von Neumann functional equation

$$2f(x)+2f(y)=f(xy)+f(xy^{-1}), \quad \forall x,y \in G.$$

Fechner [3] and Gilányi [5] have proved the generalized Hyers-Ulam stability of the functional inequality (1.1). Park et. al. [14] have investigated the generalized Hyers-Ulam stability of the following inequalities associated with Jordon-Von Neumann type additive functional equations:

$$\|f(x)+f(y)+f(z)\| \leqslant \left\|2f\left(\frac{x+y+z}{2}\right)\right\|,$$
$$\|f(x)+f(y)+f(z)\| \leqslant \|f(x+y+z)\|,$$
$$\|f(x)+f(y)+2f(z)\| \leqslant \left\|2f\left(\frac{x+y}{2}+z\right)\right\|.$$

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In 2013, Kim, Jun and Son [10] considered the following quadratic functional inequality

$$\|f(x-y)+f(y-z)+f(x-z)-3f(x)-3f(y)-3f(z)\| \leqslant \|f(x+y+z)\|. \tag{1.2}$$

They established the general solution of the quadratic functional inequality (1.2), and then investigated the generalized Hyers-Ulam stability of this inequality in Banach spaces and in non-Archimedean Banach spaces. Recently, the stability results of several functional equations and inequalities were investigated [11, 12, 15, 16, 17] in matrix normed spaces, matrix paranormed spaces and matrix fuzzy normed spaces.

In 2015, Park [13] considered the following functional inequalities:

$$\|f(x+y)-f(x)-f(y)\| \leqslant \left\|f\left(\frac{x+y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y)\right\|, \tag{1.3}$$

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \leqslant \left\|f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y)\right\|. \tag{1.4}$$

Using the direct method, he proved Hyers-Ulam stability of the functional inequalities (1.3) and (1.4) in Banach spaces. The main purpose of this paper is to apply the fixed point method to investigate the Hyers-Ulam stability of functional inequalities (1.3) and (1.4) in matrix Banach spaces, respectively.

Next, we will also use the following notations:

The set of all $m\times n$-matrices in X will be denoted by $M_{m,n}(X)$. When $m=n$, the matrix $M_{m,n}(X)$ will be written as $M_n(X)$. The symbols $e_j \in M_{1,n}(\mathbb{C})$ will denote the row vector whose jth component is 1 and the other components are 0. Similarly, $E_{ij}\in M_n(\mathbb{C})$ will denote the $n\times n$ matrix whose (i,j)-component is 1 and the other components are 0. The $n\times n$ matrix whose (i,j)-component is x and the other components are 0 will be denoted by $E_{ij}\otimes x\in M_n(X)$.

For $x\in M_n(X)$, $y\in M_k(X)$,

$$x\oplus y=\begin{pmatrix} x & 0\\ 0 & y\end{pmatrix}.$$

Let $(X,\|\cdot\|)$ be a normed space. Note that $(X,\{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X),\|\cdot\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \leqslant \|A\|\|B\|\|x\|_n$ holds for $A\in M_{k,n}$, $x=[x_{ij}]\in M_n(X)$ and $B\in M_{n,k}$. And that $(X,\{\|\cdot\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X,\{\|\cdot\|_n\})$ is a matrix normed space.

A matrix normed space $(X,\|\cdot\|_n)$ is called an L^∞-matrix normed space if $\|x\oplus y\|_{n+k}=\max\{\|x\|_n,\|y\|_k\}$ holds for all $x\in M_n(X)$ and all $y\in M_k(X)$.

Let E,F be vector spaces. For a given mapping $h:E\to F$ and a given positive integer n, define $h_n: M_n(E)\to M_n(F)$ by

$$h_n([x_{ij}])=[h(x_{ij})]$$

for all $[x_{ij}]\in M_n(E)$.

Let S be a set. A function $d: S\times S\to[0,\infty]$ is called a generalized metric on S if d satisfies

(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$, $\forall x,y\in S$;
(3) $d(x,z)\leqslant d(x,y)+d(y,z)$, $\forall x,y,z\in S$.

LEMMA 1.1. (cf. [2]) *Let (S,d) be a complete generalized metric space and $J:S\to S$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each fixed element $x\in S$, either*

$$d(J^n x, J^{n+1}x)=\infty \quad \forall n\geqslant 0,$$

or

$$d(J^n x, J^{n+1}x)<\infty \quad \forall n\geqslant n_0,$$

for some natural number n_0. Moreover, if the second alternative holds then:
(i) The sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;
(ii) y^* is the unique fixed point of J in the set $S^*:=\{y\in S \mid d(J^{n_0}x,y)<+\infty\}$ and $d(y,y^*)\leqslant \frac{1}{1-L}d(y,Jy)$, $\forall x,y\in S^*$.

Throughout this paper, let $(X,\{\|\cdot\|_n\})$ be a matrix normed space and $(Y,\{\|\cdot\|_n\})$ a matrix Banach space.

2. Hyers-Ulam stability of the functional inequality (1.3) in matrix Banach spaces

In this section, we prove the Hyers-Ulam stability of the additive functional inequality (1.3) in matrix Banach spaces by using the fixed point method. We need the following Lemmas:

LEMMA 2.1. (cf. [11, 12, 15, 16]) *Let $(X,\{\|\cdot\|_n\})$ be a matrix normed space. Then*
(1) $\|E_{kl}\otimes x\|_n=\|x\|$ *for* $x\in X$;
(2) $\|x_{kl}\|\leqslant\|[x_{ij}]\|_n\leqslant\sum_{i,j=1}^{n}\|x_{ij}\|$ *for* $[x_{ij}]\in M_n(X)$;
(3) $\lim_{n\to\infty}x_n=x$ *if and only if* $\lim_{n\to\infty}x_{ijn}=x_{ij}$ *for* $x_n=[x_{ijn}]$, $x=[x_{ij}]\in M_k(X)$.

LEMMA 2.2. (cf. [13]) *A mapping $f:X\to Y$ satisfies*

$$\|f(a+b)-f(a)-f(b)\|\leqslant\left\|f\left(\frac{a+b}{2}\right)-\frac{1}{2}f(a)-\frac{1}{2}f(b)\right\|$$

for all $a,b\in X$ if and only if $f:X\to Y$ is additive.

THEOREM 2.1. *Let $\varphi:X^2\to[0,\infty)$ be a function such that there exists an $\alpha<1$ with*

$$\varphi(a,b)\leqslant 2\alpha\varphi\left(\frac{a}{2},\frac{b}{2}\right) \tag{2.1}$$

for all $a, b \in X$. *Suppose that* $f : X \to Y$ *is a mapping satisfying*

$$\begin{aligned}\|f_n([x_{ij}]+[y_{ij}]) - f_n([x_{ij}]) - f_n([y_{ij}])\|_n \\ \leqslant \left\| f_n\Big(\frac{[x_{ij}]+[y_{ij}]}{2}\Big) - \frac{1}{2}f_n([x_{ij}]) - \frac{1}{2}f_n([y_{ij}])\right\|_n + \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij})\end{aligned} \tag{2.2}$$

for all $x = [x_{ij}]$, $y = [y_{ij}] \in M_n(X)$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\|f_n([x_{ij}]) - A_n([x_{ij}])\|_n \leqslant \sum_{i,j=1}^{n} \frac{1}{2(1-\alpha)} \varphi(x_{ij}, x_{ij}) \tag{2.3}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. When $n = 1$, (2.2) is equivalent to

$$\|f(a+b) - f(a) - f(b)\| \leqslant \left\| f\Big(\frac{a+b}{2}\Big) - \frac{1}{2}f(a) - \frac{1}{2}f(b)\right\| + \varphi(a,b) \tag{2.4}$$

for all $a, b \in X$. Letting $b = a$ in (2.4), we get

$$\|f(2a) - 2f(a)\| \leqslant \varphi(a,a) \tag{2.5}$$

for all $a \in X$. So

$$\|f(a) - \frac{1}{2}f(2a)\| \leqslant \frac{1}{2}\varphi(a,a) \tag{2.6}$$

for all $a \in X$.

Let $S_1 := \{g_1 : X \to Y\}$, and introduce a generalized metric d_1 on S_1 as follows:

$$d_1(g_1, h_1) := \inf\left\{ \lambda \in \mathbb{R}_+ \,\middle|\, \|g_1(a) - h_1(a)\| \leqslant \lambda\varphi(a,a), \forall a \in X \right\}.$$

It is easy to prove that (S_1, d_1) is a complete generalized metric space [8, 9].

Now we consider the mapping $\mathscr{J}_1 : S_1 \to S_1$ defined by

$$\mathscr{J}_1 g_1(a) := \frac{1}{2} g_1(2a), \text{ for all } g_1 \in S_1 \text{ and } a \in X. \tag{2.7}$$

Let $g_1, h_1 \in S_1$ and let $\lambda \in \mathbb{R}_+$ be an arbitrary constant with $d_1(g_1, h_1) \leqslant \lambda$. From the definition of d_1, we get

$$\|g_1(a) - h_1(a)\| \leqslant \lambda\varphi(a,a)$$

for all $a \in X$. Therefore, using (2.1), we get

$$\|\mathscr{J}_1 g_1(a) - \mathscr{J}_1 h_1(a)\| = \left\| \frac{1}{2} g_1(2a) - \frac{1}{2} h_1(2a) \right\| \leqslant \frac{\lambda}{2}\varphi(2a,2a) \leqslant \alpha\lambda\varphi(a,a) \tag{2.8}$$

for some $\alpha < 1$ and for all $a \in X$. Hence, it holds that $d_1(\mathscr{J}_1 g_1, \mathscr{J}_1 h_1) \leqslant \alpha\lambda$, that is, $d_1(\mathscr{J}_1 g_1, \mathscr{J}_1 h_1) \leqslant \alpha d_1(g_1, h_1)$ for all $g_1, h_1 \in S_1$.

It follows from (2.6) that $d_1(f, \mathscr{J}_1 f) \leqslant \frac{1}{2}$. Therefore according to Lemma 1.1, the sequence $\mathscr{J}_1^n f$ converges to a fixed point A of $\mathscr{J}_1$, that is,

$$A : X \to Y, \quad \lim_{n\to\infty} \frac{1}{2^n} f(2^n a) = A(a)$$

for all $a \in X$, and

$$A(2a) = 2A(a) \tag{2.9}$$

for all $a \in X$. Also A is the unique fixed point of $\mathcal{J}_1$ in the set $S_1^* = \{g_1 \in S_1 : d_1(f, g_1) < \infty\}$. This implies that A is a unique mapping satisfying (2.9) such that there exists a $\lambda \in \mathbb{R}_+$ such that

$$\|f(a) - A(a)\| \leqslant \lambda \varphi(a, a)$$

for all $a \in X$. Also,

$$d_1(f, A) \leqslant \frac{1}{1-\alpha} d_1(f, \mathcal{J}_1 f) \leqslant \frac{1}{2(1-\alpha)}.$$

So

$$\|f(a) - A(a)\| \leqslant \frac{1}{2(1-\alpha)} \varphi(a, a) \tag{2.10}$$

for all $a \in X$.

It follows from (2.1) and (2.4) that

$$\begin{aligned} &\lim_{l\to\infty} \frac{1}{2^l} \|f(2^l(a+b)) - f(2^l a) - f(2^l b)\| \\ &\leqslant \lim_{l\to\infty} \left(\frac{1}{2^l} \|f\left(\frac{2^l(a+b)}{2}\right) - \frac{1}{2} f(2^l a) - \frac{1}{2} f(2^l b)\| + \frac{1}{2^l} \varphi(2^l a, 2^l b) \right) \end{aligned} \tag{2.11}$$

for all $a, b \in X$. By (2.11), we get

$$\|A(a+b) - A(a) - A(b)\| \leqslant \left\| A\left(\frac{a+b}{2}\right) - \frac{1}{2} A(a) - \frac{1}{2} A(b) \right\|$$

for all $a, b \in X$. By Lemma 2.2, the mapping $A : X \to Y$ is additive.

By Lemma 2.1 and (2.10),

$$\|f_n([x_{ij}]) - A_n([x_{ij}])\|_n \leqslant \sum_{i,j=1}^{n} \|f(x_{ij}) - A(x_{ij})\| \leqslant \sum_{i,j=1}^{n} \frac{1}{2(1-\alpha)} \varphi(x_{ij}, x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $A : X \to Y$ is a unique additive mapping satisfying (2.3), as desired. This completes the proof of the theorem. □

COROLLARY 2.1. *Let r, θ be positive real numbers with $r < 1$. Suppose that $f : X \to Y$ is a mapping satisfying*

$$\begin{aligned} &\|f_n([x_{ij}] + [y_{ij}]) - f_n([x_{ij}]) - f_n([y_{ij}])\|_n \\ &\leqslant \left\| f_n\left(\frac{[x_{ij}] + [y_{ij}]}{2}\right) - \frac{1}{2} f_n([x_{ij}]) - \frac{1}{2} f_n([y_{ij}]) \right\|_n + \sum_{i,j=1}^{n} \theta(\|x_{ij}\|^r + \|y_{ij}\|^r) \end{aligned} \tag{2.12}$$

for all $x = [x_{ij}]$, $y = [y_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f_n([x_{ij}]) - A_n([x_{ij}])\|_n \leqslant \sum_{i,j=1}^{n} \frac{2}{2-2^r} \theta \|x_{ij}\|^r \tag{2.13}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows immediately by taking $\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)$ for all $a,b \in X$ and choosing $\alpha = 2^{r-1}$ in Theorem 2.1. □

THEOREM 2.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a,b) \leqslant \frac{\alpha}{2}\varphi(2a,2b) \tag{2.14}$$

for all $a,b \in X$. Suppose that $f : X \to Y$ is a mapping satisfying (2.2) for all $x = [x_{ij}]$, $y = [y_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f_n([x_{ij}]) - A_n([x_{ij}])\|_n \leqslant \sum_{i,j=1}^{n} \frac{\alpha}{2(1-\alpha)}\varphi(x_{ij},x_{ij}) \tag{2.15}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Let (S_1,d_1) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the mapping $\mathscr{J}_1 : S_1 \to S_1$ defined by

$$\mathscr{J}_1 g_1(a) := 2g_1\left(\frac{a}{2}\right), \text{ for all } g_1 \in S_1 \text{ and } a \in X. \tag{2.16}$$

It follows from (2.5) that

$$\left\|f(a) - 2f\left(\frac{a}{2}\right)\right\| \leqslant \frac{\alpha}{2}\varphi(a,a) \tag{2.17}$$

for all $a \in X$. Thus $d_1(f,\mathscr{J}_1 f) \leqslant \frac{\alpha}{2}$. So

$$d_1(f,A) \leqslant \frac{1}{1-\alpha}d_1(f,\mathscr{J}_1 f) \leqslant \frac{\alpha}{2(1-\alpha)}.$$

The rest of the proof is similar to the proof of Theorem 2.1. □

COROLLARY 2.2. *Let r,θ be positive real numbers with $r > 1$. Suppose that $f : X \to Y$ is a mapping satisfying (2.12) for all $x = [x_{ij}]$, $y = [y_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f_n([x_{ij}]) - A_n([x_{ij}])\|_n \leqslant \sum_{i,j=1}^{n} \frac{2}{2^r - 2}\theta\|x_{ij}\|^r \tag{2.18}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. By choosing $\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)$ for all $a,b \in X$ and $\alpha = 2^{1-r}$ in Theorem 2.2, we obtain the inequality (2.18). □

3. Hyers-Ulam stability of the functional inequality (1.4) in matrix Banach spaces

In this section, we prove the Hyers-Ulam stability of the quadratic functional inequality (1.4) in matrix Banach spaces by using the fixed point method.

We need the following result.

LEMMA 3.1. (cf. [13]) *A mapping* $f : X \to Y$ *satisfies*

$$\|f(a+b)+f(a-b)-2f(a)-2f(b)\| \leqslant \left\|f\left(\frac{a+b}{2}\right)+f\left(\frac{a-b}{2}\right)-\frac{1}{2}f(a)-\frac{1}{2}f(b)\right\|$$

for all $a,b \in X$ *if and only if* $f : X \to Y$ *is quadratic.*

THEOREM 3.1. *Let* $\phi : X^2 \to [0,\infty)$ *be a function with* $\phi(0,0)=0$ *such that there exists an* $\alpha < 1$ *with*

$$\phi(a,b) \leqslant 4\alpha\phi\left(\frac{a}{2},\frac{b}{2}\right) \tag{3.1}$$

for all $a,b \in X$. *Suppose that* $f : X \to Y$ *is a mapping satisfying*

$$\begin{aligned}\|f_n([x_{ij}]+[y_{ij}])&+f_n([x_{ij}]-[y_{ij}])-2f_n([x_{ij}])-2f_n([y_{ij}])\|_n\\ &\leqslant \left\|f_n\left(\frac{[x_{ij}]+[y_{ij}]}{2}\right)+f_n\left(\frac{[x_{ij}]-[y_{ij}]}{2}\right)-\frac{1}{2}f_n([x_{ij}])-\frac{1}{2}f_n([y_{ij}])\right\|_n\\ &\quad+\sum_{i,j=1}^{n}\phi(x_{ij},y_{ij})\end{aligned} \tag{3.2}$$

for all $x=[x_{ij}]$, $y=[y_{ij}] \in M_n(X)$. *Then there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\|f_n([x_{ij}])-Q_n([x_{ij}])\|_n \leqslant \sum_{i,j=1}^{n}\frac{1}{4(1-\alpha)}\phi(x_{ij},x_{ij}) \tag{3.3}$$

for all $x=[x_{ij}] \in M_n(X)$.

Proof. When $n=1$, (3.2) is equivalent to

$$\begin{aligned}\|f(a+b)&+f(a-b)-2f(a)-2f(b)\|\\ &\leqslant \left\|f\left(\frac{a+b}{2}\right)+f\left(\frac{a-b}{2}\right)-\frac{1}{2}f(a)-\frac{1}{2}f(b)\right\|+\phi(a,b)\end{aligned} \tag{3.4}$$

for all $a,b \in X$. Letting $a=b=0$ in (3.4), we get $\|2f(0)\| \leqslant \|f(0)\|$. So $f(0)=0$.

Letting $b=a$ in (3.4), we get

$$\|f(2a)-4f(a)\| \leqslant \phi(a,a) \tag{3.5}$$

for all $a \in X$. So

$$\left\|f(a)-\frac{1}{4}f(2a)\right\| \leqslant \frac{1}{4}\phi(a,a) \tag{3.6}$$

for all $a \in X$.

Let $S_2 := \{g_2 : X \to Y\}$, and introduce a generalized metric d_2 on S_2 as follows:

$$d_2(g_2,h_2) := \inf\left\{\mu \in \mathbb{R}_+ \middle| \|g_2(a)-h_2(a)\| \leqslant \mu\phi(a,a), \ \ \forall a \in X\right\}.$$

It is easy to prove that (S_2,d_2) is a complete generalized metric space [8, 9].

Now we consider the mapping $\mathscr{J}_2 : S_2 \to S_2$ defined by

$$\mathscr{J}_2 g_2(a) := \frac{1}{4}g_2(2a), \ \text{ for all } g_2 \in S_2 \ \text{ and } a \in X. \tag{3.7}$$

Let $g_2,h_2 \in S_2$ and let $\mu \in \mathbb{R}_+$ be an arbitrary constant with $d_2(g_2,h_2) \leqslant \mu$. From the definition of d_2, we get

$$\|g_2(a)-h_2(a)\| \leqslant \mu\phi(a,a)$$

for all $a \in X$. Therefore, using (3.1), we get

$$\|\mathscr{J}_2 g_2(a) - \mathscr{J}_2 h_2(a)\| = \left\|\frac{1}{4}g_2(2a) - \frac{1}{4}h_2(2a)\right\| \leqslant \frac{\mu}{4}\phi(2a,2a) \leqslant \alpha\mu\phi(a,a) \tag{3.8}$$

for some $\alpha < 1$ and for all $a \in X$. Hence, it holds that $d_2(\mathscr{J}_2 g_2, \mathscr{J}_2 h_2) \leqslant \alpha\mu$, that is, $d_2(\mathscr{J}_2 g_2, \mathscr{J}_2 h_2) \leqslant \alpha d_2(g_2,h_2)$ for all $g_2,h_2 \in S_2$.

It follows from (3.6) that $d_2(f, \mathscr{J}_2 f) \leqslant \frac{1}{4}$. Therefore according to Lemma 1.1, the sequence $\mathscr{J}_2^n f$ converges to a fixed point Q of $\mathscr{J}_2$, that is,

$$Q : X \to Y, \quad \lim_{n\to\infty} \frac{1}{4^n} f(2^n a) = Q(a)$$

for all $a \in X$, and

$$Q(2a) = 4Q(a) \tag{3.9}$$

for all $a \in X$. Also Q is the unique fixed point of $\mathscr{J}_2$ in the set $S_2^* = \{g_2 \in S_2 : d_2(f,g_2) < \infty\}$. This implies that Q is a unique mapping satisfying (3.9) such that there exists a $\mu \in \mathbb{R}_+$ such that

$$\|f(a)-Q(a)\| \leqslant \mu\phi(a,a)$$

for all $a \in X$. Also,

$$d_2(f,Q) \leqslant \frac{1}{1-\alpha} d_2(f, \mathscr{J}_2 f) \leqslant \frac{1}{4(1-\alpha)}.$$

So

$$\|f(a)-Q(a)\| \leqslant \frac{1}{4(1-\alpha)}\phi(a,a) \tag{3.10}$$

for all $a \in X$.

It follows from (3.1) and (3.4) that

$$\begin{aligned} &\lim_{l\to\infty} \frac{1}{4^l}\|f(2^l(a+b)) + f(2^l(a-b)) - 2f(2^l a) - 2f(2^l b)\| \\ &\leqslant \lim_{l\to\infty}\left(\frac{1}{4^l}\left\|f\left(\frac{2^l(a+b)}{2}\right) + f\left(\frac{2^l(a-b)}{2}\right) - \frac{1}{2}f\left(2^l a\right) - \frac{1}{2}f(2^l b)\right\| + \frac{1}{4^l}\phi(2^l a, 2^l b)\right) \end{aligned} \tag{3.11}$$

for all $a,b \in X$. By (3.11), we get

$$\|Q(a+b)+Q(a-b)-2Q(a)-2Q(b)\| \leqslant \left\|Q\left(\frac{a+b}{2}\right)+Q\left(\frac{a-b}{2}\right)-\frac{1}{2}Q(a)-\frac{1}{2}Q(b)\right\|$$

for all $a,b \in X$. By Lemma 3.1, the mapping $Q: X \to Y$ is quadratic.

By Lemma 2.1 and (3.10),

$$\|f_n([x_{ij}])-Q_n([x_{ij}])\|_n \leqslant \sum_{i,j=1}^{n} \|f(x_{ij})-Q(x_{ij})\| \leqslant \sum_{i,j=1}^{n} \frac{1}{4(1-\alpha)}\phi(x_{ij},x_{ij})$$

for all $x=[x_{ij}] \in M_n(X)$. Thus $Q: X \to Y$ is a unique quadratic mapping satisfying (3.3), as desired. This completes the proof of the theorem. □

COROLLARY 3.1. *Let r,θ be positive real numbers with $r<2$. Suppose that $f: X \to Y$ is a mapping satisfying*

$$\begin{aligned}\|f_n([x_{ij}]+[y_{ij}])+f_n([x_{ij}]-[y_{ij}])-2f_n([x_{ij}])-2f_n([y_{ij}])\|_n \\ \leqslant \left\|f_n\left(\frac{[x_{ij}]+[y_{ij}]}{2}\right)+f_n\left(\frac{[x_{ij}]-[y_{ij}]}{2}\right)-\frac{1}{2}f_n([x_{ij}])-\frac{1}{2}f_n([y_{ij}])\right\|_n \\ +\sum_{i,j=1}^{n}\theta(\|x_{ij}\|^r+\|y_{ij}\|^r)\end{aligned} \tag{3.12}$$

for all $x=[x_{ij}]$, $y=[y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q: X \to Y$ such that

$$\|f_n([x_{ij}])-Q_n([x_{ij}])\|_n \leqslant \sum_{i,j=1}^{n} \frac{2}{4-2^r}\theta\|x_{ij}\|^r \tag{3.13}$$

for all $x=[x_{ij}] \in M_n(X)$.

Proof. The proof follows immediately by taking $\phi(a,b)=\theta(\|a\|^r+\|b\|^r)$ for all $a,b \in X$ and choosing $\alpha=2^{r-2}$ in Theorem 3.1. □

THEOREM 3.2. *Let $\phi: X^2 \to [0,\infty)$ be a function with $\phi(0,0)=0$ such that there exists an $\alpha<1$ with*

$$\phi(a,b) \leqslant \frac{\alpha}{4}\phi(2a,2b) \tag{3.14}$$

for all $a,b \in X$. Suppose that $f: X \to Y$ is a mapping satisfying (3.2) for all $x=[x_{ij}]$, $y=[y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q: X \to Y$ such that

$$\|f_n([x_{ij}])-Q_n([x_{ij}])\|_n \leqslant \sum_{i,j=1}^{n} \frac{\alpha}{4(1-\alpha)}\phi(x_{ij},x_{ij}) \tag{3.15}$$

for all $x=[x_{ij}] \in M_n(X)$.

Proof. Let (S_2, d_2) be the generalized metric space defined in the proof of Theorem 3.1.

Now we consider the mapping $\mathscr{J}_2 : S_2 \to S_2$ defined by

$$\mathscr{J}_2 g_2(a) := 4 g_2\left(\frac{a}{2}\right), \text{ for all } g_2 \in S_2 \text{ and } a \in X. \tag{3.16}$$

It follows from (3.5) that

$$\left\| f(a) - 4f\left(\frac{a}{2}\right) \right\| \leqslant \frac{\alpha}{4}\phi(a,a) \tag{3.17}$$

for all $a \in X$. Thus $d_2(f, \mathscr{J}_2 f) \leqslant \frac{\alpha}{4}$. So

$$d_2(f, Q) \leqslant \frac{1}{1-\alpha} d_2(f, \mathscr{J}_2 f) \leqslant \frac{\alpha}{4(1-\alpha)}.$$

The rest of the proof is similar to the proof of Theorem 3.1. □

COROLLARY 3.2. *Let r, θ be positive real numbers with $r > 2$. Suppose that $f : X \to Y$ is a mapping satisfying (3.12) for all $x = [x_{ij}]$, $y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leqslant \sum_{i,j=1}^{n} \frac{2}{2^r - 4} \theta \|x_{ij}\|^r \tag{3.18}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. By choosing $\phi(a,b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ and $\alpha = 2^{2-r}$ in Theorem 3.2, we obtain the inequality (3.18). □

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Zhihua Wang
School of Science, Hubei University of Technology
Wuhan, Hubei 430068, P. R. China
e-mail: matwzh2000@126.com