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BOUNDEDNESS ON MORREY SPACE OF MULTILINEAR SINGULAR INTEGRAL OPERATORS SATISFYING A VARIANT OF HÖRMANDER'S CONDITION

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Abstract. We prove the boundedness properties on Morrey space of the multilinear operator associated to the singular integral operator satisfying a variant of Hörmander's condition.

1. Introduction and preliminaries

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [2–4], [6–9], [12], [15], [16]). Let T be the Calderón-Zygmund singular integral operator and $b \in BMO(R^n)$, a classical result of Coifman, Rochberg and Weiss (see [6]) stated that the commutator $[b,T](f) = T(bf) - bT(f)$ is bounded on $L^p(R^n)$ for $1 < p < \infty$. In [12], Hu and Yang proved a variant sharp function estimate for the multilinear singular integral operators. In [15–16], C. Pérez, G. Pradolini and R. Trujillo-Gonzalez obtained a sharp weighted estimates for the singular integral operators and their commutators. The main purpose of this paper is to study the boundedness properties on Morrey space of the multilinear operator associated to the singular integral operator satisfying a variant of Hörmander's condition.

First, let us introduce some notations. Throughout this paper, $Q = Q(x,r)$ will denote a cube of R^n with sides parallel to the axes and center at x and edge is r. For any locally integrable function f, the sharp function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [10], [17])

$$M^{\#}(f)(x) \approx \sup_{Q \ni x} \inf_{c \in \mathbf{C}} \frac{1}{|Q|} \int_Q |f(y) - c| dy$$

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and

$$||b - b_{2^kQ}||_{BMO} \leqslant Ck||b||_{BMO} \text{ for } k \geqslant 1.$$

We say that f belongs to $BMO(R^n)$ if $M^{\#}(f)$ belongs to $L^\infty(R^n)$ and $||f||_{BMO} = ||M^{\#}(f)||_{L^\infty}$. Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)|dy,$$

and we write that $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$.

We denote the Muckenhoupt weights by A_1, that is (see [10]):

$$A_1 = \{0 < w \in L^1_{loc}(R^n) : M(w)(x) \leqslant Cw(x), a.e.\}.$$

Given a weight function w. For $1 \leqslant p < \infty$, the weighted Lebesgue space $L^p(R^n, w)$ is the space of functions f such that

$$||f||_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x)dx \right)^{1/p} < \infty.$$

Throughout this paper, φ will denote a positive, increasing function on R^+ for which there exists a constant $D > 0$ such that

$$\varphi(2t) \leqslant D\varphi(t) \text{ for } t \geqslant 0.$$

Let w be a weight function on R^n (that is w is a non-negative locally integrable function) and f be a locally integrable function on R^n. Set, for $1 \leqslant p < \infty$,

$$||f||_{L^{p,\varphi}(w)} = \sup_{x \in R^n,\ d > 0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p w(y)dy \right)^{1/p}.$$

The generalized weighted Morrey spaces are defined by

$$L^{p,\varphi}(R^n, w) = \{f \in L^1_{loc}(R^n) : ||f||_{L^{p,\varphi}(w)} < \infty\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n, w) = L^{p,\delta}(R^n, w)$, which is the classical Morrey space (see [1], [16], [17]). As the Morrey space may be considered as an extension of the Lebesgue space (the Morrey space $L^{p,\lambda}$ becomes the Lebesgue space L^p when $\lambda = 0$), it is natural and important to study the boundedness of the multilinear singular integral operator on the Morrey spaces $L^{p,\varphi}(R^n, w)$ (see [1], [7], [8]).

2. Theorem

DEFINITION 1. Let $\Phi = \{\phi_1, \dots, \phi_m\}$ be a finite family of bounded functions in R^n. For any weight function w and locally integrable function f, the Φ sharp maximal function of f is defined by

$$M^{\#}_\Phi(f)(x) = \sup_{Q \ni x} \inf_{\{c_1, \dots, c_m\}} \frac{1}{w(Q)} \int_Q |f(y) - \sum_{j=1}^m c_j \phi_j(x_Q - y)|w(y)dy,$$

where the infimum is taken over all m-tuples $\{c_1,\dots,c_m\}$ of complex numbers and x_Q is the center of Q.

REMARK. Note that $M^{\#}_{\Phi} \approx M^{\#}(f)$ if $m=1$ and $\phi_1=1$ (see [15]).

DEFINITION 2. Given a positive and locally integrable function f in R^n, we say that f satisfies the reverse Hölder's condition (write this as $f \in RH_\infty(R^n)$), if for any cube Q centered at the origin we have

$$0 < \sup_{x\in Q} f(x) \leqslant C\frac{1}{|Q|}\int_Q f(y)dy.$$

In this paper, we will study some singular integral operators as following (see [1]).

DEFINITION 3. Let $K \in L^2(R^n)$ and satisfy

$$||\hat{K}||_{L^\infty} \leqslant C,$$

$$|K(x)| \leqslant C|x|^{-n},$$

there exist functions $B_1,\dots,B_m \in L^1_{loc}(R^n-\{0\})$ and $\Phi=\{\phi_1,\dots,\phi_m\}\subset L^\infty(R^n)$ such that $|det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nm})$, and for a fixed $\delta>0$ and any $|x|>2|y|>0$,

$$|K(x-y)-\sum_{j=1}^m B_j(x)\phi_j(y)| \leqslant C\frac{|y|^\delta}{|x-y|^{n+\delta}}.$$

For $f\in C_0^\infty$, we define the singular integral operator related to the kernel K by

$$T(f)(x)=\int_{R^n} K(x-y)f(y)dy.$$

Let l and m_j be the positive integers $(j=1,\cdots,l)$, $m_1+\cdots+m_l=m$ and b_j be the functions on R^n $(j=1,\cdots,l)$. Set, for $1\leqslant j\leqslant l$,

$$R_{m_j+1}(b_j;x,y)=b_j(x)-\sum_{|\alpha|\leqslant m_j}\frac{1}{\alpha!}D^\alpha b_j(y)(x-y)^\alpha.$$

The multilinear operator associated to T is defined by

$$T^b(f)(x)=\int_{R^n}\frac{\prod_{j=1}^l R_{m_j+1}(b_j;x,y)}{|x-y|^m}K(x-y)f(y)dy.$$

Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 5 (see [11], [18]). Also note that when $m=0$, T^b is just multilinear commutators of T and b (see [15–16]). It is well-known that multilinear operator, as a non-trivial extension of commutator, is of great interest in harmonic analysis and has been widely studied by many authors (see [2–4]). The main purpose of this paper is to prove the sharp maximal inequalities for the commutator T^b. As the application, we obtain the weighted L^p-norm inequality and Morrey space boundedness for the multilinear singular integral operator T^b.

We shall prove the following theorems in Section 3.

THEOREM 1. *Let* $1 < r < \infty$, T *be the singular integral operator as Definition 3 and* $D^\alpha b_j \in BMO(R^n)$ *for all* α *with* $|\alpha| = m_j$ *and* $j = 1, \cdots, l$. *Then there exists a constant* $C > 0$ *such that for any* $f \in C_0^\infty(R^n)$, $1 < s < \infty$ *and* $\tilde{x} \in R^n$,

$$M_\Phi^\#(T^b(f))(\tilde{x}) \leqslant C \prod_{j=1}^{l} \left(\sum_{|\alpha_j| = m_j} ||D^{\alpha_j} b_j||_{BMO} \right) M_r(f)(\tilde{x}).$$

THEOREM 2. *Let* $1 < p < \infty$, $w \in A_1$, T *be the singular integral operators as Definition 3 and* $D^\alpha b_j \in BMO(R^n)$ *for all* α *with* $|\alpha| = m_j$ *and* $j = 1, \cdots, l$. *Then* T^b *is bounded on* $L^p(R^n, w)$, *that is*

$$||T^b(f)||_{L^p(w)} \leqslant C \prod_{j=1}^{l} \left(\sum_{|\alpha_j| = m_j} ||D^{\alpha_j} b_j||_{BMO} \right) ||f||_{L^p(w)}.$$

THEOREM 3. *Let* $1 < p < \infty$, $w \in A_1$, $0 < D < 2^n$, T *be the singular integral operators as Definition 3 and* $D^\alpha b_j \in BMO(R^n)$ *for all* α *with* $|\alpha| = m_j$ *and* $j = 1, \cdots, l$. *Then* T^b *is bounded on* $L^{p,\varphi}(R^n, w)$, *that is*

$$||T^b(f)||_{L^{p,\varphi}(w)} \leqslant C \prod_{j=1}^{l} \left(\sum_{|\alpha_j| = m_j} ||D^{\alpha_j} b_j||_{BMO} \right) ||f||_{L^{p,\varphi}(w)}.$$

3. Proof of Theorem

We begin with some preliminaries lemmas.

LEMMA 1. (see [4]) *Let* b *be a function on* R^n *and* $D^\alpha b \in L^q(R^n)$ *for* $|\alpha| = m$ *and some* $q > n$. *Then*

$$|R_m(b; x, y)| \leqslant C|x - y|^m \sum_{|\alpha| = m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x,y)} |D^\alpha b(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ *is the cube centered at* x *and having side length* $5\sqrt{n}|x - y|$.

LEMMA 2. (see [11], [18]) *Let* T *be the singular integral operator as Definition 3. Then* T *is bounded on* $L^p(R^n, w)$ *for* $w \in A_1$ *and* $1 < p < \infty$.

LEMMA 3. (see [18]) *Let* $1 < p < \infty$, $w \in A_1$ *and* $\Phi = \{\phi_1, \ldots, \phi_m\} \subset L^\infty(R^n)$ *such that* $|det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nm})$. *Then*

$$\int_{R^n} M(f)(x)^p w(x) dx \leqslant C \int_{R^n} M_\Phi^\#(f)(x)^p w(x) dx$$

for any smooth function f *for which the left-hand side is finite.*

LEMMA 4. *Let $1 < p < \infty$, $w \in A_1$, $0 < D < 2^n$ and $\Phi = \{\phi_1, \dots, \phi_m\} \subset L^\infty(R^n)$ such that $|det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nm})$. Then, for any smooth function f for which the left-hand side is finite,*

(a) $||M(f)||_{L^{p,\varphi}(w)} \leqslant C||M^\#_\Phi(f)||_{L^{p,\varphi}(w)}$;

(b) $||M_r(f)||_{L^{p,\varphi}(w)} \leqslant C||f||_{L^{p,\varphi}(w)}$ *for* $1 < r < p$.

Proof. (a) For any cube $Q = Q(x_0, d)$ in R^n, we know $M(w\chi_Q) \in A_1$ for any cube $Q = Q(x,d)$ by [5]. We get, by Lemma 3,

$$\int_Q M(f)(y)^p w(y)dy = \int_{R^n} M(f)(y)^p w(y)\chi_Q(y)dy$$
$$\leqslant C\int_{R^n} |M(f)(y)|^p M(w\chi_Q)(y)dy \leqslant C\int_{R^n} |M^\#_\Phi(f)(y)|^p M(w\chi_Q)(y)dy$$
$$\leqslant C\left[\int_Q |M^\#_\Phi(f)(y)|^p M(w)(y)dy + \sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^kQ} |M^\#_\Phi(f)(y)|^p \left(\sup_{Q\ni y}\frac{1}{|Q|}\int_Q w(z)dz\right)dy\right]$$
$$\leqslant C\left[\int_Q |M^\#_\Phi(f)(y)|^p M(w)(y)dy + \sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^kQ} |M^\#_\Phi(f)(y)|^p \left(\frac{1}{|2^{k+1}Q|}\int_B w(z)dz\right)dy\right]$$
$$\leqslant C\left[\int_Q |M^\#_\Phi(f)(y)|^p w(y)dy + \sum_{k=0}^\infty \int_{2^{k+1}Q} |M^\#_\Phi(f)(y)|^p \frac{w(y)}{2^{nk}}dy\right]$$
$$\leqslant C||M^\#_\Phi(f)||^p_{L^{p,\varphi}(w)} \sum_{k=0}^\infty 2^{-nk}\varphi(2^{k+1}d)$$
$$\leqslant C||M^\#_\Phi(f)||^p_{L^{p,\varphi}(w)} \sum_{k=0}^\infty (2^{-n}D)^k\varphi(d)$$
$$\leqslant C||M^\#_\Phi(f)||^p_{L^{p,\varphi}(w)}\varphi(d),$$

thus

$$||M(f)||_{L^{p,\varphi}(w)} \leqslant C||M^\#_\Phi(f)||_{L^{p,\varphi}(w)}.$$

A similar argument as in the proof of (a) will give the proof of (b), we omit the details. This finishes the proof. $\square$

Proof of Theorem 1. Without loss of generality, we may assume $l = 2$. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q |T^b(f)(x) - C_0|dx \leqslant C\prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO}\right) M_r(f)(\tilde{x}).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha b_j)_{\tilde{Q}} x^\alpha$, then $R_m(b_j; x, y) = R_m(\tilde{b}_j; x, y)$ and $D^\alpha \tilde{b}_j = D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f_1 = f\chi_{\tilde{Q}}$, $f_2 = f\chi_{R^n\setminus\tilde{Q}}$ and $C_0 = \sum_{j=1}^m c_j\phi_j(x_0 - x)$ with $c_j = \int_{R^n} \frac{K(x_0,y)}{|x_0-y|^m} B_j(x_0 -$

$y)f_2(y)dy,$

$$
\begin{aligned}
T^b(f)(x) = & \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m} K(x-y) f_1(y)dy \\
& - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1 !} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1} D^{\alpha_1}\tilde{b}_1(y)}{|x-y|^m} K(x-y) f_1(y)dy \\
& - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2 !} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2} D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K(x-y) f_1(y)dy \\
& + \sum_{\substack{|\alpha_1|=m_1 \\ |\alpha_2|=m_2}} \frac{1}{\alpha_1 ! \alpha_2 !} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K(x-y) f_1(y)dy \\
& + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m} K(x-y) f_2(y)dy \\
= & \, T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_1 \right) \\
& - T\left(\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1 !} \frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1} D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m} f_1 \right) \\
& - T\left(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2 !} \frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2} D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1 \right) \\
& + T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1 ! \alpha_2 !} \frac{(x-\cdot)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1 \right) \\
& + T^{\tilde{b}}(f_2)(x),
\end{aligned}
$$

then

$$
\begin{aligned}
\left| T^b(f)(x) - C_0 \right| \leqslant & \left| T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_1 \right) \right| \\
& + \left| T\left(\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1 !} \frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1} D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m} f_1 \right) \right| \\
& + \left| T\left(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2 !} \frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2} D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1 \right) \right| \\
& + \left| T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1 ! \alpha_2 !} \frac{(x-\cdot)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1 \right) \right| \\
& + |T^{\tilde{b}}(f_2)(x) - C_0| \\
= & \, L_1(x) + L_2(x) + L_3(x) + L_4(x) + L_5(x)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q \left|T^b(f)(x)-C_0\right| dx\\
&\leqslant \frac{1}{|Q|}\int_Q L_1(x)dx+\frac{1}{|Q|}\int_Q L_2(x)dx+\frac{1}{|Q|}\int_Q L_3(x)dx+\frac{1}{|Q|}\int_Q L_4(x)dx+\frac{1}{|Q|}\int_Q L_5(x)dx\\
&= L_1+L_2+L_3+L_4+L_5.
\end{aligned}
$$

Now, for L_1, if $x\in Q$ and $y\in 2Q$, by using Lemma 1, we get

$$R_m(\tilde{b};x,y)\leqslant C|x-y|^m\sum_{|\alpha|=m}||D^\alpha b||_{BMO},$$

thus, by the L^s boundedness of T (see Lemma 2) and Hölder's inequality, we obtain

$$
\begin{aligned}
L_1 &\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\frac{1}{|Q|}\int_Q |T(f_1)(x)|dx\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\left(\frac{1}{|Q|}\int_{R^n}|T(f_1)(x)|^r dx\right)^{1/r}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\left(\frac{1}{|Q|}\int_{R^n}|f_1(x)|^r dx\right)^{1/r}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^r dx\right)^{1/r}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)M_r(f)(\tilde{x}).
\end{aligned}
$$

For L_2, denoting $r=uv$ for $1<u,v<\infty$ and $1/v+1/v'=1$, we have

$$
\begin{aligned}
L_2 &\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}\sum_{|\alpha|=m}\frac{1}{|Q|}\int_Q |T(D^{\alpha_1}\tilde{b}_1 f_1)(x)|dx\\
&\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{b}_1 f_1)(x)|^u dx\right)^{1/u}\\
&\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|D^{\alpha_1}\tilde{b}_1(x)||f_1(x)|^u dx\right)^{1/u}
\end{aligned}
$$

$$\leqslant C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} b_2||_{BMO} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x)|^{uv} dx \right)^{1/uv}$$

$$\times \sum_{|\alpha_1|=m_1} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_1} \tilde{b}_1(x) - (D^{\alpha_1} \tilde{b}_1)_{\tilde{Q}}|^{uv'} dx \right)^{1/uv'}$$

$$\leqslant C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO} \right) M_r(f)(\tilde{x}).$$

For L_3, similar to the proof of L_2, we get

$$L_3 \leqslant C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO} \right) M_r(f)(\tilde{x}).$$

Similarly, for L_4, denoting $r = uw$ for $1 < u, v_1, v_2, w < \infty$ and $1/v_1 + 1/v_2 + 1/w = 1$, we obtain, by Hölder' inequality,

$$L_4 \leqslant C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|} \int_Q |T(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1)(x)| dx$$

$$\leqslant C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1)(x)|^u dx \right)^{1/u}$$

$$\leqslant C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1/u} \left(\int_{R^n} |D^{\alpha_1} \tilde{b}_1(x) D^{\alpha_2} \tilde{b}_2(x) f_1(x)|^u dx \right)^{1/u}$$

$$\leqslant C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_1} \tilde{b}_1(x)|^{uv_1} dx \right)^{1/uv_1} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_2} \tilde{b}_2(x)|^{uv_2} dx \right)^{1/uv_2}$$

$$\times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x)|^{uw} dx \right)^{1/uw}$$

$$\leqslant C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO} \right) M_r(f)(\tilde{x}).$$

For L_5, we write

$$\begin{aligned} & T^{\tilde{b}}(f_2)(x) - C_0 \\ &= \int_{R^n} \left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m} - \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x_0, y)}{|x_0-y|^m} \right) K(x-y) f_2(y) dy \\ &\quad + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x_0, y)}{|x_0-y|^m} \left(K(x-y) - \sum_{j=1}^{m} B_j(x_0-y)\phi_j(x_0-x) \right) f_2(y) dy \\ &= L_5^{(1)}(x) + L_5^{(2)}(x). \end{aligned}$$

By Lemma 1 and the following inequality (see [17])

$$|b_{Q_1}-b_{Q_2}|\leqslant C\log(|Q_2|/|Q_1|)||b||_{BMO} \text{ for } Q_1\subset Q_2,$$

we know that, for $x\in Q$ and $y\in 2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}$,

$$\begin{aligned}|R_{m_j}(\tilde{b}_j;x,y)| &\leqslant C|x-y|^{m_j}\sum_{|\alpha|=m_j}(||D^\alpha b_j||_{BMO}+|(D^\alpha b_j)_{\tilde{Q}(x,y)}-(D^\alpha b_j)_{\tilde{Q}}|)\\ &\leqslant Ck|x-y|^{m_j}\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}.\end{aligned}$$

By the formula (see [4]):

$$R_{m_j}(\tilde{b}_j;x,y)-R_{m_j}(\tilde{b}_j;x_0,y)=\sum_{|\beta|<m_j}\frac{1}{\beta!}R_{m_j-|\beta|}(D^\beta\tilde{b}_j;x,x_0)(x-y)^\beta$$

and Lemma 1, we have

$$|R_{m_j}(\tilde{b}_j;x,y)-R_{m_j}(\tilde{b}_j;x_0,y)|\leqslant C\sum_{|\beta|<m_j}\sum_{|\alpha|=m_j}|x-x_0|^{m_j-|\beta|}|x-y|^{|\beta|}||D^\alpha b_j||_{BMO},$$

Note that $|x-y|\sim|x_0-y|$ for $x\in Q$ and $y\in R^n\setminus\tilde{Q}$, thus, by the conditions on K, we obtain, for $1<s_1,s_2<\infty$ with $1/r+1/s_1+1/s_2=1$,

$$\begin{aligned}|L_5^{(1)}(x)| \leqslant{}& \int_{R^n}\left|\frac{1}{|x-y|^m}-\frac{1}{|x_0-y|^m}\right||\prod_{j=1}^{2}R_{m_j}(\tilde{b}_j;x,y)||K(x-y)||f_2(y)|dy\\
&+\int_{R^n}\left|R_{m_1}(\tilde{b}_1;x,y)-R_{m_1}(\tilde{b}_1;x_0,y)\right|\frac{|R_{m_2}(\tilde{b}_2;x,y)|}{|x_0-y|^m}|K(x-y)||f_2(y)|dy\\
&+\int_{R^n}\left|R_{m_2}(\tilde{b}_2;x,y)-R_{m_2}(\tilde{b}_2;x_0,y)\right|\frac{|R_{m_1}(\tilde{b}_1;x_0,y)|}{|x_0-y|^m}|K(x-y)||f_2(y)|dy\\
&+\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\frac{1}{\alpha_1!}\int_{R^n}\left|\frac{(x-y)^{\alpha_1}}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}}{|x_0-y|^m}\right||R_{m_1}(\tilde{b}_1;x,y)||D^{\alpha_2}\tilde{b}_2(y)|\\
&\times|K(x-y)||f_2(y)|dy\\
&+\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\frac{1}{\alpha_1!}\int_{R^n}\left|\frac{(x_0-y)^{\alpha_1}}{|x_0-y|^m}\right||R_{m_1}(\tilde{b}_1;x,y)-R_{m_1}(\tilde{b}_1;x_0,y)||D^{\alpha_2}\tilde{b}_2(y)|\\
&\times|K(x-y)||f_2(y)|dy\\
&+\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\frac{1}{\alpha_2!}\int_{R^n}\left|\frac{(x-y)^{\alpha_2}}{|x-y|^m}-\frac{(x_0-y)^{\alpha_2}}{|x_0-y|^m}\right||R_{m_2}(\tilde{b}_2;x,y)||D^{\alpha_1}\tilde{b}_1(y)|\\
&\times|K(x-y)||f_2(y)|dy\\
&+\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\frac{1}{\alpha_2!}\int_{R^n}\left|\frac{(x_0-y)^{\alpha_2}}{|x_0-y|^m}\right||R_{m_2}(\tilde{b}_2;x,y)-R_{m_2}(\tilde{b}_2;x_0,y)||D^{\alpha_1}\tilde{b}_1(y)|\end{aligned}$$

$$
\begin{aligned}
&\times|K(x-y)||f_2(y)|dy\\
&+\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left|\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m}\right||D^{\alpha_1}\tilde b_1(y)||D^{\alpha_2}\tilde b_2(y)|\\
&\times|K(x-y)||f_2(y)|dy\\
\leqslant&\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde Q\setminus 2^k\tilde Q}\left|\frac{1}{|x-y|^m}-\frac{1}{|x_0-y|^m}\right||\prod_{j=1}^{2}R_{m_j}(\tilde b_j;x,y)||K(x-y)||f(y)|dy\\
&+\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde Q\setminus 2^k\tilde Q}\left|R_{m_1}(\tilde b_1;x,y)-R_{m_1}(\tilde b_1;x_0,y)\right|\frac{|R_{m_2}(\tilde b_2;x,y)|}{|x_0-y|^m}|K(x-y)||f(y)|dy\\
&+\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde Q\setminus 2^k\tilde Q}\left|R_{m_2}(\tilde b_2;x,y)-R_{m_2}(\tilde b_2;x_0,y)\right|\frac{|R_{m_1}(\tilde b_1;x_0,y)|}{|x_0-y|^m}|K(x-y)||f(y)|dy\\
&+C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde Q\setminus 2^k\tilde Q}\left|\frac{(x-y)^{\alpha_1}}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}}{|x_0-y|^m}\right||R_{m_1}(\tilde b_1;x,y)||D^{\alpha_2}\tilde b_2(y)|\\
&\times|K(x-y)||f(y)|dy\\
&+C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde Q\setminus 2^k\tilde Q}\left|\frac{(x_0-y)^{\alpha_1}}{|x_0-y|^m}\right||R_{m_1}(\tilde b_1;x,y)-R_{m_1}(\tilde b_1;x_0,y)||D^{\alpha_2}\tilde b_2(y)|\\
&\times|K(x-y)||f(y)|dy\\
&+C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde Q\setminus 2^k\tilde Q}\left|\frac{(x-y)^{\alpha_2}}{|x-y|^m}-\frac{(x_0-y)^{\alpha_2}}{|x_0-y|^m}\right||R_{m_2}(\tilde b_2;x,y)||D^{\alpha_1}\tilde b_1(y)|\\
&\times|K(x-y)||f(y)|dy\\
&+C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde Q\setminus 2^k\tilde Q}\left|\frac{(x_0-y)^{\alpha_2}}{|x_0-y|^m}\right||R_{m_2}(\tilde b_2;x,y)-R_{m_2}(\tilde b_2;x_0,y)||D^{\alpha_1}\tilde b_1(y)|\\
&\times|K(x-y)||f(y)|dy\\
&+C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde Q\setminus 2^k\tilde Q}\left|\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m}\right||D^{\alpha_1}\tilde b_1(y)||D^{\alpha_2}\tilde b_2(y)|\\
&\times|K(x-y)||f(y)|dy\\
\leqslant&\,C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\sum_{k=0}^{\infty}k^2\int_{2^{k+1}\tilde Q\setminus 2^k\tilde Q}\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|dy\\
&+C\sum_{|\alpha_1|=m_1}||D^{\alpha_1}b_1||_{BMO}\sum_{|\alpha_2|=m_2}\sum_{k=0}^{\infty}k\int_{2^{k+1}\tilde Q\setminus 2^k\tilde Q}\frac{|x-x_0|}{|x_0-y|^{n+1}}|D^{\alpha_2}\tilde b_2(y)||f(y)|dy\\
&+C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=0}^{\infty}k\int_{2^{k+1}\tilde Q\setminus 2^k\tilde Q}\frac{|x-x_0|}{|x_0-y|^{n+1}}|D^{\alpha_1}\tilde b_1(y)||f(y)|dy
\end{aligned}
$$

$$+C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+1}}|D^{\alpha_1}\tilde{b}_1(y)||D^{\alpha_2}\tilde{b}_2(y)||f(y)|dy$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\sum_{k=1}^{\infty}k^2 2^{-k}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^r dy\right)^{1/r}$$

$$+C\sum_{|\alpha_1|=m_1}||D^{\alpha_1}b_1||_{BMO}\sum_{|\alpha_2|=m_2}\sum_{k=1}^{\infty}k2^{-k}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_2}\tilde{b}_2(y)|^{r'}dy\right)^{1/r'}$$

$$\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^r dy\right)^{1/r}$$

$$+C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k2^{-k}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(y)|^{r'}dy\right)^{1/r'}$$

$$\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^r dy\right)^{1/r}$$

$$+C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\sum_{k=1}^{\infty}2^{-k}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^r dy\right)^{1/r}$$

$$\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(y)|^{s_1}dy\right)^{1/s_1}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_2}\tilde{b}_2(y)|^{s_2}dy\right)^{1/s_2}$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\sum_{k=1}^{\infty}k^2 2^{-k}M_r(f)(\tilde{x})$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)M_r(f)(\tilde{x}).$$

For $L_5^{(2)}$, similar to the proof of $L_5^{(1)}$ and by the conditions on K, we get, for $1<t_1,t_2<\infty$ with $1/r+1/t_1+1/t_2=1$,

$$|L_5^{(2)}(x)|\leqslant\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{\prod_{j=1}^{2}|R_{m_j}(\tilde{b}_j;x_0,y)|}{|x_0-y|^m}|K(x-y)-\sum_{j=1}^{m}B_j(x_0-y)\phi_j(x_0-x)||f(y)|dy$$

$$+C\sum_{|\alpha_2|=m_2}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|R_{m_1}(\tilde{b}_1;x_0,y)||D^{\alpha_2}\tilde{b}_2(y)||(x_0-y)^{\alpha_2}|}{|x_0-y|^m}$$

$$\times|K(x-y)-\sum_{j=1}^{m}B_j(x_0-y)\phi_j(x_0-x)||f(y)|dy$$

$$+C\sum_{|\alpha_1|=m_1}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|R_{m_2}(\tilde{b}_2;x_0,y)||D^{\alpha_1}\tilde{b}_1(y)||(x_0-y)^{\alpha_1}|}{|x_0-y|^m}$$

$$
\begin{aligned}
&\times|K(x-y)-\sum_{j=1}^{m}B_j(x_0-y)\phi_j(x_0-x)||f(y)|dy\\
&+C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|D^{\alpha_1}\tilde{b}_1(y)||D^{\alpha_2}\tilde{b}_2(y)||(x_0-y)^{\alpha_1+\alpha_2}|}{|x_0-y|^m}\\
&\times|K(x-y)-\sum_{j=1}^{m}B_j(x_0-y)\phi_j(x_0-x)||f(y)|dy\\
\leqslant{}& C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\sum_{k=0}^{\infty}k^2\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|^\delta}{|x_0-y|^{n+\delta}}|f(y)|dy\\
&+C\sum_{|\alpha_1|=m_1}||D^{\alpha_1}b_1||_{BMO}\sum_{|\alpha_2|=m_2}\sum_{k=0}^{\infty}k\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|^\delta}{|x_0-y|^{n+\delta}}|D^{\alpha_2}\tilde{b}_2(y)||f(y)|dy\\
&+C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=0}^{\infty}k\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|^\delta}{|x_0-y|^{n+\delta}}|D^{\alpha_1}\tilde{b}_1(y)||f(y)|dy\\
&+C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|^\delta}{|x_0-y|^{n+\delta}}|D^{\alpha_1}\tilde{b}_1(y)||D^{\alpha_2}\tilde{b}_2(y)||f(y)|dy\\
\leqslant{}& C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\sum_{k=1}^{\infty}k^2 2^{-\delta k}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^r dy\right)^{1/r}\\
&+C\sum_{|\alpha_1|=m_1}||D^{\alpha_1}b_1||_{BMO}\sum_{|\alpha_2|=m_2}\sum_{k=1}^{\infty}k2^{-\delta k}\\
&\times\left(\frac{1}{|2^{k+1}Q|}\int_{2^k\tilde{Q}}|D^{\alpha_2}\tilde{b}_2(y)|^{r'}dy\right)^{1/r'}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^r dy\right)^{1/r}\\
&+C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k2^{-\delta k}\\
&\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(y)|^{r'}dy\right)^{1/r'}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^r dy\right)^{1/r}dy\\
&+C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\sum_{k=1}^{\infty}2^{-\delta k}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^r dy\right)^{1/r}\\
&\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(y)|^{t_1}dy\right)^{1/t_1}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_2}\tilde{b}_2(y)|^{t_1}dy\right)^{1/t_2}
\end{aligned}
$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\sum_{k=1}^{\infty}k^2 2^{-\delta k}M_r(f)(\tilde{x})$$
$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)M_r(f)(\tilde{x}).$$

Thus

$$L_5\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)M_r(f)(\tilde{x}).$$

This completes the proof of Theorem 1. $\square$

Proof of Theorem 2. Taking $1<r<p$ in Theorem 1 and by Lemma 3, we obtain

$$\begin{aligned}||T^b(f)||_{L^p(w)} &\leqslant ||M(T^b(f))||_{L^p(w)}\leqslant C||M_\Phi^{\#}(T^b(f))||_{L^p(w)}\\ &\leqslant C\prod_{j=1}^{l}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)||M_r(f)||_{L^p(w)}\\ &\leqslant C\prod_{j=1}^{l}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)||f||_{L^p(w)}.\end{aligned}$$

This finishes the proof. $\square$

Proof of Theorem 3. Taking $1<r<p$ in Theorem 1 and by Lemma 4, we obtain

$$\begin{aligned}||T^b(f)||_{L^{p,\varphi}(w)} &\leqslant ||M(T^b(f))||_{L^{p,\varphi}(w)}\leqslant C||M_\Phi^{\#}(T^b(f))||_{L^{p,\varphi}(w)}\\ &\leqslant C\prod_{j=1}^{l}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)||M_r(f)||_{L^{p,\varphi}(w)}\\ &\leqslant C\prod_{j=1}^{l}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)||f||_{L^{p,\varphi}(w)}.\end{aligned}$$

This finishes the proof. $\square$

REFERENCES

[1] F. CHIARENZA AND M. FRASCA, *Morrey spaces and Hardy-Littlewood maximal function*, Rend. Mat. **7** (1987), 273–279.

[2] J. COHEN, *A sharp estimate for a multilinear singular integral on R^n*, Indiana Univ. Math. J. **30** (1981), 693–702.

[3] J. COHEN AND J. GOSSELIN, *On multilinear singular integral operators on R^n*, Studia Math. **72** (1982), 199–223.

[4] J. COHEN AND J. GOSSELIN, *A BMO estimate for multilinear singular integral operators*, Illinois J. Math. **30** (1986), 445–465.

[5] R. R. COIFMAN AND R. ROCHBERG, *Another characterization of BMO*, Proc. Amer. Math. Soc. **79** (1980), 249–254.

[6] R. R. COIFMAN, R. ROCHBERG AND G. WEISS, *Fractorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (1976), 611–635.

[7] G. DI FAZIO AND M. A. RAGUSA, *Commutators and Morrey spaces*, Boll. Un. Mat. Ital. (7) 5-A (1991), 323–332.

[8] G. DI FAZIO AND M. A. RAGUSA, *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Func. Anal. **112** (1993), 241–256.

[9] Y. DING AND S. Z. LU, *Weighted boundedness for a class rough multilinear operators*, Acta Math Sinica **17** (2001), 517–526.

[10] J. GARCIA-CUERVA AND J. L. RUBIO DE FRANCIA, *Weighted norm inequalities and related topics*, North-Holland Math. **116**, Amsterdam, 1985.

[11] D. J. GRUBB AND C. N. MOORE, *A variant of Hörmander's condition for singular integrals*, Colloq. Math. **73** (1997), 165–172.

[12] G. E. HU AND D. C. YANG, *A variant sharp estimate for multilinear singular integral operators*, Studia Math. **141** (2000), 25–42.

[13] J. PEETRE, *On convolution operators leaving $L^{p,\lambda}$ -spaces invariant*, Ann. Mat. Pura. Appl. **72** (1966), 295–304.

[14] J. PEETRE, *On the theory of $L^{p,\lambda}$ -spaces*, J. Func. Anal. **4** (1969), 71–87.

[15] C. PÉREZ AND G. PRADOLINI, *Sharp weighted endpoint estimates for commutators of singular integral operators*, Michigan Math. J. **49** (2001), 23–37.

[16] C. PÉREZ AND R. TRUJILLO-GONZALEZ, *Sharp weighted estimates for multilinear commutators*, J. London Math. Soc. **65** (2002), 672–692.

[17] E. M. STEIN, *Harmonic analysis: real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton NJ, 1993.

[18] R. TRUJILLO-GONZALEZ, *Weighted norm inequalities for singular integral operators satisfying a variant of Hörmander's condition*, Comment. Math. Univ. Carolin. **44** (2003), 137–152.

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