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SHARP BOUNDS FOR THE TOADER–QI MEAN IN TERMS OF HARMONIC AND GEOMETRIC MEANS

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Abstract. In the article, we present the greatest values α and λ, and the least values β and μ in $[0,1/2]$ such that the double inequalities

$$H[\alpha a+(1-\alpha)b,\alpha b+(1-\alpha)a] < TQ(a,b) < H[\beta a+(1-\beta)b,\beta b+(1-\beta)a],$$

$$G[\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a] < TQ(a,b) < G[\mu a+(1-\mu)b,\mu b+(1-\mu)a]$$

hold for all $a,b>0$ with $a\neq b$, where $H(a,b)=2ab/(a+b)$, $G(a,b)=\sqrt{ab}$ and $TQ(a,b)=\frac{2}{\pi}\int_0^{\pi/2}a^{\cos^2\theta}b^{\sin^2\theta}d\theta$ are respectively the harmonic, geometric and Toader-Qi means of a and b.

1. Introduction

For $a,b>0$, the Toader-Qi mean $TQ(a,b)$ [1], harmonic mean $H(a,b)$ and geometric mean $G(a,b)$ are defined by

$$TQ(a,b)=\frac{2}{\pi}\int_0^{\pi/2}a^{\cos^2\theta}b^{\sin^2\theta}d\theta,$$

$$H(a,b)=\frac{2ab}{a+b},\quad G(a,b)=\sqrt{ab},$$

respectively.

Recently, the Toader-Qi mean $TQ(a,b)$ have attracted the attention of several researchers. In particular, many remarkable inequalities for the Toader-Qi mean $TQ(a,b)$ can be found in the literature [2, 3].

In [2], Qi et al. proved that the identity

$$TQ(a,b)=\sqrt{ab}\,I_0\left(\frac{1}{2}\log\frac{b}{a}\right) \tag{1.1}$$

and the inequalities

$$L(a,b)<TQ(a,b)<\frac{A(a,b)+G(a,b)}{2}<\frac{2A(a,b)+G(a,b)}{3}<I(a,b) \tag{1.2}$$

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hold for all $a,b>0$ with $a\neq b$, where

$$I_0(t)=\sum_{n=0}^{\infty}\frac{t^{2n}}{2^{2n}(n!)^2} \tag{1.3}$$

is the modified Bessel function of the first kind [4], $L(a,b)=(b-a)/(\log b-\log a)$, $A(a,b)=(a+b)/2$ and $I(a,b)=(b^b/a^a)^{1/(b-a)}/e$ are respectively the classical logarithmic, arithmetic and identric means of a and b.

In [3], Yang proved that the double inequalities

$$\sqrt{\frac{2A(a,b)L(a,b)}{\pi}}<TQ(a,b)<\sqrt{A(a,b)L(a,b)},$$

$$A^{1/4}(a,b)L^{3/4}(a,b)<TQ(a,b)<\frac{1}{4}A(a,b)+\frac{3}{4}L(a,b)$$

hold for all $a,b>0$ with $a\neq b$.

It is well-known that the inequalities

$$H(a,b)<G(a,b)<L(a,b)<I(a,b)<A(a,b) \tag{1.4}$$

hold for all $a,b>0$ with $a\neq b$.

Let $x\in[0,1/2]$, $f(x)=H[xa+(1-x)b,xb+(1-x)a]$ and $g(x)=G[xa+(1-x)b,xb+(1-x)a]$. Then we clearly see that both f and g are continuous and strictly increasing on the interval $[0,1/2]$ for all fixed $a,b>0$ with $a\neq b$.

Note that

$$f(0)=H(a,b),\ \ g(0)=G(a,b),\ \ f\left(\frac{1}{2}\right)=g\left(\frac{1}{2}\right)=A(a,b). \tag{1.5}$$

It follows from (1.2), (1.4) and (1.5) that

$$f(0)<TQ(a,b)<f\left(\frac{1}{2}\right),\ \ g(0)<TQ(a,b)<g\left(\frac{1}{2}\right). \tag{1.6}$$

Motivated by (1.6) and the monotonicity of f and g on the interval $[0,1/2]$, it is natural to ask what are the greatest values α and λ, and the least values β and μ in $[0,1/2]$ such that the double inequalities

$$H[\alpha a+(1-\alpha)b,\alpha b+(1-\alpha)a]<TQ(a,b)<H[\beta a+(1-\beta)b,\beta b+(1-\beta)a],$$

$$G[\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a]<TQ(a,b)<G[\mu a+(1-\mu)b,\mu b+(1-\mu)a]$$

hold for all $a,b>0$ with $a\neq b$? The main purpose of this paper is to answer this question.

2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

LEMMA 2.1. (See [5]) *For $n \in \mathbb{N}$, the Wallis ratio*

$$W_n = \frac{(2n-1)!!}{(2n)!!}$$

satisfies the double inequality

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < W_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}.$$

LEMMA 2.2. (See [3]) *Let $I_0(t)$ be defined by (1.3). Then the identity*

$$I_0^2(t) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^4} t^{2n}$$

holds for all $t \in \mathbb{R}$.

LEMMA 2.3. (See [3]) *Let $s_n = (2n)!(2n+1)!/[2^{4n}(n!)^4]$. Then the sequence $\{s_n\}_{n=0}^{\infty}$ is strictly decreasing and*

$$\lim_{n\to\infty} s_n = \frac{2}{\pi}. \tag{2.1}$$

LEMMA 2.4. (See [6, 7]) *Let $A(t) = \sum_{k=0}^{\infty} a_k t^k$ and $B(t) = \sum_{k=0}^{\infty} b_k t^k$ be two real power series converging on $(-r,r)$ $(r>0)$ with $b_k > 0$ for all k. If the non-constant sequence $\{a_k/b_k\}$ is increasing (decreasing) for all k, then the function $A(t)/B(t)$ is strictly increasing (decreasing) on $(0,r)$.*

LEMMA 2.5. (See [8]) *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two real sequences with $b_n > 0$ and $\lim_{n\to\infty} a_n/b_n = s$. Then the power series $\sum_{n=0}^{\infty} a_n t^n$ is convergent for all $t \in \mathbb{R}$ and*

$$\lim_{t\to\infty} \frac{\sum_{n=0}^{\infty} a_n t^n}{\sum_{n=0}^{\infty} b_n t^n} = s$$

if the power series $\sum_{n=0}^{\infty} b_n t^n$ is convergent for all $t \in \mathbb{R}$.

LEMMA 2.6. *Let $I_0(t)$ be defined by (1.3). Then the function*

$$\varphi(t) = \frac{\cosh(t)[\cosh(t) - I_0(t)]}{\sinh^2(t)}$$

is strictly increasing from $(0,\infty)$ onto $(1/4, 1)$.

Proof. We clearly see that

$$\varphi(t) = \frac{\cosh(t)}{\cosh(t)+1} \times \frac{\cosh(t) - I_0(t)}{\cosh(t) - 1}$$

and $\cosh(t)/[\cosh(t)+1]$ is strictly increasing from $(0,\infty)$ onto $(1/2,1)$. Therefore, it suffices to prove that the function

$$\varphi_1(t)=\frac{\cosh(t)-I_0(t)}{\cosh(t)-1} \tag{2.2}$$

is strictly increasing from $(0,\infty)$ onto $(1/2,1)$.

Let

$$a_n=\frac{2^n n!-(2n-1)!!}{2^n n!(2n)!}, \qquad b_n=\frac{1}{(2n)!}. \tag{2.3}$$

Then simple computations lead to

$$\frac{a_n}{b_n}=1-\frac{(2n-1)!!}{(2n)!!}, \tag{2.4}$$

$$\frac{a_{n+1}}{b_{n+1}}-\frac{a_n}{b_n}=\frac{(2n-1)!!}{2^{n+1}(n+1)!}>0 \tag{2.5}$$

for all $n\geqslant 1$.

From Lemma 2.1 and (2.4) we get

$$1-\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}<\frac{a_n}{b_n}<1-\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}},$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=1. \tag{2.6}$$

It follows from (1.3) and (2.2) together with (2.3) that

$$\varphi_1(t)=\frac{\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}-\sum_{n=0}^{\infty}\frac{t^{2n}}{2^{2n}(n!)^2}}{\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}-1}=\frac{\sum_{n=1}^{\infty}a_n t^{2n}}{\sum_{n=1}^{\infty}b_n t^{2n}}. \tag{2.7}$$

Lemma 2.4, (2.5) and (2.7) lead to the conclusion that $\varphi_1(t)$ is strictly increasing on $(0,\infty)$.

From Lemma 2.5, (2.4), (2.6) and (2.7) we have

$$\lim_{t\to 0}\varphi_1(t)=\frac{a_1}{b_1}=\frac{1}{2},\quad \lim_{t\to\infty}\varphi_1(t)=\frac{a_n}{b_n}=1. \quad \square$$

LEMMA 2.7. *Let $I_0(t)$ be defined by (1.3). Then the function*

$$\phi(t)=\frac{\cosh^2(t)-I_0^2(t)}{\sinh^2(t)} \tag{2.8}$$

is strictly increasing from $(0,\infty)$ onto $(1/2,1)$.

Proof. Let s_n be defined as in Lemma 2.3 and

$$c_n=\frac{(2n)!}{2^{2n-1}(n!)^4}, \qquad d_n=\frac{2^{2n}}{(2n)!}. \tag{2.9}$$

Then simple computations lead to

$$\frac{c_n}{d_n} = \frac{2s_n}{2n+1}, \tag{2.10}$$

$$\frac{c_{n+1}}{d_{n+1}} - \frac{c_n}{d_n} = -\frac{(4n+3)s_n}{2(n+1)^2(2n+1)} < 0 \tag{2.11}$$

for all $n \geqslant 1$.

It follows from (2.8) and (2.9) together with Lemma 2.2 and the power series formula $\cosh(t) = \sum_{n=0}^{\infty} t^{2n}/(2n)!$ that

$$\phi(t) = \frac{1+\cosh(2t) - 2I_0^2(t)}{\cosh(2t)-1} = 1 - \frac{\sum_{n=1}^{\infty} c_n t^{2n}}{\sum_{n=1}^{\infty} d_n t^{2n}}. \tag{2.12}$$

Lemma 2.4, (2.11) and (2.12) lead to the conclusion that $\phi(t)$ is strictly increasing on $(0,\infty)$.

From Lemma 2.5, (2.1), (2.9), (2.10) and (2.12) we clearly see that

$$\lim_{t\to 0} \phi(t) = 1 - \frac{c_1}{d_1} = \frac{1}{2},$$

$$\lim_{t\to\infty} \phi(t) = 1 - \lim_{n\to\infty} \frac{c_n}{d_n} = 1 - \lim_{n\to\infty} \frac{2s_n}{2n+1} = 1. \qquad \square$$

3. Main results

THEOREM 3.1. *Let $\alpha, \beta \in [0,1/2]$. Then the double inequality*

$$H[\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a] < TQ(a,b) < H[\beta a + (1-\beta)b, \beta b + (1-\beta)a]$$

holds for all $a,b>0$ with $a \neq b$ if and only if $\alpha = 0$ and $\beta \geqslant 1/4$.

Proof. Since both the Toader-Qi mean $TQ(a,b)$ and harmonic mean $H(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $b > a > 0$. Let $p \in [0,1/2]$, $t = (\log b - \log a)/2 > 0$ and $v = (a-b)/(a+b) = \tanh(t) \in (0,1)$. Then (1.1) leads to

$$\begin{aligned} &TQ(a,b) - H[pa + (1-p)b, pb + (1-p)a] \\ &= \sqrt{ab}\, I_0(t) - \frac{a+b}{2}\left[1 - (1-2p)^2 v^2\right] \\ &= \sqrt{ab}\, I_0(t) - \sqrt{ab}\cosh(t)\left[1 - (1-2p)^2 \tanh^2(t)\right] \\ &= \frac{\sqrt{ab}\sinh^2(t)}{\cosh(t)}\left[(1-2p)^2 - \varphi(t)\right], \end{aligned} \tag{3.1}$$

where $\varphi(t)$ is defined as in Lemma 2.6.

Therefore, Theorem 3.1 follows easily from Lemma 2.6 and (3.1). $\square$

From Theorem 3.1 we get Corollary 3.1 immediately.

COROLLARY 3.1. *The double inequality*

$$\frac{1}{\cosh(t)} < I_0(t) < \frac{3\cosh(t)}{4} + \frac{1}{4\cosh(t)}$$

holds for all $t > 0$.

THEOREM 3.2. *Let* $\lambda, \mu \in [0,1/2]$. *Then the double inequality*

$$G[\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a] < TQ(a,b) < G[\mu a + (1-\mu)b, \mu b + (1-\mu)a]$$

holds for all $a,b > 0$ *with* $a \neq b$ *if and only if* $\lambda = 0$ *and* $\mu \geqslant 1/2 - \sqrt{2}/4$.

Proof. Without loss of generality, we assume that $b > a > 0$. Let $q \in [0,1/2]$, $t = (\log b - \log a)/2 > 0$ and $v = (a-b)/(a+b) = \tanh(t) \in (0,1)$. Then (1.1) leads to

$$\begin{aligned} & TQ(a,b) - G[qa + (1-q)b, qb + (1-q)a] \\ = & \sqrt{ab}\, I_0(t) - \frac{a+b}{2}\sqrt{1-(1-2q)^2 v^2} \\ = & \sqrt{ab}\, I_0(t) - \sqrt{ab}\cosh(t)\sqrt{1-(1-2q)^2\tanh^2(t)} \\ = & \frac{\sqrt{ab}\sinh^2(t)}{\cosh(t)\sqrt{1-(1-2q)^2\tanh^2(t)} + I_0(t)}\left[(1-2q)^2 - \phi(t)\right], \end{aligned} \tag{3.2}$$

where $\phi(t)$ is defined by (2.8).

Therefore, Theorem 3.2 follows easily from Lemma 2.7 and (3.2). □

From Theorem 3.2 we get Corollary 3.2 immediately.

COROLLARY 3.2. *The inequality*

$$I_0(t) < \frac{\sqrt{2\left[1+\cosh^2(t)\right]}}{2}$$

holds for all $t > 0$.

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