A NOTE ON A WIELANDT TYPE NORM INEQUALITY

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Abstract. As a continuation of recent study on a Wielandt type norm inequality due to Lin [13, Conjecture 3.4], we prove the following result: Let $A \in M_n(C)$ satisfying $0 < m \leq A \leq M$, and let $X$ and $Y$ be $n \times k$ matrices such that $X^*X = Y^*Y = I_k$ and $X^*Y = 0$. Then for every 2-positive unital linear map $\Phi$, we have

$$\|\langle \Phi(X^*AY) \Phi(Y^*AY)^{-1} \Phi(Y^*AX)^{-\frac{p}{2}} \Phi(X^*AX)^{-\frac{p}{2}} \rangle \| \leq \begin{cases} \left( \frac{M - m}{M + m} \right)^{\frac{p}{2}} \left( \frac{M^p + m^p}{M_4 m^4} \right)^{\frac{2}{p}} & 1 < p < 2 \\ \left( \frac{M - m}{M + m} \right)^{\frac{p}{2}} \left( \frac{M^p + m^p}{M_4 m^4} \right)^{\frac{2}{p}} & p \geq 2. \end{cases}$$

1. Introduction

Let $M, m$ be scalars. $M_n(C)$ denotes the set of all $n \times n$ complex matrices. $A^* \in M_n(C)$ stands for the adjoint of $A$. For a Hermitian matrix $A \in M_n(C)$, we use the notation $A \geq 0$ to mean that $A$ is positive semidefinite, and $A > 0$ to mean it is positive definite. A linear map $\Phi : M_n(C) \to M_k(C)$ is called (strictly) positive if $\Phi(A) \geq 0$ ($\Phi(A) > 0$) whenever $A \geq 0$ ($A > 0$). It is said to be unital if $\Phi(I_n) = I_k$. We say that $\Phi$ is 2-positive if whenever the $2 \times 2$ matrix $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ is positive, then so is $\begin{bmatrix} \Phi(A) & \Phi(B) \\ \Phi(B^*) & \Phi(C) \end{bmatrix}$. We use $\| \cdot \|$ for operator norm.

The Wielandt inequality [10, p. 443] is as follows: if $0 < mI \leq A \leq MI$, and $x, y \in H$ with $x \perp y$, then

$$|\langle x, Ay \rangle| \leq \left( \frac{m - M}{M + m} \right)^{\frac{2}{4}} \langle x,Ay \rangle \langle y, Ay \rangle. \quad (1.1)$$

Wielandt’s inequality plays an important role in different contexts. For example, it has a variety of applications in numerical methods, especially eigenvalue estimation [6]. It is also applied in multivariate analysis [2, 5, 7, 10]. For the latest study on the Wielandt and generalized Wielandt inequality, readers are referred to [12].

The operator version of (1.1) was proved by Bhatia and Davis [3] (independently by Wang and Ip [16]) as follows: Let $0 < m \leq A \leq M$, and let $X, Y$ be two partial
isometries on a Hilbert space $H$ whose final spaces are orthogonal to each other. Then for every 2-positive linear map $\Phi$,

$$\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX) \leq \left( \frac{M-m}{M+m} \right)^2 \Phi(X^*AX).$$

(1.2)

Under the same condition, Lin [13, Conjecture 3.4] conjectured the following assertion could be true:

$$\|\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX)\Phi(X^*AX)^{-1}\| \leq \left( \frac{M-m}{M+m} \right)^2.$$  \hspace{1cm} (1.3)

Recently, the authors [6] obtained the following result in the finite-dimensional case: Let $A \in M_n(\mathbb{C})$ satisfying $0 < m \leq A \leq M$, and let $X$ and $Y$ be $n \times k$ matrices such that $X^*X = Y^*Y = I_k$ and $X^*Y = 0$. Then for every 2-positive unital linear map $\Phi$, we have

$$\|\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX)\Phi(X^*AX)^{-1}\| \leq \frac{1}{4} \left( \frac{M-m}{M+m} \right)^2$$

$$\left( \frac{M}{M+m} + \frac{1}{m} \right)^2,$$

(1.4)

which was a step closer to the conjecture (1.3).

In this note, we obtain the following result in the finite-dimensional case: Let $A \in M_n$ with $0 < ml_n \leq A \leq MI_n$, and let $X$ and $Y$ be $n \times k$ matrices such that $X^*X = Y^*Y = I_k$ (i.e. isometries) and $X^*Y = 0$. Then for every 2-positive unital linear map $\Phi$, we have

$$\|\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX)\Phi(X^*AX)^{-1}\| \leq \begin{cases} \left( \frac{M-m}{M+m} \right)^p \left( \frac{M}{M+m} \right)^{\frac{p}{2}} \left( \frac{M}{M+m} \right)^{\frac{p}{2}} \frac{1}{m} & 1 < p < 2 \\ \left( \frac{M-m}{M+m} \right)^p \frac{1}{4M^2m^2} & p \geq 2 \end{cases},$$

(1.4)

which is tighter than (1.4).

### 2. Main result

We need two lemmas which play a very important role in the proof of the main theorem of this paper. The first Lemma is Ando-Zhan’s celebrated result.

**Lemma 1.** [1] *Let $A$ and $B$ be positive operators. Then for $1 \leq r < \infty$*

$$\|A^r + B^r\| \leq \|(A + B)^r\|.$$  \hspace{1cm} (2.1)

The next lemma holds for positive definite matrices but a careful observation shows that it is true for positive definite operators on a Hilbert space.
LEMMA 2. [4] Let $A, B > 0$. Then the following norm inequality holds:

$$||AB|| \leq \frac{1}{4}||A + B||^2. \quad (2.2)$$

Now we are devoted to presenting the main result which is a refinement of (1.4) in the finite-dimensional case.

THEOREM 3. Let $A \in M_n$ with $0 < mI_n \leq A, B \leq MI_n$ and let $X$ and $Y$ be $n \times k$ matrices such that $X^*X = Y^*Y = I_k$ and $X^*Y = 0$. Then for every 2-positive unital linear map $\Phi$,

$$||((\Phi(X^*AY)\Phi(Y^*AY))^{-1}\Phi(Y^*AX))^{\frac{p}{2}}\Phi(X^*AX)^{-\frac{p}{2}}||$$

$$\leq \begin{cases} 
\frac{1}{4} \left[ \left( \frac{M-m}{M+m} \right)^2 Mm\Phi(X^*AX)^{-1} \right]^{\frac{p}{2}} & 1 < p < 2 \\
\frac{1}{4} \left[ \left( \frac{M-m}{M+m} \right)^2 Mm\Phi(X^*AX)^{-1} \right]^{\frac{p}{2}} & p \geq 2.
\end{cases} \quad (2.3)$$

Proof. Firstly, consider the case of $p \geq 2$. Compute

$$\left| \left( \frac{M-m}{M+m} \right)^2 \Phi(X^*AX)^{-1} \right|^{\frac{p}{2}} \leq \frac{1}{4} \left[ \left( \frac{M-m}{M+m} \right)^2 Mm\Phi(X^*AX)^{-1} \right]^{\frac{p}{2}}$$

(by (2.2))

$$\leq \frac{1}{4} \left( \frac{M-m}{M+m} \right)^2 \Phi(X^*AX) + \left( \frac{M-m}{M+m} \right)^2 Mm\Phi(X^*AX)^{-1} \right|^{\frac{p}{2}} \quad (by \, (2.1))$$

$$\leq \frac{1}{4} \left( \frac{M-m}{M+m} \right)^2 \Phi(X^*AX) + \left( \frac{M-m}{M+m} \right)^2 Mm\Phi(X^*AX)^{-1} \right|^{\frac{p}{2}} \quad (by \, (1.2))$$

$$= \frac{1}{4} \left( \frac{M-m}{M+m} \right)^2 \left\| \Phi(X^*AX) + Mm\Phi(X^*AX)^{-1} \right\|$$

$$\leq \frac{1}{4} \left( \frac{M-m}{M+m} \right)^2 \left\| \Phi(X^*AX) + Mm\Phi(X^*AX)^{-1} \right\|$$

The last inequality above is obtained: Since $0 < mI_n \leq A \leq MI_n$, $mI_k \leq \Phi(X^*AX) \leq MI_k$ and $\frac{1}{M} \leq \Phi(X^*AX)^{-1} \leq \frac{1}{m}$, we have

$$\Phi(X^*AX)(m - \Phi(X^*AX))\Phi(X^*AX)^{-1} \leq 0,$$

which implies

$$Mm\Phi(X^*AX)^{-1} + \Phi(X^*AX) \leq M + m.$$
So
\[ \left\| \left( \Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX) \right)^{\frac{p}{2}} \Phi(X^*AX)^{-\frac{p}{2}} \right\| \leq \frac{(M-m)^p}{4M^2m^2}. \]

Next consider the case of \( 1 < p < 2 \). Compute
\[
\left\| \left( \Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX) \right)^{\frac{p}{2}} \left( \left( \frac{M-m}{M+m} \right)^2 Mm\Phi(X^*AX)^{-1} \right)^{\frac{p}{2}} \right\|
\leq \frac{1}{4} \left\| \left( \Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX) \right)^{\frac{p}{2}} + \left( \frac{M-m}{M+m} \right)^p M^2m^2 \Phi(X^*AX)^{-\frac{p}{2}} \right\|^2
\]
(by (2.2))
\[
= \frac{1}{4} \left( \frac{M-m}{M+m} \right)^p \left( \Phi(X^*AX)^{\frac{p}{2}} + M^2m^2 \Phi(X^*AX)^{-\frac{p}{2}} \right)^2
\]
(by Löwner-Heinz inequality and (1.2))
\[
= \frac{1}{4} \left( \frac{M-m}{M+m} \right)^2 \left( M^2 + m^2 \right)^2.
\]
The last inequality above holds as follows: By using \( 0 < mL_n \leq A \leq ML_n, \ m^\frac{p}{2} \leq \Phi(X^*AX)^{\frac{p}{2}} \leq M^\frac{p}{2} \) and \( M^{-\frac{p}{2}} \leq \Phi(X^*AX)^{-\frac{p}{2}} \leq m^{-\frac{p}{2}} \), we have
\[
(M^\frac{p}{2} - \Phi(X^*AX)^{\frac{p}{2}})(m^\frac{p}{2} - \Phi(X^*AX)^{\frac{p}{2}})\Phi(X^*AX)^{-\frac{p}{2}} \leq 0,
\]
which means
\[
M^\frac{p}{2}m^\frac{p}{2} \Phi(X^*AX)^{-\frac{p}{2}} + \Phi(X^*AX)^{\frac{p}{2}} \leq M^\frac{p}{2} + m^\frac{p}{2}.
\]
That is,
\[
\left\| \left( \Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX) \right)^{\frac{p}{2}} \Phi(X^*AX)^{-\frac{p}{2}} \right\|
\leq \left( \frac{M-m}{M+m} \right)^p \frac{(M^\frac{p}{2} + m^\frac{p}{2})^2}{4M^2m^2}.
\]
\[ \square \]

**Remark 4.** If \( p = 2 \), the right side of the inequality (2.3) is \( \frac{(M-m)^2}{4Mm} \). Obviously, the below inequality holds
\[
\frac{(M-m)^2}{4Mm} \leq \frac{M}{m} \left( \frac{M-m}{M+m} \right)^2 \leq \frac{1}{4} \left( \left( \frac{M-m}{M+m} \right)^2 + \frac{1}{m} \right)^2,
\]
which shows that the bound of (2.3) is smaller than that of (1.4). Thus, (2.3) is a refinement of (1.4) for \( p = 2 \).
Remark 5. When \( p = 2 \), the author [9, (2.7)] obtained a stronger result than the inequality (2.3). However, if we present \( p (p > 2) \) power of (2.7) in [9] through the similar method of the proof of Theorem 3, we will find that the result for \( p > 2 \) is very complicated and not continuous at \( p = 2 \).

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