Journal of Mathematical Inequalities
Volume 11, Number 1 (2017), 135–141 doi:10.7153/jmi-11-13

A LYAPUNOV–TYPE INEQUALITY FOR A FRACTIONAL DIFFERENTIAL EQUATION WITH HADAMARD DERIVATIVE

QINGHUA MA, CHAO MA AND JINXUN WANG

(*Communicated by J. Pečarić*)

Abstract. In this work, a Lyapunov-type inequality is obtained for the case when working with a fractional boundary value problem with the Hadamard derivative.

1. Introduction

In the last few decades, fractional differential equations have gained considerable importance and attention due to their applications in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physical, mechanics, chemistry, aerodynamics electro dynamics of complex medium, etc. The literature on such differential equations and their applications is vast; see the monographs of Kilbas, Srivastava and Trujillo [1], Miller and Ross [2], Podludny [3] and the references given therein.

It is well known that various types of integral inequalities play a dominant role in the study of quantitative properties of solutions of differential and integral equations. One of them is the Lyapunov-type inequality which has been proved to be very useful in studying the zeros of solutions of differential equations. For example, see [4] and [5]. The famous Lyapunov theorem [6] can be stated as follows:

If a nontrivial solution to the boundary value problem

$$\begin{cases} y''(t)+q(t)y(t)=0, & a<t<b, \\ y(a)=0=y(b), \end{cases} \tag{1.1}$$

exists, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \frac{4}{b-a}. \tag{1.2}$$

Recently, some authors began to study the Lyapunov-type inequality for fractional boundary value problems. In [7], Ferreira investigated a Lyapunov-type inequality for the Riemann-Liouville fractional boundary problem

$$\begin{cases} {}_a^RD^{\alpha}y(t)+q(t)y(t)=0, & a<t<b, \\ y(a)=0=y(b), \end{cases} \tag{1.3}$$

Mathematics subject classification (2010): Please write....

Keywords and phrases: Lyapunov inequality, Hadamard fractional derivative, boundary value problem, Green's function.

Paper JMI-11-13

where ${}_{a}^{R}D^{\alpha}$ is the Riemann-Liouville derivative of order $\alpha, 1<\alpha\leqslant 2$, and $q:[a,b]\rightarrow$ is a continuous function. It has been proved that if (1.3) has a nontrivial solution then

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}, \tag{1.4}$$

which can lead to Lyapunov's classical inequality (1.2) when $\alpha=2$. In [8], the same author established a Lyapunov-type inequality depended on a fractional boundary problem with Caputo fractional derivative. In both of his works, the author has given some interesting applications to localize the real zeros of certain Mittag-Leffler functions.

In [9], Jleli and Samet considered the fractional differential equation

$${}_{a}^{C}D^{\alpha}y(t)+q(t)y(t)=0, \quad a<t<b, \quad 1<\alpha\leqslant 2, \tag{1.5}$$

with the mixed boundary conditions

$$y(a)=0=y'(b) \tag{1.6}$$

or

$$y'(a)=0=y(b) \tag{1.7}$$

Under boundary conditions (1.6) and (1.7), two Lyapunov-type inequalities were established as follows

$$\int_a^b (b-s)^{\alpha-2}|q(s)|ds \geqslant \frac{\Gamma(\alpha)}{\max\{\alpha-1,2-\alpha\}(b-a)}, \tag{1.8}$$

and

$$\int_a^b (b-s)^{\alpha-1}|q(s)|ds \geqslant \Gamma(\alpha) \tag{1.9}$$

respectively.

Very recently, Rong and Bai [10] considered (1.5) under boundary condition involving the Caputo fractional derivative as follows:

$$y(a)=0, \quad {}_{a}^{C}D^{\beta}y(t)=0, \quad 0<\beta\leqslant 1 \tag{1.10}$$

and an interesting Lyapunov-type inequality was established

$$\int_a^b (b-s)^{\alpha-\beta-1}|q(s)|ds \geqslant \frac{(b-a)^{-\beta}}{\max\{\frac{1}{\Gamma(\alpha)}-\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{2-\alpha}{\alpha-1}\cdot\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\}}, \tag{1.11}$$

which has also generalized Jleli and Samet's main results in [9].

However, we have noted that all of the results mentioned above are concerned with the Riemann-Liouville fractional derivative or the Caputo fractional derivative. To the best of the authors' knowledge, there is no work on the Lyapunov-type inequality on fractional differential equations involving the Hadamard fractional derivative, which is represented as a quite different kind of weakly singular kernel. For some recent results on the Hadamard fractional differential equations, one can see [11–16].

In this paper we will consider a Lyapunov-type inequality for the Hadamard fractional boundary value problem

$$\begin{cases} {}_{1}^{H}D^{\alpha}y(t)-q(t)y(t)=0, & 1<t<e, \\ y(1)=0=y(e), \end{cases} \tag{1.12}$$

where ${}_{1}^{H}D^{\alpha}$ is the fractional derivative in the sense of the Hadamard of order $1<\alpha\leqslant 2$, $t\in I=[1,e]$. The interest of this article does not lie only on the fact that the problem is quite different to the existing results, but also is a useful supplement to this type of inequality.

2. Main results

In this section, we begin to cite some definitions and establish some useful lemmas in the discussion of our proof as follows:

DEFINITION 2.1. ([1]) The Hadamard fractional integral of order $\alpha \in \mathrm{R}_+$ of function $f(t)$, for all $t \geqslant 1$, is defined by:

$$ {}_HD_{1,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s} $$

where $\Gamma(\cdot)$ is the Euler Gamma function.

DEFINITION 2.2. ([1]) The Hadamard derivative of order $\alpha \in [n-1,n)$, $n \in Z_+$ of a function $f(t)$ is given by

$$ {}_HD_{1,t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\ln \frac{t}{s}\right)^{n-\alpha-1} f(s) \frac{ds}{s}. $$

LEMMA 2.3. *If $y \in C[1,e]$ is a solution of the boundary value problem (1.12), then y satisfies the integral equation*

$$ y(t) = \int_1^e G(t,s) q(s) y(s) ds, $$

where

$$ G(t,s) = \begin{cases} G_1(t,s) = G_2(t,s) + \frac{1}{\Gamma(\alpha)} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{1}{s}, & 1 \leqslant s \leqslant t \leqslant e \\ G_2(t,s) = -\frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)} \left(\ln \frac{e}{s}\right)^{\alpha-1} \frac{1}{s}, & 1 \leqslant t \leqslant s \leqslant e. \end{cases} \tag{2.1} $$

Proof. By the result in [1], the solution of the Hadamard differential equation in (1.12) can be written as

$$ y(t) = c_1 (\ln t)^{\alpha-1} + c_2 (\ln t)^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} q(s) y(s) \frac{ds}{s}. \tag{2.2} $$

Using the given boundary conditions, we get $c_2 = 0$, and

$$ c_1 = -\frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{e}{s}\right)^{\alpha-1} q(s) y(s) \frac{ds}{s}. $$

Substituting the values of c_1 and c_2 in (2.2), we obtain

$$ \begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} q(s) y(s) \frac{ds}{s} - \frac{1}{\Gamma(\alpha)} \int_1^e (\ln t)^{\alpha-1} \left(\ln \frac{e}{s}\right)^{\alpha-1} q(s) y(s) \frac{ds}{s} \\ &= \frac{1}{\Gamma(\alpha)} \int_1^t \left[\left(\ln \frac{t}{s}\right)^{\alpha-1} - (\ln t)^{\alpha-1} \left(\ln \frac{e}{s}\right)^{\alpha-1}\right] q(s) y(s) \frac{ds}{s} \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_t^e (\ln t)^{\alpha-1} \left(\ln \frac{e}{s}\right)^{\alpha-1} q(s) y(s) \frac{ds}{s}, \end{aligned} $$

which concludes the proof. □

LEMMA 2.4. *Let function G be defined as in Lemma 2.3, then we have*

$$|G| \leqslant \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)}(1-\lambda)^{\alpha-1}\exp(-\lambda) \tag{2.3}$$

where $\lambda = \frac{2\alpha-1-\sqrt{(2\alpha-2)^2+1}}{2}$.

Proof. Let $D_1 : 1 \leqslant s \leqslant t \leqslant e$ and $D_2 : 1 \leqslant t \leqslant s \leqslant e$ and $G_i(t,s),\ i = 1,2$ be defined as in (2.1).

For function G_2, it is obvious to see that $G_2 \leqslant 0$ and $G_2(1,s) = 0$ which is the maximum value of G_2. So we will consider function G_2's minimum value and G_1's maximum and minimum values respectively.

For $t \leqslant s$, we observe that

$$G_2 \geqslant -\frac{(\ln s)^{\alpha-1}}{\Gamma(\alpha)}\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{1}{s} = -\frac{1}{\Gamma(\alpha)}\frac{(\ln s - \ln^2 s)^{\alpha-1}}{s}. \tag{2.4}$$

It follows that we only need to get the maximum value of the function

$$f(s) = \frac{(\ln s - \ln^2 s)^{\alpha-1}}{s}.$$

By computation, we have

$$f'(s) = \frac{(\ln s - \ln^2 s)^{\alpha-2}}{s^2}[\ln^2 s - (2\alpha-1)\ln s + \alpha - 1]$$

which implies that $f'(s) = 0 \Leftrightarrow \ln^2 s - (2\alpha-1)\ln s + \alpha - 1 = 0.$

$$\Leftrightarrow \ln s = \frac{2\alpha-1 \pm \sqrt{(2\alpha-2)^2+1}}{2}.$$

However, since $\dfrac{2\alpha-1+\sqrt{(2\alpha-2)^2+1}}{2} > \dfrac{2-1+\sqrt{(2-2)^2+1}}{2} = 1$ and

$$0 \leqslant \frac{2\alpha-1-\sqrt{(2\alpha-2)^2+1}}{2} < \frac{2\alpha-1-\sqrt{(2\alpha-2)^2}}{2} = \frac{1}{2},$$

we have only one solution

$$\ln s = \frac{2\alpha-1-\sqrt{(2\alpha-2)^2+1}}{2}$$

of the equation $f'(s) = 0$ on D_2.

By (2.4) and discussion above, we can conclude that the minimum value of the function G_2 reaches on

$$t = s = \exp\frac{2\alpha-1-\sqrt{(2\alpha-2)^2+1}}{2}. \tag{2.5}$$

For the function $G_1 = \dfrac{1}{\Gamma(\alpha)s}[(\ln t - \ln s)^{\alpha-1} - (\ln t(1-\ln s))^{\alpha-1}]$, it is easy to see that $G_1 \leqslant 0$, which follows that the maximum value of G_1 is $G_1(e,e) = 0$. To obtain the minimum value of G_1, for fixed s, we set

$$g(t) = (\ln t - \ln s)^{\alpha-1} - (\ln t(1-\ln s))^{\alpha-1}.$$

By computation,

$$g'(t)=\frac{\alpha-1}{t}[(\ln t-\ln s)^{\alpha-2}-(\ln t-\ln t\ln s)^{\alpha-2}(1-\ln s)]. \tag{2.6}$$

Observing that $\ln t-\ln s\leqslant \ln t-\ln t\ln s$ and $\alpha-2\leqslant 0$, it follows that

$$(\ln t-\ln s)^{\alpha-2}\geqslant(\ln t-\ln t\ln s)^{\alpha-2}\geqslant(\ln t-\ln t\ln s)^{\alpha-2}(1-\ln s). \tag{2.7}$$

By (2.6) and (2.7), we get

$$g'(t)\geqslant 0$$

for $t\geqslant s$, which follows that when $t=s$ we can get the minimum value of $g(t)$. So, the function $G_1(t,s)=\frac{1}{\Gamma(\alpha)s}g(t)$ reaches the minimum value

$$-\frac{1}{\Gamma(\alpha)}\frac{(\ln s-\ln^2 s)^{\alpha-1}}{s} \tag{2.8}$$

when $t=s$. By (2.4) and (2.8), we can conclude that

$$\min G_1(t,s)=\min G_2(t,s)=G_1(e^{\lambda},e^{\lambda})=-\frac{\lambda^{\alpha-1}}{\Gamma(\alpha)}(1-\lambda)^{\alpha-1}\exp(-\lambda),$$

where λ is defined as in Lemma 2.4. □

THEOREM 2.1. *If the following Hadarmard fractional value problem (HFBVP)*

$$\begin{cases} {}_1^H D^{\alpha}y(t)-q(t)y(t)=0, & 1<t<e,\\ y(1)=0=y(e), \end{cases} \tag{2.9}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_1^e |q(s)|ds>\Gamma(\alpha)\lambda^{1-\alpha}(1-\lambda)^{1-\alpha}\exp\lambda \tag{2.10}$$

where $\lambda=\frac{2\alpha-1-\sqrt{(2\alpha-2)^2+1}}{2}$.

Proof. Let $\mathbf{B}=C[1,e]$ be the Banach space endowed with norm $\|y\|=\sup_{t\in[1,e]}|y(t)|$. By Lemma 2.3, the solution of the HFBVP can be written as

$$y(t)=\int_1^e G(t,s)q(s)y(s)ds.$$

Hence,

$$\|y\|\leqslant\max\int_1^e|G(t,s)||q(s)|ds\|y\|,\qquad \text{i.e.,}\qquad 1\leqslant\max\int_1^e|G(t,s)||q(s)|ds.$$

An application of Lemma 2.4 to the last inequality leads to

$$1<\frac{\lambda^{\alpha-1}}{\Gamma(\alpha)}(1-\lambda)^{\alpha-1}\exp(-\lambda)\int_1^e|q(s)|ds,$$

from which the inequality (2.10) follows. □

REMARK 2.1. As stated in [8], the best strategy known so far to deduce the Lyapunov inequality with the fractional differential setting seems to be converting the boundary value problem (1.12) into an equivalent integral equation and then find the maximum value of its Green's function. Along with this strategy, we have obtained a Lyapunov inequality with the Hadamard derivative in the special interval $[1,e]$. However, when we consider the same problems on a general interval $[a,b]$ $(1 \leqslant a < b)$, we will face some insurmountable obstacles to find the maximum value of the corresponding Green's function. So we state the following as an open problem for readers:

How to get the Lyapunov inequality for the following the Hadarmard fractional value problem (HFBVP)

$$\begin{cases} {}_a^H D^\alpha y(t) - q(t)y(t) = 0, \quad 1 \leqslant a < t < b, \quad 1 < \alpha \leqslant 2, \\ y(a) = 0 = y(b), \end{cases}$$

where the Hadamard derivative ${}_a^H D^\alpha$ of a function $f(t)$ is defined by

$${}_a^H D^\alpha f(t) = \frac{1}{\Gamma(2-\alpha)} \left(t\frac{d}{dt} \right)^2 \int_a^t \left(\ln \frac{t}{s} \right)^{1-\alpha} f(s) \frac{ds}{s}.$$

Acknowledgement. The authors are very grateful to the editor and referee for their helpful and useful comments for improving the paper. The research was supported by NNSF of China (Project No. 71471045), NSF of Guangdong Province of China (Project No. 2015A030313574) and The Humanities and Social Science Research Foundation of the Ministry of Education of China (Project No. 15YJAZH051).

REFERENCES

[1] A. A. KILBAS, H. M. SRIVASTAVA AND J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Vol. **204**, Elsevier, Amsterdam, 2006.

[2] K. S. MILLER AND B. ROSS, *An Introduction the the Fractional Calclus and Fractional Equations*, Wiley, New York, 1993.

[3] I. PODLUBNY, *Fractional Differential Equations*, Academic Press, New York, 1993.

[4] R. C. BROWN, D. B. HINTON, *Lyapunov inequalities and their applications*, In: Survey on Classical Inequalities (Ed. T. M. Rassias), Math. Appl. **517**, Kluwer Acad. Publ., Dordrecht–London (2000), 1–25.

[5] A. TIRYAKI, *Recent developments of Lyapunov-type inequalities*, Adv. Dyn. Syst. Appl. **5** (2) (2010), 231–248.

[6] A. M. LYAPUNOV, *Probleme général de la stabilité du mouvement*, (French Transl. of a Russian paper dated 1893). Ann. Fac. Sci. Univ. Toulouse **2** (1907), 27–247; Reprinted in: Ann. Math. Studies, no. 17, Princeton (1947).

[7] R. A. C. FERREIRA, *A Lyapunov type inequality for a fractional boundary value problem*, Fract. Calc. Appl. Anal. **16** (2013), no. 4, 978–984.

[8] R. A. C. FERREIRA, *On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function*, J. Math. Anal. Appl. **412** (2014), 1058–1063.

[9] M. JLELI, B. SAMET, *Lyapunov type inequalities for a fractional differential equations with mixed boundary value problems*, Math. Inequal. Appl., **18** (2) (2015), 443–451.

[10] J. RONG, C. Z. BAI, *Lyapunov-type inequality for a fractional differential equations with fractional boundary value problems*, Adv. Difference. Equ., (2015), 2015:82, 10 pp.

[11] J. R. WANG, Y. ZHOU AND M. MEDVEĎ, *Existence and stability of fractional differential equations with Hadamard derivative*, Topol. Methods in Nonlinear Anal., **41** (2013), 113–133.

[12] Q. H. MA, J. W. WANG, R. N. WANG AND X. H. KE, *Study on some qualitative properties for solutions of a certain two-dimensional fractional differential system with Hadamard derivative*, Appl. Math. Letters, **36** (2014), 7–13.

[13] B. AHMAD, S. K. NTOUYAS, *On Hadamard fractional integro-differential boundary value problems*, J. Appl. Math. Comput., **47** (2015), 119–131.

[14] J. R. WANG, Y. R. ZHANG, *On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives*, Appl. Math. Letters, **39** (2015), 85–90.

[15] Q. H. MA, R. N. WANG, J. W. WANG AND Y. C. MA, *Qualitative analysis for solutions of a certain more generalized two-dimensional fractional differential system with Hadamard derivative*, Appl. Math. Comput., **257** (2015), 436–445.

[16] RICARDO ALMEIDA, DELFIM F. M. TORRES, *Computing Hadamard type operators of variable fractional order*, Appl. Math. Comput., **257** (2015), 74–88.

(Received December 9, 2015)

Qinghua Ma
Applied Mathematics Department
Guangdong University of Foreign Studies
Guangzhou 510420, P. R. China
e-mail: gdqhma@21cn.com

Chao Ma
Applied Mathematics Department, Guangdong
University of Foreign Studies
Guangzhou 510420, P. R. China
e-mail: 335364745@qq.com

Jinxun Wang
Applied Mathematics Department, Guangdong
University of Foreign Studies
Guangzhou 510420, P. R. China
e-mail: wangjx08@163.com