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AN UPPER BOUND OF A DERIVATIVE FOR SOME CLASS OF POLYNOMIALS

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Abstract. In [S. Kumar and R. Lal, Generalizations of some polynomial inequalities, Int. Electron. J. Pure Appl. Math., 3, 2 (2011), 111–117.], Kumar and Lal provided an upper bound of a derivative for polynomial degree n having some of zeros at the origin and rest of zeros lying on or outside the boundary of a prescribed disk. In this paper, we present an upper bound of a derivative for polynomials $p(z)=(z-z_m)^{t_m}(z-z_{m-1})^{t_{m-1}}\cdots(z-z_0)^{t_0}\left(a_0+\sum_{\nu=\mu}^{n-(t_m+\cdots+t_0)}a_\nu z^\nu\right)$ of degree n having zeros $z_0,\ldots,z_m$ with $|z_j|<1$ for $0\leqslant j\leqslant m$ and the remaining $n-(t_m+\cdots+t_0)$ zeros are outside $\{z:|z|<k\}$ where $k\geqslant 1$.

1. Introduction

Let $p(z)$ be a polynomial of degree n. Then we have the Bernstein's inequality (see [2])

$$\max_{|z|=1}|p'(z)|\leqslant n\max_{|z|=1}|p(z)|. \tag{1}$$

Equality holds in (1) if and only if $p(z)$ has all of its zeros at the origin.

For a positive real number k, we let $D(0,k)$ and $C(0,k)$ denote the sets $\{z:|z|<k\}$ and $\{z:|z|=k\}$, respectively.

If we restrict ourselves to the class of polynomials having no zero in $D(0,1)$, the inequality (1) can be sharpened. In fact, it was conjectured by Erdös and later proved by Lax [8] that if $p(z)$ has no zero in $D(0,1)$, then

$$\max_{|z|=1}|p'(z)|\leqslant \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{2}$$

The equality holds if all zeros of $p(z)$ lie on $C(0,1)$, for example, $p(z)=\alpha+\beta z^n$, $|\alpha|=|\beta|$.

Aziz and Dawood [1] improved the inequality (2) under the same hypothesis and obtained that

$$\max_{|z|=1}|p'(z)|\leqslant \frac{n}{2}\left[\max_{|z|=1}|p(z)|-\min_{|z|=1}|p(z)|\right]. \tag{3}$$

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Equality in (3) holds for $p(z) = \beta + \alpha z^n$, $|\beta| \geqslant |\alpha|$.

For the class of polynomials $p(z)$ of degree n having no zero in $D(0,k)$, $k \geqslant 1$, Malik [9] proved that

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{4}$$

Inequality (4) was further improved by Govil [6] under the same hypothesis as

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{1+k} \left[\max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right]. \tag{5}$$

Inequalities (4) and (5) are sharp and extremal polynomial is $p(z) = (z+k)^n$.

Chan and Malik [3] considered the class of polynomials as in [9] and obtained the following generalization of (4).

THEOREM 1. [3] *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n having no zero in $D(0,k)$, $k \geqslant 1$, then*

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{1+k^\mu} \max_{|z|=1} |p(z)|. \tag{6}$$

The result is best possible and extremal polynomial is $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

The following theorem was proved by Pukhta [10], which is an improvement of Theorem 1 and a generalization of the inequality (5).

THEOREM 2. [10] *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n having no zero in $D(0,k)$, $k \geqslant 1$, then*

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{1+k^\mu} \left[\max_{|z|=1} |p(z)| - \min_{|z|=k} |p(z)| \right].$$

The result is best possible and extremal polynomial is $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

For polynomials having all its zeros on $C(0,k)$, $k \leqslant 1$, Govil [5] proved that

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{k^n + k^{n-1}} \max_{|z|=1} |p(z)|. \tag{7}$$

Dewan and Hans [4] generalized the inequality (7) for the polynomials of the type $p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leqslant \mu \leqslant n$ and proved the following theorem.

THEOREM 3. [4] *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n having all its zeros on $C(0,k)$, $k \leqslant 1$, then*

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{k^{n-2\mu+1} + k^{n-\mu+1}} \max_{|z|=1} |p(z)|. \tag{8}$$

Kumar and Lal [7] investigated the polynomials of degree n having some zeros at the origin and the rest of zeros lying on or outside the boundary of a prescribed disk. They obtained a generalized results of some well-known results.

THEOREM 4. [7] *If* $p(z) = z^s\left(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu\right)$, $1 \leqslant \mu \leqslant n-s$, $0 \leqslant s \leqslant n-1$, *is a polynomial of degree* n *having zeros of order* s *at the origin and the remaining* $n-s$ *zeros are outside* $D(0,k)$, $k \geqslant 1$, *then*

$$\max_{|z|=1}|p'(z)| \leqslant \frac{n+sk^\mu}{1+k^\mu}\max_{|z|=1}|p(z)| - \frac{(n-s)}{k^s(1+k^\mu)}\min_{|z|=k}|p(z)|. \tag{9}$$

2. Main results

THEOREM 5. (Main) *If* $p(z) = (z-z_0)^s\left(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu\right)$, $1 \leqslant \mu \leqslant n-s$, $0 \leqslant s \leqslant n-1$, *is a polynomial of degree* n *having zero of order* s *at* z_0 *with* $|z_0| < 1$ *and the remaining* $n-s$ *zeros are outside* $D(0,k)$, $k \geqslant 1$, *then*

$$\max_{|z|=1}|p'(z)| \leqslant \left[\frac{s}{(1-|z_0|)} + \frac{A}{(1-|z_0|)^s}\right]\max_{|z|=1}|p(z)| - \frac{A}{(k+|z_0|)^s}\min_{|z|=k}|p(z)|,$$

where $A = \dfrac{(1+|z_0|)^{s+1}(n-s)}{(1+k^\mu)(1-|z_0|)}$.

Proof. Let $p(z) = (z-z_0)^s\phi(z)$ where $\phi(z) = a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu$ be a polynomial of degree $n-s$ having no zero in $D(0,k)$, $k \geqslant 1$.

Then $p'(z) = (z-z_0)^s\phi'(z) + s(z-z_0)^{s-1}\phi(z)$ and $(z-z_0)p'(z) = sp(z) + (z-z_0)^{s+1}\phi'(z)$.

Therefore, $\max_{|z|=1}|z-z_0||p'(z)| \leqslant s\max_{|z|=1}|p(z)| + \max_{|z|=1}|z-z_0|^{s+1}|\phi'(z)|$.

Since $|z-z_0| \geqslant |z| - |z_0| = 1-|z_0|$ and $|z-z_0| \leqslant |z| + |z_0| = 1+|z_0|$ for $|z| = 1$, we obtain $(1-|z_0|)\max_{|z|=1}|p'(z)| \leqslant s\max_{|z|=1}|p(z)| + (1+|z_0|)^{s+1}\max_{|z|=1}|\phi'(z)|$.

Now let $m = \min_{|z|=k}|\phi(z)|$. Then $m \leqslant |\phi(z)|$ for $|z| = k$.

If $\phi(z)$ has a zero on $C(0,k)$, then $m = 0$. From now on, we assume that all $n-s$ zeros of $\phi(z)$ lie outside $D(0,k)$, $k \geqslant 1$. Therefore, for every complex number α such that $|\alpha| < 1$, it follows from Rouche's Theorem that all zeros of the polynomial $\phi(z) - \alpha m$ of degree $n-s$ lie outside $D(0,k)$, $k \geqslant 1$.

Applying the relation (6) to the polynomial $\phi(z) - \alpha m$, we get

$$\max_{|z|=1}|\phi'(z)| \leqslant \frac{n-s}{1+k^\mu}\max_{|z|=1}|\phi(z) - \alpha m|. \tag{10}$$

Now choosing α such that

$$|\phi(z) - \alpha m| = |\phi(z)| - |\alpha|m \tag{11}$$

and letting $|\alpha| \to 1$, we get from (10) in view of (11) that

$$\max_{|z|=1}|\phi'(z)| \leqslant \frac{n-s}{1+k^\mu}\max_{|z|=1}(|\phi(z)| - m). \tag{12}$$

Combining the relation (11) and the relation (12), we obtain that

$$(1-|z_0|)\max_{|z|=1}|p'(z)| \leqslant s\max_{|z|=1}|p(z)| + \left[(1+|z_0|)^{s+1}\,\frac{n-s}{1+k^{\mu}}\right]\max_{|z|=1}|\phi(z)| - \left[(1+|z_0|)^{s+1}\,\frac{n-s}{1+k^{\mu}}\right]m. \tag{13}$$

The relation between $\phi(z)$ and $p(z)$ implies that

$$\max_{|z|=1}|\phi(z)| = \max_{|z|=1}\left[\frac{1}{|z-z_0|^s}\,|p(z)|\right] \leqslant \frac{1}{(1-|z_0|)^s}\,\max_{|z|=1}|p(z)|.$$

Applying this relation in the relation (13), we have

$$(1-|z_0|)\max_{|z|=1}|p'(z)| \leqslant \left[s+\frac{(1+|z_0|)^{s+1}(n-s)}{(1+k^{\mu})(1-|z_0|)^s}\right]\max_{|z|=1}|p(z)| - \left[\frac{(n-s)(1+|z_0|)^{s+1}}{1+k^{\mu}}\right]m. \tag{14}$$

Again, the relation between $\phi(z)$ and $p(z)$ yields

$$m = \min_{|z|=k}|\phi(z)| = \min_{|z|=k}\left[\frac{1}{|z-z_0|^s}\,|p(z)|\right] \geqslant \frac{1}{(k+|z_0|)^s}\,\min_{|z|=k}|p(z)|.$$

Applying this relation in the relation (14), we have

$$(1-|z_0|)\max_{|z|=1}|p'(z)| \leqslant \left[s+\frac{(1+|z_0|)^{s+1}(n-s)}{(1+k^{\mu})(1-|z_0|)^s}\right]\max_{|z|=1}|p(z)| - \left[\frac{(1+|z_0|)^{s+1}(n-s)}{(1+k^{\mu})(k+|z_0|)^s}\right]\min_{|z|=k}|p(z)|.$$

Consequently,

$$\max_{|z|=1}|p'(z)| \leqslant \left[\frac{s}{(1-|z_0|)}+\frac{A}{(1-|z_0|)^s}\right]\max_{|z|=1}|p(z)| - \frac{A}{(k+|z_0|)^s}\,\min_{|z|=k}|p(z)|,$$

where $A = \dfrac{(1+|z_0|)^{s+1}(n-s)}{(1+k^{\mu})(1-|z_0|)}$. □

REMARK 1. By letting $z_0 = 0$ in Theorem 5, we get $A = \dfrac{n-s}{1+k^{\mu}}$ and

$$\begin{aligned}\max_{|z|=1}|p'(z)| &\leqslant (s+A)\,\max_{|z|=1}|p(z)| - \frac{A}{k^s}\,\min_{|z|=k}|p(z)|\\ &= \frac{n+sk^{\mu}}{(1+k^{\mu})}\,\max_{|z|=1}|p(z)| - \frac{(n-s)}{k^s(1+k^{\mu})}\,\min_{|z|=k}|p(z)|\end{aligned}$$

which is the relation (9).

In particular, Theorem 5 is an extension of Theorem 4.

REMARK 2. It is not shown in [7] that the upper bound (9) in Theorem 4 is best possible. Next, we give an example to show that the bound is best possible.

Consider the polynomial $p(z) = z^s(z+k)^{n-s}$ where k is a real number with $k \geqslant 1$. Since $p'(z) = z^s(n-s)(z+k)^{n-s-1} + (z+k)^{n-s}sz^{s-1}$, we have

$$\begin{aligned}\max_{|z|=1}|p'(z)| &\leqslant (n-s)\max_{|z|=1}|z|^s|z+k|^{n-s-1} + s\max_{|z|=1}|z+k|^{n-s}|z|^{s-1}\\ &= (n+sk)(1+k)^{n-s-1}\end{aligned} \tag{15}$$

and $\max_{|z|=1}|p(z)| = \max_{|z|=1}|z|^s|z+k|^{n-s} = \max_{|z|=1}|z+k|^{n-s} = (1+k)^{n-s}$.

The right-hand side of the relation (9) is

$$\frac{n+sk^\mu}{1+k^\mu}\max_{|z|=1}|p(z)| - \frac{(n-s)}{k^s(1+k^\mu)}\min_{|z|=k}|p(z)| = \frac{n+sk}{1+k}(1+k)^{n-s} = (n+sk)(1+k)^{n-s-1}$$

which is equal to $\max_{|z|=1}|p'(z)|$ in (15).

Thus, the bound in Theorem 4 is best possible.

COROLLARY 1. *If* $p(z) = (z-z_1)^{t_1}(z-z_0)^{t_0}\left(a_0 + \sum_{\nu=\mu}^{n-(t_1+t_0)} a_\nu z^\nu\right)$, $1 \leqslant \mu \leqslant n-(t_1+t_0)$, $0 \leqslant t_1+t_0 \leqslant n-1$, *is a polynomial of degree* n *having zeros* z_0, z_1 *with* $|z_0| < 1$, $|z_1| < 1$ *and the remaining* $n-(t_1+t_0)$ *zeros are outside* $D(0,k)$, $k \geqslant 1$, *then*

$$\begin{aligned}\max_{|z|=1}|p'(z)| \leqslant \Bigg[&\frac{t_1(1+|z_1|)^{t_1-1}}{(1-|z_1|)^{t_1}} + \frac{(1+|z_1|)^{t_1}t_0}{(1-|z_0|)(1-|z_1|)^{t_1}}\\ &+ \frac{(1+|z_1|)^{t_1}A}{(1-|z_0|)^{t_0}(1-|z_1|)^{t_1}}\Bigg]\max_{|z|=1}|p(z)|\\ &- \left[\frac{(1+|z_1|)^{t_1}A}{(k+|z_0|)^{t_0}(k+|z_1|)^{t_1}}\right]\min_{|z|=k}|p(z)|,\end{aligned}$$

where $A = \dfrac{(1+|z_0|)^{t_0+1}(n-(t_0+t_1))}{(1+k^\mu)(1-|z_0|)}$.

Proof. Let $p_0(z) = (z-z_0)^{t_0}\left(a_0 + \sum_{\nu=\mu}^{n-(t_0+t_1)} a_\nu z^\nu\right)$.

Then $p(z) = (z-z_1)^{t_1}p_0(z)$ and $p'(z) = (z-z_1)^{t_1}p_0'(z) + t_1(z-z_1)^{t_1-1}p_0(z)$. Theorem 5 implies that

$$\begin{aligned}\max_{|z|=1}|p'(z)| \leqslant &\left[t_1(1+|z_1|)^{t_1-1} + \frac{(1+|z_1|)^{t_1}t_0}{(1-|z_0|)} + \frac{(1+|z_1|)^{t_1}A}{(1-|z_0|)^{t_0}}\right]\max_{|z|=1}|p_0(z)|\\ &- \frac{(1+|z_1|)^{t_1}A}{(k+|z_0|)^{t_0}}\min_{|z|=k}|p_0(z)|,\end{aligned}$$

where $A = \dfrac{(1+|z_0|)^{t_0+1}(n-(t_0+t_1))}{(1+k^{\mu})(1-|z_0|)}$.

Since

$$\max_{|z|=1}|p_0(z)| = \max_{|z|=1}\left(\frac{1}{|z-z_1|^{t_1}}\;|p(z)|\right) \leqslant \frac{1}{(1-|z_1|)^{t_1}}\;\max_{|z|=1}|p(z)|$$

and

$$\min_{|z|=k}|p_0(z)| = \min_{|z|=k}\left(\frac{1}{|z-z_1|^{t_1}}\;|p(z)|\right) \geqslant \frac{1}{(k+|z_1|)^{t_1}}\;\min_{|z|=k}|p(z)|,$$

we obtain that

$$\begin{aligned}\max_{|z|=1}|p'(z)| \leqslant \Bigg[&\frac{t_1(1+|z_1|)^{t_1-1}}{(1-|z_1|)^{t_1}} + \frac{(1+|z_1|)^{t_1}t_0}{(1-|z_0|)(1-|z_1|)^{t_1}} \\ &+ \frac{(1+|z_1|)^{t_1}A}{(1-|z_0|)^{t_0}(1-|z_1|)^{t_1}}\Bigg]\max_{|z|=1}|p(z)| \\ &- \left[\frac{(1+|z_1|)^{t_1}A}{(k+|z_0|)^{t_0}(k+|z_1|)^{t_1}}\right]\min_{|z|=k}|p(z)|. \qquad \square\end{aligned}$$

REMARK 3. By using Theorem 5, we can obtain an upper bound of $\max\limits_{|z|=1}|p'(z)|$ for a polynomial

$$p(z) = (z-z_m)^{t_m}(z-z_{m-1})^{t_{m-1}}\cdots(z-z_0)^{t_0}\left(a_0 + \sum_{\nu=\mu}^{n-(t_m+\cdots+t_0)} a_\nu z^\nu\right)$$

of degree n having zeros $z_0,\ldots,z_m$ with $|z_j|<1$ for $0\leqslant j\leqslant m$ and the remaining $n-(t_m+\cdots+t_0)$ zeros are outside $D(0,k), k\geqslant 1$.

Let $p_0(z) = (z-z_0)^{t_0}\left(a_0 + \sum\limits_{\nu=\mu}^{n-(t_m+\cdots+t_0)} a_\nu z^\nu\right)$ and $p_j(z) = (z-z_j)^{t_j}p_{j-1}(z)$ for $1\leqslant j\leqslant m$. We obtain an upper bound of $\max_{|z|=1}|p_0'(z)|$ by Theorem 5.

By substitution this upper bound with the facts that

$$\max_{|z|=1}|p_0(z)| \leqslant \frac{1}{(1-|z_1|)^{t_1}}\;\max_{|z|=1}|p_1(z)| \quad\text{and}\quad \min_{|z|=k}|p_0(z)| \geqslant \frac{1}{(k+|z_1|)^{t_1}}\;\min_{|z|=k}|p_1(z)|,$$

we obtain an upper bound of $\max_{|z|=1}|p_1'(z)|$ as in Corollary 1.

Next, we can find an upper bound of $\max_{|z|=1}|p_j'(z)|$ for $1\leqslant j\leqslant m$ by similar process with using an upper bound of $\max_{|z|=1}|p_{j-1}'(z)|$ from the previous process and the facts that $\max\limits_{|z|=1}|p_{j-1}(z)| \leqslant \dfrac{1}{(1-|z_j|)^{t_j}}\;\max\limits_{|z|=1}|p_j(z)|$ and $\min\limits_{|z|=k}|p_{j-1}(z)| \geqslant \dfrac{1}{(k+|z_j|)^{t_j}}\;\min\limits_{|z|=k}|p_j(z)|$ for $1\leqslant j\leqslant m$.

THEOREM 6. *If* $p(z) = (z - z_0)^s \left(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu\right)$, $1 \leqslant \mu \leqslant n-s$, $0 \leqslant s \leqslant n-1$, *is a polynomial of degree* n *having zero* z_0 *with* $|z_0| < 1$ *and the remaining* $n-s$ *zeros are on* $C(0,k)$, $k \geqslant 1$, *then*

$$\max_{|z|=1} |p'(z)| \leqslant \left[\frac{s}{(1-|z_0|)} + \frac{(1+|z_0|)^{s+1}(n-s)}{(k^{n-s-2\mu+1}+k^{n-s-\mu+1})(1-|z_0|)^{s+1}}\right] \max_{|z|=1} |p(z)|.$$

Proof. Let $p(z) = (z-z_0)^s \phi(z)$ where $\phi(z) = a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu$ is a polynomial of degree $n-s$ having all its zeros on $C(0,k)$, $k \geqslant 1$.

Applying the relation (8) to $\phi(z)$, we obtain that

$$\max_{|z|=1} |\phi'(z)| \leqslant \frac{n-s}{(k^{n-s-2\mu+1}+k^{n-s-\mu+1})} \max_{|z|=1} |\phi(z)|. \tag{16}$$

As in the proof of Theorem 5, one can show that

$$(1-|z_0|) \max_{|z|=1} |p'(z)| \leqslant s \max_{|z|=1} |p(z)| + (1+|z_0|)^{s+1} \max_{|z|=1} |\phi'(z)|. \tag{17}$$

Applying the relation (16) in the inequality (17), we have

$$(1-|z_0|) \max_{|z|=1} |p'(z)| \leqslant s \max_{|z|=1} |p(z)| + \frac{(1+|z_0|)^{s+1}(n-s)}{(k^{n-s-2\mu+1}+k^{n-s-\mu+1})} \max_{|z|=1} |\phi(z)|.$$

Using the fact that $\max_{|z|=1} |\phi(z)| \leqslant \frac{1}{(1-|z_0|)^s} \max_{|z|=1} |p(z)|$, we get

$$(1-|z_0|) \max_{|z|=1} |p'(z)| \leqslant \left[s + \frac{(1+|z_0|)^{s+1}(n-s)}{(k^{n-s-2\mu+1}+k^{n-s-\mu+1})(1-|z_0|)^s}\right] \max_{|z|=1} |p(z)|.$$

Therefore,

$$\max_{|z|=1} |p'(z)| \leqslant \left[\frac{s}{1-|z_0|} + \frac{(1+|z_0|)^{s+1}(n-s)}{(k^{n-s-2\mu+1}+k^{n-s-\mu+1})(1-|z_0|)^{s+1}}\right] \max_{|z|=1} |p(z)|. \quad \square$$

3. Conclusion

This paper gives an upper bound of a derivative for polynomials

$$p(z) = (z-z_m)^{t_m}(z-z_{m-1})^{t_{m-1}} \cdots (z-z_0)^{t_0} \left(a_0 + \sum_{\nu=\mu}^{n-(t_m+\cdots+t_0)} a_\nu z^\nu\right)$$

of degree n having zeros $z_0, \ldots, z_m$ with $|z_j| < 1$ for $0 \leqslant j \leqslant m$ and the remaining $n-(t_m+\cdots+t_0)$ zeros are outside $\{z : |z| < k\}$, $k \geqslant 1$.

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