AN UPPER BOUND OF A DERIVATIVE
FOR SOME CLASS OF POLYNOMIALS

KEAITSUDA MANEERUK NAKPRASIT AND JIRAPHORN SOMSUWAN

(Communicated by J. Pečarić)

Abstract. In [S. Kumar and R. Lal, Generalizations of some polynomial inequalities, Int. Electron. J. Pure Appl. Math., 3, 2 (2011), 111–117.], Kumar and Lal provided an upper bound of a derivative for polynomial degree $n$ having some of zeros at the origin and rest of zeros lying on or outside the boundary of a prescribed disk. In this paper, we present an upper bound of a derivative for polynomials $p(z) = (z - z_m)^{t_m}(z - z_{m-1})^{t_{m-1}} \cdots (z - z_0)^{t_0} \left( a_0 + \sum_{\nu=\mu}^{n - (t_m + \cdots + t_0)} a_\nu z^\nu \right)$ of degree $n$ having zeros $z_0, \ldots, z_m$ with $|z_j| < 1$ for $0 \leq j \leq m$ and the remaining $n - (t_m + \cdots + t_0)$ zeros are outside $\{z : |z| < k\}$ where $k \geq 1$.

1. Introduction

Let $p(z)$ be a polynomial of degree $n$. Then we have the Bernstein’s inequality (see [2])

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|.$$

Equality holds in (1) if and only if $p(z)$ has all of its zeros at the origin.

For a positive real number $k$, we let $D(0,k)$ and $C(0,k)$ denote the sets $\{z : |z| < k\}$ and $\{z : |z| = k\}$, respectively.

If we restrict ourselves to the class of polynomials having no zero in $D(0,1)$, the inequality (1) can be sharpened. In fact, it was conjectured by Erdős and later proved by Lax [8] that if $p(z)$ has no zero in $D(0,1)$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|.$$

The equality holds if all zeros of $p(z)$ lie on $C(0,1)$, for example, $p(z) = \alpha + \beta z^n$, $|\alpha| = |\beta|$.

Aziz and Dawood [1] improved the inequality (2) under the same hypothesis and obtained that

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \left[ \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right].$$


Keywords and phrases: Polynomial, derivative, inequality.
Equality in (3) holds for $p(z) = \beta + \alpha z^n$, $|\beta| \geq |\alpha|$.

For the class of polynomials $p(z)$ of degree $n$ having no zero in $D(0, k)$, $k \geq 1$, Malik [9] proved that

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k} \max_{|z|=1} |p(z)|.$$  \hfill (4)

Inequality (4) was further improved by Govil [6] under the same hypothesis as

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k} \left[ \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right].$$  \hfill (5)

Inequalities (4) and (5) are sharp and extremal polynomial is $p(z) = (z + k)^n$.

Chan and Malik [3] considered the class of polynomials as in [9] and obtained the following theorem was proved by Pukhta [10], which is an improvement of Theorem 1 and a generalization of the inequality (5).

**Theorem 1.** [3] If $p(z) = a_0 + \sum_{v=\mu}^n a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having no zero in $D(0, k)$, $k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^\mu} \max_{|z|=1} |p(z)|.$$  \hfill (6)

The result is best possible and extremal polynomial is $p(z) = (z^\mu + k^\mu)^\frac{n}{\mu}$, where $n$ is a multiple of $\mu$.

The following theorem was proved by Pukhta [10], which is an improvement of Theorem 1 and a generalization of the inequality (5).

**Theorem 2.** [10] If $p(z) = a_0 + \sum_{v=\mu}^n a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having no zero in $D(0, k)$, $k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^\mu} \left[ \max_{|z|=1} |p(z)| - \min_{|z|=k} |p(z)| \right].$$  \hfill (7)

The result is best possible and extremal polynomial is $p(z) = (z^\mu + k^\mu)^\frac{n}{\mu}$, where $n$ is a multiple of $\mu$.

For polynomials having all its zeros on $C(0, k)$, $k \leq 1$, Govil [5] proved that

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{k^n + k^{n-1}} \max_{|z|=1} |p(z)|.$$  \hfill (8)

Dewan and Hans [4] generalized the inequality (7) for the polynomials of the type $p(z) = c_n z^n + \sum_{v=\mu}^n c_{n-v} z^{n-v}$, $1 \leq \mu \leq n$ and proved the following theorem.

**Theorem 3.** [4] If $p(z) = a_0 + \sum_{v=\mu}^n a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having all its zeros on $C(0, k)$, $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{k^{n-2\mu+1} + k^{n-\mu+1}} \max_{|z|=1} |p(z)|.$$  \hfill (9)
Kumar and Lal [7] investigated the polynomials of degree \( n \) having some zeros at the origin and the rest of zeros lying on or outside the boundary of a prescribed disk. They obtained a generalized results of some well-known results.

**Theorem 5. Main** If \( p(z) = a_0 + \sum_{v=1}^{n-s} a_v z^v \), \( 1 \leq \mu \leq n-s \), \( 0 \leq s \leq n-1 \), is a polynomial of degree \( n \) having zeros of order \( s \) at the origin and the remaining \( n-s \) zeros are outside \( D(0,k), \ k \geq 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n+sk^\mu}{1+k^\mu} \max_{|z|=1} |p(z)| - \frac{(n-s)}{ks(1+k^\mu)} \min_{|z|=k} |p(z)|.
\] (9)

2. Main results

**Theorem 4.** [7] If \( p(z) = (z-z_0)^s \left( a_0 + \sum_{v=s+1}^{n-s} a_v z^v \right) \), \( 1 \leq \mu \leq n-s \), \( 0 \leq s \leq n-1 \), is a polynomial of degree \( n \) having zero of order \( s \) at \( z_0 \) with \( |z_0| < 1 \) and the remaining \( n-s \) zeros are outside \( D(0,k), \ k \geq 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{s}{(1-|z_0|)} + \frac{A}{(1-|z_0|)^s} \max_{|z|=1} |p(z)| - \frac{A}{(k+|z_0|)^s} \min_{|z|=k} |p(z)|,
\]

where \( A = \frac{(1+|z_0|)^{s+1}(n-s)}{(1+k^\mu)(1-|z_0|)} \).

**Proof.** Let \( p(z) = (z-z_0)^s \phi(z) \) where \( \phi(z) = a_0 + \sum_{v=s+1}^{n-s} a_v z^v \) be a polynomial of degree \( n-s \) having no zero in \( D(0,k), \ k \geq 1 \).

Then \( p'(z) = (z-z_0)^s \phi'(z) + s(z-z_0)^{s-1} \phi(z) \) and \( (z-z_0)p'(z) = sp(z) + (z-z_0)^{s+1} \phi(z) \).

Therefore, \( \max_{|z|=1} |z-z_0||p'(z)| \leq s \max_{|z|=1} |p(z)| + \max_{|z|=1} |z-z_0|^{s+1} |\phi'(z)|. \)

Since \( |z-z_0| \geq |z| - |z_0| = 1 - |z_0| \) and \( |z-z_0| \leq |z| + |z_0| = 1 + |z_0| \) for \( |z| = 1 \), we obtain \( (1-|z_0|) \max_{|z|=1} |p'(z)| \leq s \max_{|z|=1} |p(z)| + (1 + |z_0|)^{s+1} \max_{|z|=1} |\phi'(z)|. \)

Now let \( m = \min_{|z|=k} |\phi(z)|. \) Then \( m \leq |\phi(z)| \) for \( |z| = k \).

If \( \phi(z) \) has a zero on \( C(0,k) \), then \( m = 0 \). From now on, we assume that all \( n-s \) zeros of \( \phi(z) \) lie outside \( D(0,k), \ k \geq 1 \). Therefore, for every complex number \( \alpha \) such that \( |\alpha| < 1 \), it follows from Rouche’s Theorem that all zeros of the polynomial \( \phi(z) - \alpha m \) of degree \( n-s \) lie outside \( D(0,k), \ k \geq 1 \).

Applying the relation (6) to the polynomial \( \phi(z) - \alpha m \), we get

\[
\max_{|z|=1} |\phi'(z)| \leq \frac{n-s}{1+k^\mu} \max_{|z|=1} |\phi(z) - \alpha m|.
\] (10)

Now choosing \( \alpha \) such that

\[
|\phi(z) - \alpha m| = |\phi(z)| - |\alpha|m
\] (11)

and letting \( |\alpha| \to 1 \), we get from (10) in view of (11) that

\[
\max_{|z|=1} |\phi'(z)| \leq \frac{n-s}{1+k^\mu} \max(|\phi(z)| - m).
\] (12)
Combining the relation (11) and the relation (12), we obtain that
\[
(1 - |z_0|) \max_{|z|=1} |p'(z)| \leq s \max_{|z|=1} |p(z)| + \left[ (1 + |z_0|)^{s+1} \frac{n-s}{1+k\mu} \right] \max_{|z|=1} |\phi(z)| - \left[ (1 + |z_0|)^{s+1} \frac{n-s}{1+k\mu} \right] m. \tag{13}
\]

The relation between \( \phi(z) \) and \( p(z) \) implies that
\[
\max_{|z|=1} |\phi(z)| = \max_{|z|=1} \left[ \frac{1}{|z-z_0|^s} |p(z)| \right] \leq \frac{1}{(1 - |z_0|)^s} \max_{|z|=1} |p(z)|. \]

Applying this relation in the relation (13), we have
\[
(1 - |z_0|) \max_{|z|=1} |p'(z)| \leq \left[ s + \frac{(1 + |z_0|)^{s+1}(n-s)}{(1 + k\mu)(1 - |z_0|)^s} \right] \max_{|z|=1} |p(z)| - \left[ \frac{(n-s)(1 + |z_0|)^{s+1}}{1 + k\mu} \right] m. \tag{14}
\]

Again, the relation between \( \phi(z) \) and \( p(z) \) yields
\[
m = \min_{|z|=k} |\phi(z)| = \min_{|z|=k} \left[ \frac{1}{|z-z_0|^s} |p(z)| \right] \geq \frac{1}{(k + |z_0|)^s} \min_{|z|=k} |p(z)|.
\]

Applying this relation in the relation (14), we have
\[
(1 - |z_0|) \max_{|z|=1} |p'(z)| \leq \left[ s + \frac{(1 + |z_0|)^{s+1}(n-s)}{(1 + k\mu)(1 - |z_0|)^s} \right] \max_{|z|=1} |p(z)| - \left[ \frac{(n-s)(1 + |z_0|)^{s+1}}{(1 + k\mu)(k + |z_0|)^s} \right] \min_{|z|=k} |p(z)|.
\]

Consequently,
\[
\max_{|z|=1} |p'(z)| \leq \left[ \frac{s}{(1 - |z_0|)} + \frac{A}{(1 - |z_0|)^s} \right] \max_{|z|=1} |p(z)| - \frac{A}{(k + |z_0|)^s} \min_{|z|=k} |p(z)|,
\]
where
\[
A = \frac{(1 + |z_0|)^{s+1}(n-s)}{(1 + k\mu)(1 - |z_0|)}.
\]

\[\square\]

**Remark 1.** By letting \( z_0 = 0 \) in Theorem 5, we get \( A = \frac{n-s}{1+k\mu} \) and
\[
\max_{|z|=1} |p'(z)| \leq (s + A) \max_{|z|=1} |p(z)| - \frac{A}{k^s} \min_{|z|=k} |p(z)|
\]
\[
= \frac{n+s k\mu}{(1 + k\mu)} \max_{|z|=1} |p(z)| - \frac{(n-s)}{k^s (1 + k\mu)} \min_{|z|=k} |p(z)|
\]

which is the relation (9).

In particular, Theorem 5 is an extension of Theorem 4.
REMARK 2. It is not shown in [7] that the upper bound (9) in Theorem 4 is best possible. Next, we give an example to show that the bound is best possible.

Consider the polynomial \( p(z) = z^s(z + k)^{n-s} \) where \( k \) is a real number with \( k \geq 1 \). Since \( p'(z) = z^{n-s} \cdot \frac{d}{dz} (z + k)^{n-s} = z^{n-s} \cdot (n-s)(z + k)^{n-s-1} \), we have

\[
\max_{|z|=1} |p'(z)| \leq (n-s) \max_{|z|=1} |z|^s |z + k|^{n-s-1} + s \max_{|z|=1} |z + k|^{n-s} |z|^{s-1} = (n + sk)(1 + k)^{n-s-1}
\]

and

\[
\max_{|z|=1} |p(z)| = \max_{|z|=1} |z|^s |z + k|^{n-s} = \max_{|z|=1} |z + k|^{n-s} = (1 + k)^{n-s}.
\]

The right-hand side of the relation (9) is

\[
\frac{n + sk^\mu}{1 + k^\mu} \max_{|z|=1} |p(z)| - \frac{(n-s)}{k^\mu (1 + k^\mu)} \min_{|z|=k} |p(z)| = \frac{n + sk}{1 + k} (1 + k)^{n-s} = (n + sk)(1 + k)^{n-s-1}
\]

which is equal to \( \max_{|z|=1} |p'(z)| \) in (15).

Thus, the bound in Theorem 4 is best possible.

COROLLARY 1. If \( p(z) = (z - z_1)^{t_1} (z - z_0)^{t_0} \left( a_0 + \sum_{\nu=\mu}^{n-(t_0+t_1)} a_{\nu} z^\nu \right) \), \( 1 \leq \mu \leq n - (t_1 + t_0) \), \( 0 \leq t_1 + t_0 \leq n - 1 \), is a polynomial of degree \( n \) having zeros \( z_0, z_1 \) with \( |z_0| < 1, \ |z_1| < 1 \) and the remaining \( n - (t_1 + t_0) \) zeros are outside \( D(0,k), k \geq 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \left[ \frac{t_1 (1 + |z_1|)^{t_1-1}}{(1 - |z_1|)^{t_1}} + \frac{(1 + |z_1|)^{t_1} t_0}{(1 - |z_0|)(1 - |z_1|)^{t_1}} \right] \max_{|z|=1} |p(z)|
\]

\[\begin{align*}
&+ \left[ \frac{(1 + |z_1|)^{t_1} A}{(1 - |z_0|)^{t_0} (1 - |z_1|)^{t_1}} \right] \min_{|z|=k} |p(z)|, \\
&- \left[ \frac{(1 + |z_1|)^{t_1} A}{(k + |z_0|)^{t_0} (k + |z_1|)^{t_1}} \right] \min_{|z|=k} |p(z)|,
\end{align*}\]

where \( A = \frac{(1 + |z_0|)^{t_0+1} (n - (t_0 + t_1))}{(1 + k^\mu)(1 - |z_0|)} \).

Proof. Let \( p_0(z) = (z - z_0)^{t_0} \left( a_0 + \sum_{\nu=\mu}^{n-(t_0+t_1)} a_{\nu} z^\nu \right) \).

Then \( p(z) = (z - z_1)^{t_1} p_0(z) \) and \( p'(z) = (z - z_1)^{t_1} p'_0(z) + t_1 (z - z_1)^{t_1-1} p_0(z) \).

Theorem 5 implies that

\[
\max_{|z|=1} |p'(z)| \leq \left[ t_1 (1 + |z_1|)^{t_1-1} + \frac{(1 + |z_1|)^{t_1} t_0}{(1 - |z_0|)} + \frac{(1 + |z_1|)^{t_1} A}{(1 - |z_0|)^{t_0}} \right] \max_{|z|=1} |p_0(z)|
\]

\[\begin{align*}
&- \left[ \frac{(1 + |z_1|)^{t_1} A}{(k + |z_0|)^{t_0} (k + |z_1|)^{t_1}} \right] \min_{|z|=k} |p_0(z)|, \\
&- \left[ \frac{(1 + |z_1|)^{t_1} A}{(k + |z_0|)^{t_0} (k + |z_1|)^{t_1}} \right] \min_{|z|=k} |p_0(z)|,
\end{align*}\]
where \( A = \frac{(1 + |z_0|)^{\gamma_0} + 1 (n - (t_0 + t_1))}{(1 + k^{\mu})(1 - |z_0|)} \).

Since
\[
\max_{|z| = 1} |p_0(z)| = \max_{|z| = 1} \left( \frac{1}{|z - z_1|^{\gamma_1}} |p(z)| \right) \leq \frac{1}{(1 - |z_1|)^{\gamma_1}} \max_{|z| = 1} |p(z)|
\]
and
\[
\min_{|z| = k} |p_0(z)| = \min_{|z| = k} \left( \frac{1}{|z - z_1|^{\gamma_1}} |p(z)| \right) \geq \frac{1}{(k + |z_1|)^{\gamma_1}} \min_{|z| = k} |p(z)|,
\]
we obtain that
\[
\max_{|z| = 1} |p'(z)| \leq \left[ \frac{t_1 (1 + |z_1|)^{\gamma_1 - 1}}{(1 - |z_1|)^{\gamma_1}} + \frac{(1 + |z_1|)^{\gamma_1} t_0}{(1 - |z_0|)(1 - |z_1|)^{\gamma_1}} \right] \max_{|z| = 1} |p(z)|
\]
\[
- \left[ \frac{(1 + |z_1|)^{\gamma_1} A}{(k + |z_0|)^{\gamma_0} (k + |z_1|)^{\gamma_1}} \right] \min_{|z| = k} |p(z)|. \quad \Box
\]

**Remark 3.** By using Theorem 5, we can obtain an upper bound of \( \max_{|z| = 1} |p'(z)| \) for a polynomial
\[
p(z) = (z - z_m)^m (z - z_{m-1})^{m-1} \cdots (z - z_0)^0 \left( a_0 + \sum_{v=\mu}^{n-(t_m+\cdots+t_0)} a_v z^v \right)
\]
of degree \( n \) having zeros \( z_0, \ldots, z_m \) with \( |z_j| < 1 \) for \( 0 \leq j \leq m \) and the remaining \( n - (t_m + \cdots + t_0) \) zeros are outside \( D(0,k), k \geq 1 \).

Let \( p_0(z) = (z - z_0)^{\gamma_0} \left( a_0 + \sum_{v=\mu}^{n-(t_m+\cdots+t_0)} a_v z^v \right) \) and \( p_j(z) = (z - z_j)^{\gamma_j} p_{j-1}(z) \) for \( 1 \leq j \leq m \). We obtain an upper bound of \( \max_{|z| = 1} |p_0(z)| \) by Theorem 5.

By substitution this upper bound with the facts that
\[
\max_{|z| = 1} |p_0(z)| \leq \frac{1}{(1 - |z_1|)^{\gamma_1}} \max_{|z| = 1} |p_1(z)| \quad \text{and} \quad \min_{|z| = k} |p_0(z)| \geq \frac{1}{(k + |z_1|)^{\gamma_1}} \min_{|z| = k} |p_1(z)|,
\]
we obtain an upper bound of \( \max_{|z| = 1} |p_j'(z)| \) as in Corollary 1.

Next, we can find an upper bound of \( \max_{|z| = 1} |p_j'(z)| \) for \( 1 \leq j \leq m \) by similar process with using an upper bound of \( \max_{|z| = 1} |p_{j-1}(z)| \) from the previous process and the facts that \( \max_{|z| = 1} |p_{j-1}(z)| \leq \frac{1}{(1 - |z_j|)^{\gamma_j}} \max_{|z| = 1} |p_j(z)| \) and \( \min_{|z| = k} |p_{j-1}(z)| \geq \frac{1}{(k + |z_j|)^{\gamma_j}} \min_{|z| = k} |p_j(z)| \) for \( 1 \leq j \leq m \).
Theorem 6. If \( p(z) = (z - z_0)^s (a_0 + \sum_{v=\mu}^{n-s} a_v z^v) \), \( 1 \leq \mu \leq n - s \), \( 0 \leq s \leq n - 1 \), is a polynomial of degree \( n \) having zero \( z_0 \) with \( |z_0| < 1 \) and the remaining \( n - s \) zeros are on \( C(0, k) \), \( k \geq 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \left[ \frac{s}{1 - |z_0|} + \frac{(1 + |z_0|)^{s+1}(n-s)}{(kn-s-2\mu+1 + kn-s-\mu+1)(1 - |z_0|)^s+1} \right] \max_{|z|=1} |p(z)|.
\]

Proof. Let \( p(z) = (z - z_0)^s \phi(z) \) where \( \phi(z) = a_0 + \sum_{v=\mu}^{n-s} a_v z^v \) is a polynomial of degree \( n - s \) having all its zeros on \( C(0, k) \), \( k \geq 1 \).

Applying the relation (8) to \( \phi(z) \), we obtain that

\[
\max_{|z|=1} |\phi'(z)| \leq \frac{n-s}{(kn-s-2\mu+1 + kn-s-\mu+1)} \max_{|z|=1} |\phi(z)| .
\]  \tag{16}

As in the proof of Theorem 5, one can show that

\[
(1 - |z_0|) \max_{|z|=1} |p'(z)| \leq s \max_{|z|=1} |p(z)| + (1 + |z_0|)^{s+1} \max_{|z|=1} |\phi'(z)| .
\]  \tag{17}

Applying the relation (16) in the inequality (17), we have

\[
(1 - |z_0|) \max_{|z|=1} |p'(z)| \leq s \max_{|z|=1} |p(z)| + \frac{(1 + |z_0|)^{s+1}(n-s)}{(kn-s-2\mu+1 + kn-s-\mu+1)} \max_{|z|=1} |\phi(z)|.
\]

Using the fact that \( \max_{|z|=1} |\phi(z)| \leq \frac{1}{(1 - |z_0|)^s} \max_{|z|=1} |p(z)| \), we get

\[
(1 - |z_0|) \max_{|z|=1} |p'(z)| \leq \left[ s + \frac{(1 + |z_0|)^{s+1}(n-s)}{(kn-s-2\mu+1 + kn-s-\mu+1)(1 - |z_0|)^s} \right] \max_{|z|=1} |p(z)| .
\]

Therefore,

\[
\max_{|z|=1} |p'(z)| \leq \left[ \frac{s}{1 - |z_0|} + \frac{(1 + |z_0|)^{s+1}(n-s)}{(kn-s-2\mu+1 + kn-s-\mu+1)(1 - |z_0|)^s+1} \right] \max_{|z|=1} |p(z)| . \quad \square
\]

3. Conclusion

This paper gives an upper bound of a derivative for polynomials

\[
p(z) = (z - z_m)^{t_m}(z - z_{m-1})^{t_{m-1}} \cdots (z - z_0)^{t_0} \left( a_0 + \sum_{v=\mu}^{n-(t_m+\cdots+t_0)} a_v z^v \right)
\]

of degree \( n \) having zeros \( z_0, \ldots, z_m \) with \( |z_j| < 1 \) for \( 0 \leq j \leq m \) and the remaining \( n - (t_m + \cdots + t_0) \) zeros are outside \( \{ z : |z| < k \} \), \( k \geq 1 \).

Acknowledgements. The first author is supported by National Research Council of Thailand and Khon Kaen University, Thailand (Grant number: kku fmis (580010)). The second author is supported in part by Development and Promotion of Science and Talents Project (DPST).
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(Received December 17, 2015)

Keaitsuda Maneeruk Nakprasit
Department of Mathematics, Faculty of Science
Khon Kaen University
Khon Kaen, 40002, Thailand
e-mail: kmaneeruk@hotmail.com

Jiraphorn Somsuwan
Department of Mathematics, Faculty of Science
Khon Kaen University
Khon Kaen, 40002, Thailand
e-mail: jira.somsu@hotmail.com