Journal of Mathematical Inequalities
Volume 11, Number 1 (2017), 163–166 doi:10.7153/jmi-11-16

A NOTE ON INEQUALITIES DUE TO CLAUSING AND LEVIN–STEČKIN

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(*Communicated by Josip Pečarić*)

Abstract. A single argument yields inequalities due to Clausing and Levin-Stečkin.

The following pair of inequalities appears in Section 4.1(b) of A. Clausing's paper [1].

THEOREM 1. *Let* ϕ *be continuous on* $[0,1]$ *and increasing on* $[0,1/2]$, *with* $\phi(x) = \phi(1-x)$ *for* $x \in [0,1]$. *Then for* f *concave and positive on* $(0,1)$, *we have*

$$\int_0^1 f(x)dx \int_0^1 \phi(x)dx \leqslant \int_0^1 f(x)\phi(x)dx \leqslant \int_0^1 f(x)dx \int_0^1 \widehat{\phi}(x)dx,$$

where

$$\widehat{\phi}(x) = 4\min\{x, 1-x\}\,\phi(x).$$

The right-hand side is new in [1] and is proved there, using some rather heavy machinery. The left-hand side had already been obtained by Levin and Stečkin [2, 4, 5]; no proof is given in [1]. In the left-hand side, f can be replaced with $f+C$, so evidently f need not be positive.

In this note we offer a simple argument which yields both of these inequalities together.

Proof of Theorem 1. Let

$$C = \left\{ f \in \mathscr{C}(0,1) : \ f > 0, f \text{ is concave, and } \int_0^1 f(x)dx \leqslant 1 \right\}.$$

Then C is a compact convex subset of $\mathscr{C}(0,1)$, with local uniform convergence. Therefore, by Bauer's Maximum Principle, the linear functional on C given by

$$f \mapsto \int_0^1 f(x)\phi(x)dx$$

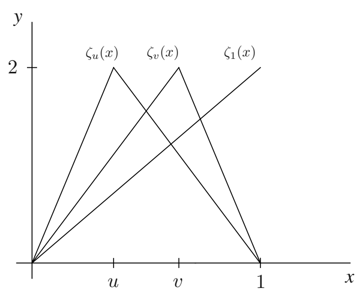
Mathematics subject classification (2010): 26D15.
Keywords and phrases: convexity; Levin-Steckin.

attains its maximum and minimum values at extreme points of C. As shown in [3], the extreme points of C are the functions

$$\zeta_u(x) = \begin{cases} 2(1-x) & \text{if } u = 0 \\ 2\min\left\{\dfrac{x}{u}, \dfrac{1-x}{1-u}\right\} & \text{if } u \in (0,1) \\ 2x & \text{if } u = 1. \end{cases}$$

□

Each of these functions ζ_u has $\int_0^1 \zeta_u = 1$. Here, for example, are three of their graphs:



So we consider the integrals

$$I(u) = \int_0^1 \zeta_u(x)\phi(x)dx, \quad \text{over } u \in [0,1].$$

Since $I(u) = I(1-u)$, it suffices to consider $u \in [0,1/2]$. We claim that that I is increasing there. This would be enough: For $\int_0^1 f = 1$, we would have

$$I(0) \leqslant \int_0^1 f(x)\phi(x)dx \leqslant I(1/2).$$

Then by the symmetry of ϕ,

$$I(0) = I(1) = \int_0^1 \zeta_1(x)\phi(x)dx = \int_0^1 2x\phi(x)dx = \int_0^1 \phi(x)dx$$

and

$$I(1/2) = \int_0^1 \zeta_{1/2}(x)\phi(x)dx = 4\int_0^1 \min\{x, 1-x\}\,\phi(x)dx,$$

so the proof would be complete.

To verify the claim, we compute $I'(u)$ then show that $I'(u) \geqslant 0$ on $[0,1/2]$. We have

$$\begin{aligned}\frac{I(u)-I(v)}{u-v} &= \frac{2}{u-v}\int_0^1 \left(\min\left\{\frac{x}{u},\frac{1-x}{1-u}\right\} - \min\left\{\frac{x}{v},\frac{1-x}{1-v}\right\}\right)\phi(x)dx \\ &= \frac{2}{u-v}\int_0^u \frac{x(v-u)}{uv}\phi(x)dx + \frac{2}{u-v}\int_u^v \frac{v-x-x(v-u)}{v(1-u)}\phi(x)dx \\ &\quad + \frac{2}{u-v}\int_v^1 \frac{x(v-u)+(u-v)}{(1-u)(1-v)}\phi(x)dx.\end{aligned}$$

Letting $v \to u$ (and using the Mean Value Theorem for Integrals on the second integral), we get

$$I'(u) = 2\int_0^u \frac{-x}{u^2}\phi(x)dx + 2\int_u^1 \frac{-x+1}{(1-u)^2}\phi(x)dx.$$

Now we show that $I'(u) \geqslant 0$ on $[0,1/2]$. Since $\phi(x) = \phi(1-x)$, we have

$$2\int_u^1 \frac{-x+1}{(1-u)^2}\phi(x)dx = 2\int_0^{1-u} \frac{x}{(1-u)^2}\phi(x)dx,$$

so we show that

$$2\int_0^u \frac{x}{u^2}\phi(x)dx \leqslant 2\int_0^{1-u} \frac{x}{(1-u)^2}\phi(x)dx.$$

This reads

$$\frac{\int_0^u x\phi(x)dx}{\int_0^u xdx} \leqslant \frac{\int_0^{1-u} x\phi(x)dx}{\int_0^{1-u} xdx} = \frac{\int_0^u x\phi(x)dx + \int_u^{1-u} x\phi(x)dx}{\int_0^u xdx + \int_u^{1-u} xdx}.$$

And for this, it suffices to show that

$$\frac{\int_0^u x\phi(x)dx}{\int_0^u xdx} \leqslant \frac{\int_u^{1-u} x\phi(x)dx}{\int_u^{1-u} xdx}.$$

But this is immediate, since ϕ is increasing on $[0,1/2]$ and $\phi(u) = \phi(1-u)$:

$$\frac{\int_0^u x\phi(x)dx}{\int_0^u xdx} \leqslant \frac{\phi(u)\int_0^u xdx}{\int_0^u xdx} = \phi(u) = \frac{\phi(u)\int_u^{1-u} xdx}{\int_u^{1-u} xdx} \leqslant \frac{\int_u^{1-u} x\phi(x)dx}{\int_u^{1-u} xdx}. \qquad \square$$

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(Received January 15, 2016)

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